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CONTENTS

CHAPTER 1

DEFORMATION QUANTIZATION FROM FUNCTIONAL INTEGRALS

The aim of this Chapter is to explain how to obtain Kontsevich's formula [**7**] from the perturbative computation of the functional integral of a topological field theory known as the Poisson sigma model. We start with an introduction to the perturbative evaluation of functional integrals. We describe next how to do it in the presence of symmetries generated by the free action of a Lie algebra. This allows the full treatment of the Poisson sigma model for an affine Poisson structure. For the general case, we refer to [**5**].

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1.1. Motivation: the Kontsevich formula

Kontsevich proposed in [**7**] a formula for the deformation quantization of a Poisson structure on $\mathbb{R}^n$. This formula is given by a sum over certain graphs of a bidifferential operator and a real number associated to each graph. A typical graph is displayed in figure 1. The vertices are free to move in the upper half plane but have to avoid each

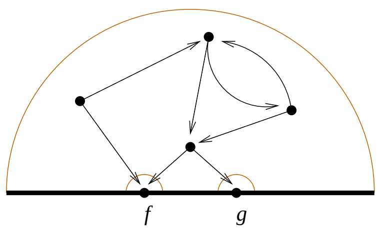

FIGURE 1. A term in the Kontsevich formula.

other and the two points on the boundary where one places the functions whose star product is wished. The number associated to the graph is obtained by integrating certain one-forms associated to the edges over the positions of the vertices. Deforming the integration domain as in figure 2, where the three boundary segments correspond

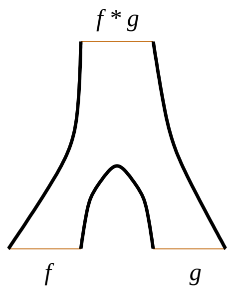

FIGURE 2. The Kontsevich formula as a string evolution.

to the half circles around the two boundary points and infinity in the previous picture, we see that Kontsevich formula is related to some sort of open string theory. The associativity of the star product can then be expressed by saying that the two string evolutions in figure 3 are equivalent. Thus, the dynamics of this peculiar string theory

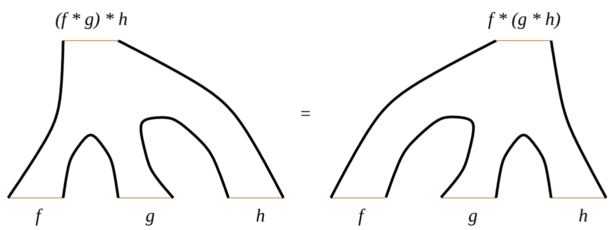

FIGURE 3. Associativity.

must be invariant under diffeomorpshisms of the worldsheet, and this invariance has to persist at the quantum level. A string or field theory with this property is usually referred to as topological.

The aim of this chapter is to give a crash course on the perturbative quantization of field theories via path integrals, with special attention for the case of the topological string theory leading to the Kontsevich formula: the Poisson sigma model.

1.2. Functional integrals and expectation values

A field theory is defined by specifying a space of fields $\mathcal{M}$ (usually a space of maps or, more generally, of sections) and an action function S (a function on $\mathcal{M}$ which is defined by integrating over the source manifold a function of the fields and their jets).

For a given function $\mathcal{O}$ on $\mathcal{M}$, one considers the expectation value

$$\langle\, \mathcal{O}\,\rangle := \frac{\int_{\mathcal{M}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\mathcal{O}}{\int_{\mathcal{M}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}}. \tag{1.2.1}$$

The integrals appearing in the above fraction are called path integrals or functional integrals (the latter term is more general, the former should be reserved for the case of quantum mechanics where the fields are actually paths) and are computed w.r.t. a fictitious measure on the space of fields. Functions on $\mathcal{M}$ whose expectation value is well-defined are called observables.

The perturbative evaluation of such integrals consists in expanding S around a nondegenerate critical point and in defining the integral as a formal power series in the expansion parameter $\hbar$ with coefficients given by Gaussian expectation values (see 1.2.1 and 1.2.3). As a simple application, we will recover in 1.2.2 the Moyal formula from the path-integral formulation of quantum mechanics on $T^*\mathbb{R}^n$.

Often, however, (and in particular in the case of interest for us) the critical point is degenerate due to symmetries, viz., a distribution of vector fields on $\mathcal{M}$ under which the action function S is invariant. When these symmetries are given by the infinitesimal free action of a Lie algebra, there is a nice algebraic way, the BRST formalism [**4, 9**], to describe the perturbative expansion of the functional integral on the quotient space (see Sect. 1.3). Observables in this setting will be understood as *invariant* functions whose expectation values may be defined. This method will allow us to compute Kontsevich's formula in the case of affine Poisson structures (see Sect. 1.5). A more general method, the BV formalism [**3**], allows one to deal with more general distributions of vector fields describing the symmetries; this method (in the version described in [**1**]) is needed, e.g., to obtain Kontsevich's formula for general Poisson structures, see [**5**].

1.2.1. Gaussian integrals. — Let A be a positive-definite symmetric matrix on $\mathbb{R}^n$, which we assume to be endowed with the Lebesgue measure $\mathrm{d}^n x$ and the Euclidean inner product $(\ ,\ )$. Then

$$I(\lambda) := \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{\lambda}{2}(x,\mathsf{A}x)}\,\mathrm{d}^n x = \frac{(2\pi)^{\frac{n}{2}}}{\lambda^{\frac{n}{2}}}\frac{1}{\sqrt{\det \mathsf{A}}},\quad \lambda > 0.$$

We may continue I to the whole complex plane minus the negative real axis. Then we get, e.g.,

$$I(-\mathrm{i}) = (2\pi)^{\frac{n}{2}}\,\mathrm{e}^{\frac{\mathrm{i}\pi n}{4}}\frac{1}{\sqrt{\det \mathsf{A}}},\quad I(\mathrm{i}) = (2\pi)^{\frac{n}{2}}\,\mathrm{e}^{\frac{-\mathrm{i}\pi n}{4}}\frac{1}{\sqrt{\det \mathsf{A}}}.$$

These formulae allow us to define the Gaussian integral when A is negative definite:

$$\int_{\mathbb{R}^n} \mathrm{e}^{\frac{\mathrm{i}}{2}(x,\mathsf{A}x)}\,\mathrm{d}^n x = \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{\mathrm{i}}{2}(-(x,\mathsf{A}x))}\,\mathrm{d}^n x = (2\pi)^{\frac{n}{2}}\,\mathrm{e}^{\frac{-\mathrm{i}\pi n}{4}}\frac{1}{\sqrt{|\det \mathsf{A}|}}.$$

When A is nondegenerate—but not necessarily positive or negative definite—by combining the above we get

$$\int_{\mathbb{R}^n} \mathrm{e}^{\frac{\mathrm{i}}{2}(x,\mathsf{A}x)}\,\mathrm{d}^n x = (2\pi)^{\frac{n}{2}}\,\mathrm{e}^{\frac{\mathrm{i}\pi\,\mathrm{sign}\,\mathsf{A}}{4}}\frac{1}{\sqrt{|\det \mathsf{A}|}}, \tag{1.2.2}$$

where $\operatorname{sign} \mathsf{A}$ denotes the signature of A.

Expectation values w.r.t. a Gaussian distribution, which we will denote by $\langle \ \rangle_0$, are easy to compute. The first step is to define the generating function

$$Z(J) = \int_{\mathbb{R}^n} \mathrm{e}^{\frac{\mathrm{i}}{2}(x,\mathsf{A}x)+(J,x)}\, \mathrm{d}^n x = (2\pi)^{\frac{n}{2}}\, \mathrm{e}^{\frac{\mathrm{i}\pi \operatorname{sign} \mathsf{A}}{4}} \frac{1}{\sqrt{|\det \mathsf{A}|}} \mathrm{e}^{\frac{\mathrm{i}}{2}(J,\mathsf{A}^{-1}J)}.$$

Then we can write

$$(1.2.3) \quad \left\langle\, x^{i_1} \dots x^{i_k} \,\right\rangle_0 = \frac{\int_{\mathbb{R}^n} \mathrm{e}^{\frac{\mathrm{i}}{2}(x,\mathsf{A}x)}\, x^{i_1} \dots x^{i_k}\, \mathrm{d}^n x}{\int_{\mathbb{R}^n} \mathrm{e}^{\frac{\mathrm{i}}{2}(x,\mathsf{A}x)}\, \mathrm{d}^n x} = \\ = \frac{\frac{\partial}{\partial J^{i_1}} \cdots \frac{\partial}{\partial J^{i_k}} Z(J)|_{J=0}}{Z(0)} = \frac{\partial}{\partial J^{i_1}} \cdots \frac{\partial}{\partial J^{i_k}} \mathrm{e}^{\frac{\mathrm{i}}{2}(J,\mathsf{A}^{-1}J)}|_{J=0}.$$

Observe then that $\left\langle\, x^{i_1} \dots x^{i_k} \,\right\rangle_0$ vanishes if k is odd and is a sum of products of matrix elements of the inverse of A if k is even. For example, for $k = 2$ we have $\left\langle\, x^i x^j \,\right\rangle_0 = \mathrm{i}(A^{-1})^{ij}$, and for $k = 2s$

$$\left\langle\, x^{i_1} \dots x^{i_{2s}} \,\right\rangle_0 = \mathrm{i}^s \sum_{\sigma \in \mathfrak{S}_{2s}} \frac{1}{2^s s!} (A^{-1})^{i_{\sigma(1)} i_{\sigma(2)}} \cdots (A^{-1})^{i_{\sigma(2s-1)} i_{\sigma(2s)}},$$

where $\mathfrak{S}_{2s}$ is the symmetric group on $2s$ elements. This formula may be simplified if we sum only over pairings, viz., permutations $\sigma \in \mathfrak{S}_{2s}$ with the property that $\sigma(2i-1) < \sigma(2i)$, $i = 1, \dots, s$, and $\sigma(1) < \sigma(3) < \cdots < \sigma(2s-3) < \sigma(2s-1)$. Denoting by $P(s)$ the set of pairings of $2s$ elements, we then have the so-called Wick theorem:

$$(1.2.4) \qquad \left\langle\, x^{i_1} \dots x^{i_{2s}} \,\right\rangle_0 = \mathrm{i}^s \sum_{\sigma \in P(s)} (A^{-1})^{i_{\sigma(1)} i_{\sigma(2)}} \cdots (A^{-1})^{i_{\sigma(2s-1)} i_{\sigma(2s)}}.$$

1.2.1.1. Infinite dimensions. — The extension to the infinite-dimensional case is done by taking formula (1.2.4) verbatim. This makes sense whenever the symmetric operator A is invertible.(1) Usually A is a differential operator and in this case $G = \mathsf{A}^{-1}$ will denote the distributional kernel of its inverse, i.e., its Green function. Thus, for example, if A is a differential operator on functions on some manifold Σ, then

$$\langle\, \phi(x_1) \dots \phi(x_{2s}) \,\rangle_0 = \frac{\int \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\int_\Sigma \phi\, \mathsf{A}\phi}\, \phi(x_1) \dots \phi(x_{2s})\, \mathrm{D}\phi}{\int \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\int_\Sigma \phi\, \mathsf{A}\phi}\, \mathrm{D}\phi} := \\ := (\mathrm{i}\hbar)^s \sum_{\sigma \in P(s)} G(x_{\sigma(1)}, x_{\sigma(2)}) \cdots G(x_{\sigma(2s-1)}, x_{\sigma(2s)}),$$

where ϕ denotes a function on Σ, $\hbar$ is a parameter that we have introduced for further convenience, $\mathrm{D}\phi$ denotes the "formal Lebesgue measure" on the space of functions, and the points $x_1, \dots, x_{2s}$ are assumed to be all distinct.

(1) When it is possible to define the determinant and the signature of the invertible operator A, one can also define the "normalized" Gaussian integral by (1.2.2) dropping the irrelevant constant "$(2\pi)^{\frac{\infty}{2}}$".

1.2.1.1.1. Normal ordering. — The formula is usually extended also to the singular case when some points coincide by restricting the sum to pairings with the property that $x_{\sigma(2i-1)} \neq x_{\sigma(2i)}$ $\forall i$. This prescription often goes under the name of normal ordering as it corresponds to the usual normal ordering in the operator formulation of Gaussian field theories.

1.2.1.1.2. The propagator. — As we have seen, in Gaussian integrals all expectation values are given in terms of the expectation value of quadratic monomials. These are usually called two-point functions or propagators. In the case of a Gaussian quantum field theory defined by a differential operator, propagator is thus just another name for the Green function.

1.2.1.1.3. Derivatives. — By linearity one also extends the definition of expectation values to derivatives of the field ϕ. Namely, for multiindices $I_1, \dots, I_{2s}$ one sets

$$\begin{aligned} (1.2.5) \quad \langle\, \partial_{I_1}\phi(x_1)\dots\partial_{I_{2s}}\phi(x_{2s})\,\rangle_0 &= \\ &= (\mathrm{i}\hbar)^s \frac{\partial^{|I_1|}}{\partial x_1^{I_1}} \cdots \frac{\partial^{|I_{2s}|}}{\partial x_{2s}^{I_{2s}}} \sum_{\sigma\in P(s)} G(x_{\sigma(1)}, x_{\sigma(2)}) \cdots G(x_{\sigma(2s-1)}, x_{\sigma(2s)}), \end{aligned}$$

where the derivatives on the r.h.s. are in the distributional sense.

1.2.1.2. A special case. — Let $n = 2m$ and assume

$$\mathsf{A} = \begin{pmatrix} 0 & \mathsf{B} \\ \mathsf{B}^T & 0 \end{pmatrix},$$

where B is a nondegenerate $m\times m$-matrix and B^T denotes its transpose. By regarding $x \in \mathbb{R}^{2m}$ as $y \oplus z$, with $y, z \in \mathbb{R}^m$, we may compute

$$\int_{\mathbb{R}^{2m}} \mathrm{e}^{\mathrm{i}(y,\mathsf{B}z)}\, \mathrm{d}^m y\, \mathrm{d}^m z = \int_{\mathbb{R}^{2m}} \mathrm{e}^{\frac{\mathrm{i}}{2}(x,\mathsf{A}x)}\, \mathrm{d}^{2m} x = (2\pi)^m \frac{1}{|\det \mathsf{B}|},$$

and, for $K, L \in \mathbb{R}^m$,

$$Z(K,L) = \int_{\mathbb{R}^{2m}} \mathrm{e}^{\mathrm{i}(y,\mathsf{B}z)+(K,y)+(L,z)}\, \mathrm{d}^m y\, \mathrm{d}^m z = (2\pi)^m \frac{1}{|\det B|} \mathrm{e}^{\mathrm{i}(L,\mathsf{B}^{-1}K)}.$$

In this case, the expectation value of a monomial vanishes if the degree in y is different from the degree in z and is a sum of products of matrix elements of B^{-1} otherwise:

$$(1.2.6) \qquad \langle\, y^{i_1}\dots y^{i_s} z^{j_1} \dots z^{j_s}\,\rangle_0 = \mathrm{i}^s \sum_{\sigma\in\mathfrak{S}_s} (B^{-1})^{j_{\sigma(1)} i_1} \cdots (B^{-1})^{j_{\sigma(s)} i_s}.$$

The infinite-dimensional generalization follows the same lines as before. For an example, see 1.2.2.

1.2.1.3. Grassmann variables. — A theory of integration, and in particular of Gaussian integration, may be developed also for Grassmann variables. Let V be a vector space. Its algebra of functions is the completion of the symmetric algebra SV^*, i.e., the algebra of polynomials. Its odd counterpart is the exterior algebra ΛV^* which is regarded as the algebra of functions on the odd vector space ΠV. Choosing a basis and an orientation, one may identify $\Lambda^{\mathrm{top}} V^*$ with $\mathbb{R}$. Composing this isomorphism with the projection $\Lambda V^* \to \Lambda^{\mathrm{top}} V^*$ defines a linear map $\Lambda V^* \to \mathbb{R}$ that we denote by $\int_{\Pi V}$ and call the integral on ΠV.

Let now B be an endomorphism of V. We may regard B as an element of $V^* \otimes V$ and so as a function on $\Pi V^* \times \Pi V := \Pi(V^* \oplus V)$. Up to a sign, there is a natural identification of $\Lambda^{\mathrm{top}}(V^* \oplus V)$ with $\mathbb{R}$. Then it is not difficult to see that (up to a sign, which we fix from now on to agree with this formula)

$$\int_{\Pi V^* \times \Pi V} \mathrm{e}^{\mathsf{B}} = \det \mathsf{B}.$$

If B is nondegenerate, one defines the expectation value of a function f on $\Pi V^* \times \Pi V$ by

$$\langle\, f \,\rangle_0 := \frac{\int_{\Pi V^* \times \Pi V} \mathrm{e}^{\mathsf{B}} f}{\int_{\Pi V^* \times \Pi V} \mathrm{e}^{\mathsf{B}}}.$$

Let us now choose a basis $\{e_i\}$ of V and denote by $\{\bar{e}^i\}$ its dual basis. Then $\Lambda(V^* \oplus V)$ may be identified with the Grassmann algebra generated by the anticommuting "coordinate functions" $\bar{e}^i$ and e_j. Functions on $\Pi V^* \times \Pi V$ are then linear combinations of monomials $e_{j_1} \cdots e_{j_r} \bar{e}^{i_1} \cdots \bar{e}^{i_s}$. To the endomorphism B we associate the function $\langle\, \bar{e}\,, \mathsf{B}e \,\rangle = \bar{e}^j B^i_j e_i$, where $\langle\ \ ,\ \ \rangle$ denotes the canonical pairing between V^* and V. The above formulae are then often written as

$$\int \mathrm{e}^{\langle\, \bar{e}\,, \mathsf{B}e \,\rangle} = \det \mathsf{B},$$

$$\big\langle\, e_{j_1} \cdots e_{j_r} \bar{e}^{i_1} \cdots \bar{e}^{i_s} \,\big\rangle_0 = \frac{\int \mathrm{e}^{\langle\, \bar{e}\,, \mathsf{B}e \,\rangle}\, e_{j_1} \cdots e_{j_r} \bar{e}^{i_1} \cdots \bar{e}^{i_s}}{\int \mathrm{e}^{\bar{e}^j B^i_j e_i}}.$$

Exercise 1.2.1. — Prove that $\big\langle\, e_{j_1} \cdots e_{j_r} \bar{e}^{i_1} \cdots \bar{e}^{i_s} \,\big\rangle_0$ is zero if $r \neq s$ and that

$$\big\langle\, e_{j_1} \cdots e_{j_s} \bar{e}^{i_s} \cdots \bar{e}^{i_1} \,\big\rangle_0 = \sum_{\sigma \in \mathfrak{S}_s} \operatorname{sign} \sigma \, (B^{-1})^{i_{\sigma(1)}}_{j_1} \cdots (B^{-1})^{i_{\sigma(s)}}_{j_s}.$$

This formula is the odd counterpart to (1.2.6) and may also be extended to the infinite-dimensional case.

1.2.1.3.1. Vector fields. — A vector field on ΠV is by definition a graded derivation of the algebra ΛV^*. Namely, we say that an endomorphism X of ΛV^* is a vector field of degree $|X|$ if

$$X(fg) = X(f)g + (-1)^{|X|r} f X(g), \qquad \forall f \in \Lambda^r V^*,\ \forall g \in \Lambda V^*,\ \forall r.$$

One may also define a right vector field X of degree $|X|$ to be an endomorphism $f \mapsto (f)X$ of ΛV^* that satisfies

$$(fg)X = f(g)X + (-1)^{|X|s} (f)Xg, \qquad \forall f \in \Lambda V^*,\ \forall g \in \Lambda^s V^*,\ \forall s.$$

The vector space of all vector fields on ΠV may be identified with $\Lambda V^* \otimes V$, elements of V being constant vector fields. Integration has the natural property that

$$\int_{\Pi V} X(f) = 0, \qquad \forall f,$$

if X is a constant vector field. In general, one defines the divergence $\operatorname{div} X$ of X by the formula

$$\int_{\Pi V} X(f) = \int_{\Pi V} \operatorname{div} X\, f, \qquad \forall f.$$

Exercise 1.2.2. — Given the vector field $X = g \otimes v$, $g \in \Lambda^r V^*$, $v \in V$, show that $\operatorname{div} X = (-1)^{r+1} \iota_v g$, where ι denotes contraction.

1.2.2. The Moyal star product from path integrals. — We want to quantize $T^*\mathbb{R}^n$ using the techniques described in 1.2.1. Let q denote a point in $\mathbb{R}^n$ (in coordinates we will write $q^1, \dots q^n$) and let p denote a covector (in coordinates we will write $p_1, \dots p_n$). Classical mechanics is defined in terms of the canonical symplectic form $\omega = \mathrm{d}p_i \mathrm{d}q^i$ or, more precisely, of its potential $\theta = p_i \mathrm{d}q^i$ (sums over repeated indices are from now on understood). Given a path $\gamma\colon I \to T^*\mathbb{R}^n$—$I$ is a one-manifold—the action function is defined by $S(\gamma) = \int_I \gamma^*\theta$. If we write $\gamma(t) = (Q(t), P(t))$, $t \in I$, we may also write

$$S(Q,P) = \int_{t\in I} P_i \frac{\mathrm{d}}{\mathrm{d}t} Q^i \, \mathrm{d}t. \tag{1.2.7}$$

If a Hamiltonian function H is given, one then deforms the action function to

$$S_H(Q,P) = \int_{t\in I} \left(P_i \frac{\mathrm{d}}{\mathrm{d}t} Q^i + H(Q(t), P(t), t) \right) \mathrm{d}t, \tag{1.2.8}$$

but we will not consider this case now (see 1.2.2.6).

From now on we choose the one-manifold I to be S^1. In order to make the quadratic form nondegenerate, we also choose a base point $\infty \in S^1$ and prescribe the value of the path at the base point. Setting

$$\mathcal{M} = \{(Q,P) \in C^\infty(S^1, T^*\mathbb{R}^n)\}$$

and

$$\mathcal{M}(q,p) = \{(Q,P) \in C^\infty(S^1, T^*\mathbb{R}^n) : Q(\infty) = q,\ P(\infty) = p\},$$

we may define the path integral by imposing Fubini's theorem in the form

$$\int_{\mathcal{M}} \dots := \int_{(q,p)\in T^*\mathbb{R}^n} \mu(q,p) \int_{\mathcal{M}(q,p)} \dots \tag{1.2.9}$$

where we have chosen a measure μ on $T^*\mathbb{R}^n$. Observe that the quadratic form in S is nondegenerate when restricted to $\mathcal{M}(q,p)$. Thus, we may compute

$$\langle\, \mathcal{O} \,\rangle_0(q,p) := \frac{\int_{\mathcal{M}(q,p)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S}\, \mathcal{O}}{\int_{\mathcal{M}(q,p)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S}}$$

where $\mathcal{O}$ is some function on $\mathcal{M}$ that is a polynomial or is a formal power series in Q and P. As the denominator, though infinite, is constant, we may improve (1.2.9) to

$$\langle\, \mathcal{O} \,\rangle_0 := \frac{\int_{(q,p)\in T^*\mathbb{R}^n} \mu(q,p)\, \langle\, \mathcal{O} \,\rangle_0(q,p)}{\int_{(q,p)\in T^*\mathbb{R}^n} \mu(q,p)}$$

in the case when the functions $\langle\, \mathcal{O}\, \rangle_0(q,p)$ and 1 are integrable. The latter condition prevents us from choosing μ to be the Liouville measure $\omega^n/n!$. However, at this point we may also forget this denominator and define the expectation value to be

$$\langle\, \mathcal{O}\, \rangle_0' := \int_{(q,p)\in T^*\mathbb{R}^n} \mu(q,p)\, \langle\, \mathcal{O}\, \rangle_0(q,p)$$

so that the Liouville measure is allowed. Observe finally that we may also choose μ to be the delta measure peaked at a point $(q,p)\in T^*\mathbb{R}^n$. In this case we have $\langle\, \mathcal{O}\, \rangle_0 = \langle\, \mathcal{O}\, \rangle_0' = \langle\, \mathcal{O}\, \rangle_0(q,p)$.

Once $(q,p)\in T^*\mathbb{R}^n$ has been fixed, we make the "change of variables" $Q = q+\widetilde{Q}$ and $P = p+\widetilde{P}$, where $(\widetilde{Q},\widetilde{P})$ is a map from S^1 to $T^*\mathbb{R}^n$ that vanishes at ∞, which is the same as a map $\mathbb{R}\to T^*\mathbb{R}^n$ that vanishes at infinity. The action function then reads

$$S(Q,P) = S(q+\widetilde{Q}, p+\widetilde{P}) = \int_{\mathbb{R}} \widetilde{P}_i \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Q}^i\, \mathrm{d}t.$$

Expectation values may be computed by the formula

$$\langle\, \mathcal{O}(Q,P)\, \rangle_0(q,p) \;=\; \left\langle\, \mathcal{O}(q+\widetilde{Q}, p+\widetilde{P})\, \right\rangle_0^{\sim} \;:=\; \frac{\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\, \mathcal{O}(q+\widetilde{Q}, p+\widetilde{P})\, \mathrm{D}\widetilde{P}\mathrm{D}\widetilde{Q}}{\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\, \mathrm{D}\widetilde{P}\mathrm{D}\widetilde{Q}}.$$

1.2.2.1. The propagator. — The nondegenerate quadratic form is of the type considered in 1.2.1.2. The Green function of the skew-symmetric operator $\frac{\mathrm{d}}{\mathrm{d}t}$ is one-half of the sign function:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-1}(u,v) = \theta(u-v) = \frac{1}{2}\operatorname{sig}(u-v) = \begin{cases} \frac{1}{2} & \text{if } u>v,\\ -\frac{1}{2} & \text{if } u<v.\end{cases}$$

As a consequence, by (1.2.6), we get

$$\left\langle\, \widetilde{P}_i(u)\, \widetilde{Q}^j(v)\, \right\rangle_0^{\sim} = \mathrm{i}\hbar\theta(v-u)\delta_i^j$$

and, more generally,

$$\left\langle\, \widetilde{P}_{i_1}(u_1)\cdots\widetilde{P}_{i_s}(u_s)\, \widetilde{Q}^{j_1}(v_1)\cdots\widetilde{Q}^{j_s}(v_s)\, \right\rangle_0^{\sim} = = (\mathrm{i}\hbar)^s \sum_{\sigma\in\mathfrak{S}} \theta(v_{\sigma(1)}-u_1)\cdots\theta(v_{\sigma(s)}-u_s)\, \delta_{i_1}^{j_{\sigma(1)}}\cdots\delta_{i_s}^{j_{\sigma(s)}}. \tag{1.2.10}$$

1.2.2.1.1. Normal ordering. — The normal ordering prescription may now be implemented by setting $\theta(0)=0$, also in agreement with the skew-symmetry of $\frac{\mathrm{d}}{\mathrm{d}t}$.

1.2.2.2. Expectation values. — An example of an observable on $\mathcal{M}$ is given by the evaluation of a smooth function f on $T^*\mathbb{R}^n$ at some point in the path; viz., we set

$$\mathcal{O}_{f;u}(Q,P) := f(Q(u),P(u)), \qquad f\in C^\infty(T^*\mathbb{R}^n),\ u\in S^1\setminus\{\infty\}.$$

To compute its expectation value, we have first to introduce some notations. Given a multiindex $I=(i_1,\dots,i_r)$, we set $|I|=r$, $p_I = p_{i_1}\cdots p_{i_r}$, $q^I = q^{i_1}\cdots q^{i_r}$ (and similarly for $\widetilde{P}_I$ and $\widetilde{Q}^I$),

$$\partial_I = \frac{\partial}{\partial q^{i_1}} \cdots \frac{\partial}{\partial q^{i_r}}, \qquad \partial^I = \frac{\partial}{\partial p_{i_1}} \cdots \frac{\partial}{\partial p_{i_r}}.$$

We extend to the case $|I| = 0$ by setting $p_I = q^I = 1$ and $\partial_I = \partial^I =$ identity operator. Performing the change of variables described above and using the Taylor expansion of f, we get

$$f(Q(u), P(u)) = f(q + \widetilde{Q}(u), p + \widetilde{P}(u)) = \\ = \sum_{r,s=0}^{\infty} \frac{1}{r!s!} \sum_{|I|=r,\ |J|=s} \widetilde{P}_I(u)\widetilde{Q}^J(u)\, \partial^I \partial_J f(q,p).$$

Thus,

$$\langle\ \mathcal{O}_{f;u}\ \rangle_0(q,p) = \left\langle\ \mathcal{O}_{f,u}(q + \widetilde{Q}, p + \widetilde{P})\ \right\rangle_0^{\sim} = f(q,p),$$

for the expectation values of $\widetilde{Q}$s and $\widetilde{P}$s at coinciding points vanish by the normal ordering prescription, see 1.2.2.1.1. If f is integrable, we may also define

$$\langle\ \mathcal{O}_{f,u}\ \rangle_0' = \int_{T^*\mathbb{R}^n} \mu(q,p)\, f(q,p).$$

Observe that these expectation values do not depend on the point u. We will give in 1.2.2.4 an interpretation of this fact.

The next observable we want to consider is more interesting; viz.,

$$\mathcal{O}_{f,g;u,v} = f(Q(u), P(u))\, g(Q(v), P(v)), \\ f, g \in C^\infty(T^*\mathbb{R}^n),\ u, v \in S^1 \setminus \{\infty\} \cong \mathbb{R},\ u < v.$$

By repeating the analysis above, we may compute its expectation value as follows:

$$\begin{aligned}(1.2.11)\quad \langle\ \mathcal{O}_{f,g;u,v}\ \rangle_0(q,p) &= \left\langle\ f(q + \widetilde{Q}(u), p + \widetilde{P}(u))\, g(q + \widetilde{Q}(v), p + \widetilde{P}(v))\ \right\rangle_0^{\sim} = \\ &= \sum_{r_1,s_1,r_2,s_2=0}^{\infty} \frac{1}{r_1!s_1!r_2!s_2!} \sum_{|I_1|=r_1,\ |J_1|=s_1} \sum_{|I_2|=r_2,\ |J_2|=s_2} \\ &\quad \left\langle\ \widetilde{P}_{I_1}(u)\widetilde{Q}^{J^1}(u)\widetilde{P}_{I_2}(v)\widetilde{Q}^{J_2}(v)\ \right\rangle_0^{\sim} \partial^{I_1}\partial_{J_1} f(q,p)\, \partial^{I_2}\partial_{J_2} g(q,p) = \\ &= \sum_{r,s=0}^{\infty} \frac{1}{r!s!} \left(\frac{\mathrm{i}\hbar}{2}\right)^{r+s} (-1)^s \sum_{|I|=r,\ |J|=s} \partial^I \partial_J f(q,p)\, \partial^J \partial_I g(q,p) = f \star g(q,p),\end{aligned}$$

where we have used (1.2.10) with the normal ordering prescription and $\star$ denotes the Moyal star product for the Poisson bivector field $\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$. Observe that again the expectation value is independent of the points u and v. If $f \star g$ is integrable, we may also compute

$$\langle\ \mathcal{O}_{f,g;u,v}\ \rangle_0' = \int_{T^*\mathbb{R}^n} \mu\, f \star g.$$

If μ is the Liouville measure $\omega^n/n!$, this defines the trace

$$\operatorname{Tr}(f \star g) = \int_{T^*\mathbb{R}^n} \frac{\omega^n}{n!}\, f \star g = \int_{T^*\mathbb{R}^n} \frac{\omega^n}{n!}\, fg,$$

where the second equality simply follows by using integration by parts in the correction terms to the commutative product. We may now continue along this line and define, e.g.,

$$\mathcal{O}_{f_1,\dots,f_k;u_1,\dots,u_k} = f_1(Q(u_1),P(u_1)) \cdots f_k(Q(u_k),P(u_k)),$$
$$f_1,\dots,f_k \in C^\infty(T^*\mathbb{R}^n),\ u_1,\dots,u_k \in S^1 \setminus \{\infty\} \cong \mathbb{R},\ u_1 < \cdots < u_k.$$

We leave it as an exercise (see also 1.2.2.5) to prove that

$$\langle\, \mathcal{O}_{f_1,\dots,f_k;u_1,\dots,u_k} \,\rangle_0(q,p) = f_1 \star \cdots \star f_k(q,p).$$

1.2.2.3. Divergence of local vector fields. — An important consequence of the normal ordering prescription of 1.2.2.1.1 is that local vector fields are divergence free. Recall that a vector at $(\widetilde{Q},\widetilde{P}) \in \mathcal{M}(q,p)$ may be identified with a smooth map $\mathbb{R} \to T^*\mathbb{R}^n$ that vanishes at infinity. A vector field X on $\mathcal{M}(q,p)$ is then the assignment of a map $X(\widetilde{Q},\widetilde{P})$ to each path $(\widetilde{Q},\widetilde{P})$. We say that the vector field is local if $X(\widetilde{Q},\widetilde{P})(t)$ is a function of $\widetilde{Q}(t)$ and $\widetilde{P}(t)$ for all $t \in \mathbb{R}$.

The definition of the divergence of a vector field in a path integral mimics the one in ordinary integrals, restricted to the cases where it makes sense. Formally we would like to have

$$\int_{\mathcal{M}(q,p)} X\left(\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\mathcal{O}\right) = \int_{\mathcal{M}(q,p)} \operatorname{div} X\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\mathcal{O},$$

for all observables $\mathcal{O}$. The correct requirement can then be expressed in terms of expectation values.

Definition 1.2.3. — If there exists an observable $\operatorname{div} X$ such that for any observable $\mathcal{O}$ we have

$$\langle\, X(S)\,\mathcal{O} \,\rangle_0(q,p) = \mathrm{i}\hbar\langle\, X(\mathcal{O}) \,\rangle_0(q,p) - \mathrm{i}\hbar\langle\, \operatorname{div} X\, \mathcal{O} \,\rangle_0(q,p),$$

then we say that $\operatorname{div} X$ is the divergence of X.

Now we have the

Lemma 1.2.4. — *If X is a local vector field, then* $\operatorname{div} X = 0$.

Proof. — We write $X(\widetilde{Q},\widetilde{P})(t) = (X_q(\widetilde{Q}(t),\widetilde{P}(t)), X_p(\widetilde{Q}(t),\widetilde{P}(t)))$. Then

$$X(S) = \int \left(X_p \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Q} - X_q \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{P} \right) \mathrm{d}t.$$

By the normal ordering prescription and the locality of X, in the computation of $\langle\, X(S)\,\mathcal{O} \,\rangle_0(q,p)$ we have only to contract the $\widetilde{P}$s ($\widetilde{Q}$s) in $\mathcal{O}$ with the $\widetilde{Q}$s ($\widetilde{P}$s) in $X(S)$ and replace each pair by a propagator. Whenever a $\widetilde{P}$ ($\widetilde{Q}$) in $\mathcal{O}$ is contracted with the $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Q}$ ($\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{P}$) in $X(S)$, we get the identity operator times $\mathrm{i}\hbar$ ($-\mathrm{i}\hbar$). Summing up the various terms, we see that this is the same as taking the expectation value of $X(\mathcal{O})$ multiplied by $\mathrm{i}\hbar$, which completes the proof. $\square$

We may also consider the divergence of vector fields on $\mathcal{M}$. In particular we observe that a local vector field X on $\mathcal{M}$ may be written uniquely as the sum of a vector field X_∞ on $T^*\mathbb{R}^n$ and a section $\widetilde{X}$ of local vector fields (i.e., the assignment of a local vector field $\widetilde{X}(q,p)$ on $\mathcal{M}(q,p)$ to each $(q,p) \in T^*\mathbb{R}^n$). The above Lemma implies

$$\langle\, X(S)\,\mathcal{O}\,\rangle_0' = \mathrm{i}\hbar\langle\, X(\mathcal{O})\,\rangle_0' - \mathrm{i}\hbar\langle\, \mathrm{div}_\mu X_\infty\,\mathcal{O}\,\rangle_0',$$

where $\mathrm{div}_\mu X_\infty$ is the ordinary divergence of the vector field X_∞ w.r.t. the measure μ. We may then say that $\mathrm{div}_\mu X_\infty$ is the divergence of the local vector field X on $\mathcal{M}$.

Exercise 1.2.5. — Prove that

$$\hbar\frac{\mathrm{d}}{\mathrm{d}\hbar}(f\star g) = (p_i\partial^i f)\star g + f\star(p_i\partial^i g) - p_i\partial^i(f\star g).$$

Hint: Consider the local vector field $X(\widetilde{P},\widetilde{Q})(t) := (\widetilde{P}(t),0)$.

1.2.2.4. Independence from the evaluation points. — In the above computations we have considered observables whose definition relies on the choice of points on $S^1\setminus\{\infty\}$, but it has always turned out that expectation values are independent of this choice. In other words expectation values are invariant under (pointed) diffeomorphisms of the source manifold S^1 on which the field theory is defined. A quantum field theory with this property is usually called a topological quantum field theory TQFT as it depends only on the topological data of the source manifold. In our case we see the topological nature of the theory at the classical level as the action function (1.2.7) is invariant under diffeomorphisms. The fact that this invariance survives at the quantum level indicates that the fictitious measure in the path integral is also diffeomorphism invariant.

We may convince ourselves of the independence of the expectation values from the points where we evaluate the functions if we just recall that the propagator θ is a locally constant function. There exists however another way to obtain this result which will help our intuition in more complicated situations, e.g., in those considered in Sect. 1.5. Given $f \in C^\infty(T^*\mathbb{R}^n)$ and two points a and b in $\mathbb{R}$, we observe that

$$\begin{aligned} f(Q(b),P(b)) - f(Q(a),P(a)) &= \\ &= \int_a^b \left(\partial_i f(Q(t),P(t))\frac{\mathrm{d}}{\mathrm{d}t}Q^i(t) + \partial^i f(Q(t),P(t))\frac{\mathrm{d}}{\mathrm{d}t}P_i(t)\right)\,\mathrm{d}t = \\ \lim_{r\to\infty}\int_{-\infty}^{+\infty} &\left(\partial_i f(Q(t),P(t))\frac{\mathrm{d}}{\mathrm{d}t}Q^i(t) + \partial^i f(Q(t),P(t))\frac{\mathrm{d}}{\mathrm{d}t}P_i(t)\right)\lambda_r(t)\,\mathrm{d}t = \\ &= \lim_{r\to\infty}\widetilde{X}_{f,r}(S) \end{aligned}$$

where $\{\lambda_r\}$ is a sequence of smooth, compactly supported functions that converges almost everywhere to the characteristic function of the interval $[a,b]$ and $\widetilde{X}_{f,r}$ is the local vector field(2)

$$\widetilde{X}_{f,r}(t) = (-\partial^i f(Q(t),P(t)), \partial_i f(Q(t),P(t)))\lambda_r(t).$$

We may now restrict $\widetilde{X}_{f,r}$ to $\mathcal{M}(q,p)$ and observe that, by Lemma 1.2.4, it is divergence free. Thus,

$$\begin{aligned}\langle\, (f(Q(b),P(b)) - f(Q(a),P(a)))\, \mathcal{O}\, \rangle_0(q,p) =&\\ = \lim_{r\to\infty} \left\langle\, \widetilde{X}_{f,r}(S)\, \mathcal{O}\, \right\rangle_0&(q,p) = \mathrm{i}\hbar \lim_{r\to\infty} \left\langle\, \widetilde{X}_{f,r}(\mathcal{O})\, \right\rangle_0(q,p),\end{aligned}$$

which vanishes if $\mathcal{O}$ does not depend on $(\widetilde{Q}(t), \widetilde{P}(t))$ for $t \in [a,b]$. This means that we are free to move the evaluation point at least as long as we do not meet another evaluation point.

1.2.2.5. Associativity. — The associativity of the Moyal star product is not difficult to prove using its explicit formula. It may also be proved from its path-integral representation considering the expectation value of $\mathcal{O}_{f,g,h;u,v,w}$. In its computation, there appear three kinds of propagators, corresponding to the three possible pairings of the points u, v, w. However, the function θ does not see the difference as $\theta(w-u) = \theta(w-v) = \theta(v-u) = 1/2$. We may now decide to group the propagators by considering first only those between u and v and only thereafter the other ones. It is not difficult to see that this implies that $\langle\, \mathcal{O}_{f,g,h;u,v,w}\, \rangle_0 = (f \star g) \star h$. On the other hand, we may also group first the propagators between v and w and then the others, thus obtaining $\langle\, \mathcal{O}_{f,g,h;u,v,w}\, \rangle_0 = f \star (g \star h)$. This proves associativity.

There is also a formal argument to prove it—whose generalization to more complex cases (see, e.g., Sect. 1.5) will help our intuition—that relies on the independence of the evaluation points. This in fact implies that

$$\lim_{v\to u^+} \langle\, \mathcal{O}_{f,g,h;u,v,w}\, \rangle_0(q,p) = \lim_{v\to w^-} \langle\, \mathcal{O}_{f,g,h;u,v,w}\, \rangle_0(q,p).$$

The l.h.s. corresponds then to evaluating first the expectation value of $\mathcal{O}_{f,g;u,v}$, placing the result at u and finally computing the expectation value of $\mathcal{O}_{\langle\, \mathcal{O}_{f,g;u,v}\, \rangle_0, h; w}$. The result is then $(f \star g) \star h$. Repeating the computation on the r.h.s. and using the above identity, we get the associativity of the star product.

1.2.2.6. The evolution operator. — As an application we want now to consider the formal path integral for the full action function (1.2.8). For simplicity we restrict ourselves to considering a time-independent Hamiltonian h and let it act from the instant a to the instant b, $a < b$. Thus we consider S_H with

$$H(q,p,t) = h(q,p)\, \chi_{[a,b]}(t),$$

(2)Recall that $\mathcal{M}$ is a space of maps. So a vector field on the target $T^*\mathbb{R}^n$ generates a local vector field on $\mathcal{M}$. Let $\widetilde{X}_f$ be the local vector field corresponding to the Hamiltonian vector field X_f of f. Then $\widetilde{X}_{f,r}$ is obtained by multiplying $\widetilde{X}_f$ by λ_r.

where χ denotes the characteristic function of the interval. We want to compute the *evolution operator*

$$U(q,p,T) := \frac{\int_{\mathcal{M}(q,p)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_H}}{\int_{\mathcal{M}(q,p)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S}}$$

with $T = b - a$. First we observe that

$$U(q,p,T) \quad = \quad \left\langle \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int H(Q(t),P(t),t)\,\mathrm{d}t} \right\rangle_0 (q,p) \quad = \quad \left\langle \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_a^b h(Q(t),P(t))\,\mathrm{d}t} \right\rangle_0 (q,p).$$

Next we regard the integral as a limit of Riemann sums:

$$\int_a^b h(Q(t),P(t))\,\mathrm{d}t = \lim_{N\to\infty} \frac{T}{N} \sum_{r=1}^{N} h(Q(a+rT/N), P(a+rT/N)).$$

Thus we have

$$U(q,p,T) = \lim_{N\to\infty} \left\langle \prod_{r=1}^{N} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{T}{N} h(Q(a+rT/N),P(a+rT/N))} \right\rangle_0 (q,p) =$$
$$\lim_{N\to\infty} \left\langle \mathcal{O}_{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{T}{N}h},\dots,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{T}{N}h};a+\frac{T}{N},a+2\frac{T}{N},\dots,a+T} \right\rangle_0 (q,p) =$$
$$\lim_{N\to\infty} \left(\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{T}{N}h}\right)^{\star N} (q,p) = \exp_\star\left(\frac{\mathrm{i}}{\hbar} h\right)(q,p).$$

Observe that in the above result also negative powers of $\hbar$ appear. The final answer is however well-defined if we observe that each term in the power series expansion of the star exponential is actually a Laurent series in $\hbar$.

1.2.3. Perturbative evaluation of integrals. — Let $\mathcal{M}$ be an n-dimensional manifold and S a smooth function on $\mathcal{M}$. We want to compute the integrals appearing in eq. (1.2.1). We consider first the case when S has a unique critical point $x_0 \in \mathcal{M}$ which is moreover nondegenerate. We write

$$S(x_0 + \sqrt{\hbar}x) = S(x_0) + \frac{\hbar}{2}\mathrm{d}^2_{x_0} S(x) + R_{x_0}(\sqrt{\hbar}x),$$

where R_{x_0} is computed from the Taylor expansion of S and is then a formal power series, starting with the cubic term, in $\sqrt{\hbar}x$ with $x \in \mathrm{T}_{x_0}\mathcal{M}$. We denote by A the Hessian of S at x_0 w.r.t. to the Euclidean metric; i.e., we write $\mathrm{d}^2_{x_0}S(x) = (x, \mathsf{A}x)$. Then the saddle-point approximation to the integral $Z := \int_{\mathcal{M}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\mathrm{d}^n x$ is given by the formula

$$Z = \hbar^{\frac{n}{2}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(x_0)} \int_{\mathcal{M}} \mathrm{e}^{\frac{\mathrm{i}}{2}(x,\mathsf{A}x)} \sum_{r=0}^{\infty} \frac{1}{r!} R^r_{x_0}(\sqrt{\hbar}x)\,\mathrm{d}^n x =$$
$$= (2\pi\hbar)^{\frac{n}{2}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(x_0)} \mathrm{e}^{\frac{\mathrm{i}\pi\,\mathrm{sign}\,\mathsf{A}}{4}} \frac{1}{\sqrt{|\det \mathsf{A}|}} \sum_{r=0}^{\infty} \frac{1}{r!} \left\langle R^r_{x_0}(\sqrt{\hbar}x) \right\rangle_0,$$

where $\langle\ \ \rangle_0$ denotes the Gaussian expectation value w.r.t. the nondegenerate symmetric matrix A. This formula yields an asymptotic expansion of Z as a function of $\hbar$, but for our purposes it will be simply considered as a formal power series[(3)] in $\hbar$ times the oscillatory factor $\mathrm{e}^{\frac{i}{\hbar}S(x_0)}$ that we will shortly drop. In this setting R_{x_0} is regarded as a perturbation to the Gaussian theory defined by $\mathrm{d}^2_{x_0}S$ and the above formula is referred to as the perturbative expansion of the integral. Expectation values, denoted by $\langle\ \ \rangle$ as in (1.2.1), may also be computed perturbatively by

$$\langle\, \mathcal{O}\, \rangle = \frac{\sum_{r=0}^{\infty} \frac{1}{r!}\Big\langle\, \mathcal{O}(x_0+\sqrt{\hbar}x)\, R^r_{x_0}(\sqrt{\hbar}x)\,\Big\rangle_0}{\sum_{r=0}^{\infty} \frac{1}{r!}\Big\langle\, R^r_{x_0}(\sqrt{\hbar}x)\,\Big\rangle_0}. \tag{1.2.12}$$

As the denominator is of the form $1+O(\hbar)$, the ratio can be computed as a formal power series in $\hbar$. The computation may be arranged graphically associating a vertex of valence k to the term of degree k in R_{x_0} and a vertex of degree l to the term of degree l in $\mathcal{O}$. By the Wick theorem (1.2.4), we should then connect in pairs in all possible ways the half-edges emanating from each vertex. As a result we get a collection of graphs—the Feynman diagrams—with a weight associated to each of them, i.e., a number obtained by multiplying the matrix elements of A^{-1} in the way prescribed by the graph times appropriate combinatorial factors and powers of $\hbar$. The expectation value is then a sum over all graphs of their weights. A Feynman diagram with vertices coming only from R_{x_0}—i.e., a Feynman diagram appearing in the denominator of (1.2.12)—is called a vacuum diagram. An easy combinatorial effect of the ratio in (1.2.12) is that an expectation value is given by the sum over graphs which do not have a connected component which is a vacuum diagram. We will not enter details here. The interested reader is referred to one of the many books or lecture notes on the subject (see, e.g., [**10**]).

1.2.3.1. Infinite dimensions. — The infinite-dimensional generalization is again obtained by taking (1.2.12) verbatim when it is possible to make sense of the r.h.s.

***Definition 1.2.6* (Local functions).** — A function on a space of fields on some manifold M is called local if it is the integral on M of a function that depends at each point on finite jets of the fields at that point.

If the action is a local function, A will be a differential operator. In the computations we will need its Green function—the propagator—as explained in 1.2.1.1, and in the Gaussian expectation values of $\mathcal{O}R^r$ and of R^r we will use (1.2.5). Instead of summing over indices we will have to integrate over Cartesian products of the source manifold. The normal ordering will exclude all graphs with an edge whose endpoints are equal (the so-called tadpoles) and will restrict integration to the configuration space of the source manifold (viz., the Cartesian product minus all diagonals). In general, however, this will not be enough to make all integrals converge as the Green functions are usually very singular when their two arguments approach each other. The general technique to deal with this problem goes under the name of

(3) Recall that only monomials of even degrees have a nonvanishing Gaussian expectation value. Thus, in the final formula $\sqrt{\hbar}$ is always raised to an even power.

renormalization. We will not discuss this issue here because *a nice feature of topological field theories*—of which the Poisson sigma model discussed in the next Sections is an example—*is that configuration-space integrals associated to Feynman diagrams without tadpoles converge.*

In case the action function S has more critical points and all of them are nondegenerate, the asymptotic expansion is obtained by computing the saddle-point approximation around each critical point and then summing all contributions. The formula for the expectation value is then no longer as simple as in (1.2.12). It may happen however that a critical value dominates the others so that one can safely forget them. In physical theories this is usually done by regarding $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$ as the analytic continuation of the exponential of minus a positive definite function, the "Euclidean action," so that the dominating critical point is the absolute minimum. There are however examples, as the ones we are going to consider, where there is no way to do it (as all critical points are saddle points). Another option, which is suitable also in these instances, consists in selecting one particular critical point (the "sector") and expanding only around it. In this case, formula (1.2.12) is still the one to be taken to define the infinite-dimensional generalization.

It often happens however that the critical point is degenerate. A simple case is when the critical points are parametrized by a finite-dimensional manifold $\mathcal{M}_{\mathrm{crit}}$. Then—by Fubini's theorem in the finite-dimensional case and by decree in the infinite-dimensional one—one writes, like we did in (1.2.9),

$$\int_{\mathcal{M}} \cdots := \int_{x_0 \in \mathcal{M}_{\mathrm{crit}}} \mu(x_0) \int_{\mathcal{M}(x_0)} \cdots,$$

where $\int_{\mathcal{M}(x_0)}$ denotes the asymptotic expansion of the integral in the complement to $T_{x_0}\mathcal{M}_{\mathrm{crit}}$ of a formal neighborhood of x_0, while μ is a measure on $\mathcal{M}_{\mathrm{crit}}$ (which is determined in the finite-dimensional case and has to be chosen in the infinite-dimensional case as part of the definition). If moreover the Hessian is constant on $\mathcal{M}_{\mathrm{crit}}$, we may easily write expectation values as

$$\langle\, \mathcal{O}\, \rangle = \frac{\int_{x_0 \in \mathcal{M}_{\mathrm{crit}}} \mu(x_0) \sum_{r=0}^{\infty} \frac{1}{r!} \Big\langle\, \mathcal{O}(x_0 + \sqrt{\hbar}x)\, R^r_{x_0}(\sqrt{\hbar}x)\, \Big\rangle_0}{\int_{x_0 \in \mathcal{M}_{\mathrm{crit}}} \mu(x_0) \sum_{r=0}^{\infty} \frac{1}{r!} \Big\langle\, R^r_{x_0}(\sqrt{\hbar}x)\, \Big\rangle_0},$$

where $\langle\quad\rangle_0(x_0)$ denotes the Gaussian expectation value computed expanding around x_0 orthogonally to $T_{x_0}\mathcal{M}_{\mathrm{crit}}$. One possible choice for μ is the delta measure peaked at some point x_0. In this case, consistently with Sect. 1.2.2, the expectation value will be denoted by $\langle\quad\rangle(x_0)$.

1.2.3.2. A simple generalization. — In the perturbative expansion considered above the only expansion parameter was $\hbar$. It may happen that there is another small expansion parameter, or that some coefficient appearing in S is much smaller than $\hbar$ (or better, in our setting, is an element of $\hbar^2\mathbb{R}[[\hbar]]$). In this case, the right prescription is to define the Gaussian part using the quadratic, $\hbar$-independent term of $S/\hbar$ and consider all other terms as the perturbation R. It may thus happen that the perturbation R contains quadratic and linear terms as well. A particular case is when we are in the setting of 1.2.1.2 on page 11 and our action function has the form

$$S(y,z) = (y, \mathsf{B}z) + f(y,z), \tag{1.2.13}$$

where B is a nondegenerate matrix and f is a function quadratic in z. If we work around the critical point $y = z = 0$, we may rescale z by $\hbar$,

$$S(y, \hbar z) = \hbar(y, \mathsf{B}z) + \hbar^2 f(y,z)$$

and consider f as the perturbation to the Gaussian theory defined by B and discussed in 1.2.1.2. Infinite-dimensional generalizations of this are discussed below and in Sect. 1.5.

1.2.3.2.1. Quantum mechanics. — As we have seen in 1.2.2, the topological action (1.2.7) defines the associative Moyal product in terms of the expectation value (1.2.11) which is independent of the evaluation points. If we now introduce a Hamiltonian as in 1.2.2.6, the expectation value

$$f \widehat{\star}_{a,b;u,v} g(q,p) := \frac{\int_{\mathcal{M}(q,p)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_H} \mathcal{O}_{f,g;u,v}}{\int_{\mathcal{M}(q,p)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_H}} = \frac{\left\langle \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_a^b h \mathrm{d}t} \mathcal{O}_{f,g;u,v} \right\rangle_0 (q,p)}{\left\langle \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_a^b h \mathrm{d}t} \right\rangle_0 (q,p)}$$

will be no longer independent from u and v ($a < u < v < b$) nor will it define an associative product. By the computations in 1.2.2.6, we know that the answer is

$$f \widehat{\star}_{a,b;u,v} g = \frac{\exp_\star\left(\frac{\mathrm{i}}{\hbar}(u-a)h\right) \star f \star \exp_\star\left(\frac{\mathrm{i}}{\hbar}(v-u)h\right) \star g \star \exp_\star\left(\frac{\mathrm{i}}{\hbar}(b-v)h\right)}{\exp_\star\left(\frac{\mathrm{i}}{\hbar}(b-a)h\right)}.$$

As an application, we want now to perform this computation perturbatively for a Hamiltonian of the form

$$h(q,p) = \frac{1}{2} G^{ij}(q) p_i p_j,$$

at $p = 0$ and for f and g depending only on q. By setting $Q(t) = q + \widetilde{Q}(t)$ and $P(t) = \hbar \widetilde{P}(t)$, we get

$$S_H = \hbar \int_{-\infty}^{\infty} \widetilde{P}(t) \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{Q}(t)\, \mathrm{d}t + \frac{\hbar^2}{2} \int_a^b G^{ij}(q + \widetilde{Q}(t)) \widetilde{P}_i(t) \widetilde{P}_j(t)\, \mathrm{d}t.$$

Thus, $f \widehat{\star}_{a,b;u,v} g(q,0)$ is the ratio of the expectation value of $\mathrm{e}^{\frac{\mathrm{i}\hbar}{2}\int_a^b G^{ij}(q+\widetilde{Q})\widetilde{P}_i\widetilde{P}_j \mathrm{d}t} \mathcal{O}_{f,g;u,v}$ and the expectation value of $\mathrm{e}^{\frac{\mathrm{i}\hbar}{2}\int_a^b G^{ij}(q+\widetilde{Q})\widetilde{P}_i\widetilde{P}_j \mathrm{d}t}$. As propagators pair $\widetilde{P}$s to $\widetilde{Q}$s, in a graphical description it is better to use oriented edges (say, by an arrow going from $\widetilde{P}$ to $\widetilde{Q}$). Then Feynman diagrams will be oriented graphs with

1. a vertex at u with no outgoing arrows,
2. a vertex at v with no outgoing arrows
3. vertices between a and b with exactly two outgoing arrows.

Moreover no graphs containing tadpoles or vacuum subgraphs will be allowed. See Figs. 4, 5 and 6 for examples (tails of arrows are understood to move on the whole interval $[a,b]$). To a vertex of the first type with r incoming arrows we associate an rth derivative of f, to a vertex of the second type with r incoming arrows we associate an rth derivative of g, and to a vertex of the third type with r incoming arrows we

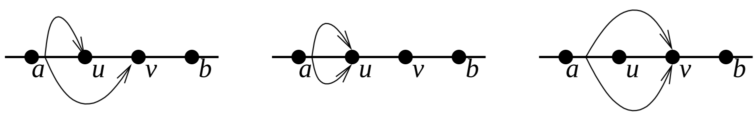

FIGURE 4. Feynman diagrams of order one.

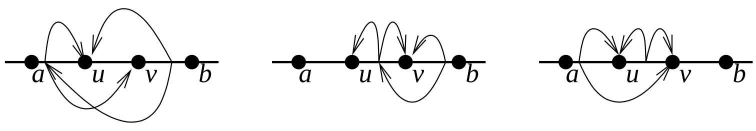

FIGURE 5. Some Feynman diagrams of order two.

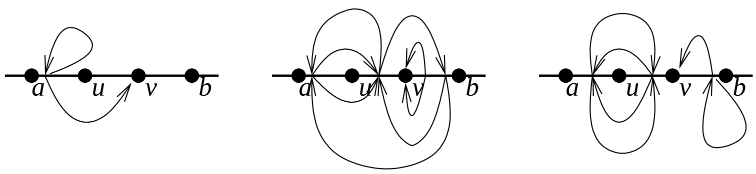

FIGURE 6. Some non allowed Feynman diagrams: tadpoles, vacuum subdiagrams, or both.

associate an rth derivative of G. The order in $\hbar$ is given by the number of vertices of the third type. So, at order 0, we just have f placed at u and g placed at v. This yields the pointwise product of f and g. At order 1, we have the graphs of figure 4. Thus, we get

$$
\begin{aligned}
f\widehat{\star}_{a,b;u,v}g(q,0) = f(q)g(q) - \frac{\mathrm{i}\hbar}{4}(b-a+2u-2v)G^{ij}(q)\partial_i f(q)\partial_j g(q) + \\
- \frac{\mathrm{i}\hbar}{4}(b-a)G^{ij}(q)[\partial_i\partial_j f(q)g(q) + f(q)\partial_i\partial_j g(q)] + O(\hbar^2)
\end{aligned}
$$

We leave to the reader the computation of some further orders.

Exercise 1.2.7. — Compute $f\widehat{\star}_{a,b;u,v}g(q,0)$ perturbatively up to the third order in $\hbar$ for the Hamiltonian

$$h(q,p) = \frac{1}{2}G^{ij}(q)p_i p_j + \lambda V(q)$$

with $\lambda \in \hbar^2\mathbb{R}[[\hbar]]$.

1.3. Symmetries and the BRST formalism

In Sect. 1.2 we have learnt how to compute integrals of the form given in (1.2.1) under the assumption that the critical point under consideration of the action function S is nondegenerate. This situation does certainly not occur when the action function is invariant under the free action of a Lie group. As this happens, e.g., in the Poisson sigma model for affine Poisson structures, we need to understand how to deal with it as well. In this Section we will explain a method that allows extending to this case the techniques of the perturbative expansion around a nondegenerate critical point.

1.3.1. The main example. — Let $\mathcal{M}$ be a finite-dimensional manifold endowed with a measure μ. Let $\mathcal{G}$ be a compact Lie group endowed with an invariant measure and with a measure-preserving free action on $\mathcal{M}$. Let us also assume that $\mathcal{M}/\mathcal{G}$ is also a manifold. Then an invariant function f is the pullback of a function $\underline{f}$ on $\mathcal{M}/\mathcal{G}$ and

$$I := \frac{\int_{\mathcal{M}} f\mu}{\mathrm{vol}(\mathcal{G})} = \int_{\mathcal{M}/\mathcal{G}} \underline{f}\,\underline{\mu},$$

where $\underline{\mu}$ is the measure induced by μ on the quotient.

If there is a section of $\pi\colon \mathcal{M} \to \mathcal{M}/\mathcal{G}$, we may also rewrite I as an integral on the image of this section. Let us assume that this image is given locally by the zero set of a function $F\colon \mathcal{M}' \to \Gamma$, with $\mathcal{M}' \subset \mathcal{M}$, $\pi(\mathcal{M}') = \mathcal{M}/\mathcal{G}$, and Γ the Lie algebra of $\mathcal{G}$ (in the physics literature the condition $F = 0$ is called gauge fixing and the function F is called the gauge-fixing function). For each $x \in \mathcal{M}'$, let $\mathsf{A}(x)$ be $\mathrm{d}F(x)$ restricted to the vertical tangent space at x. As this may also be identified with Γ, we regard $\mathsf{A}(x)$ as an endomorphism of Γ and denote by J its determinant (in the physics literature J is usually called the Faddeev–Popov determinant). Then we get

$$I = \int_{\mathcal{M}'} f\,\delta_0(F)\,J\,\mu,$$

where δ_0 is the delta-function at $0 \in \Gamma$. We want now to rewrite I in a way that is suitable for saddle-point approximation. To do so we write $\delta_0(F)$ and J in exponential form. To do so, we use the Fourier transform of the delta-function and the Grassmann integration explained in 1.2.1.3. Namely, denoting by $\langle\ ,\ \rangle$ the canonical pairing between Γ^* and Γ, we get

$$I = C \int_{\substack{x \in \mathcal{M}',\\ \lambda \in \Gamma^*,\\ c \in \Pi\Gamma,\\ \bar{c} \in \Pi\Gamma^*}} f(x)\,\mu(x)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle\,\lambda\,,\,F(x)\,\rangle}\,\omega(\lambda)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\,\bar{c}\,,\,\mathsf{A}(x)c\,\rangle},$$

where C is a constant depending on $\hbar$ and on the choice of the top form $\omega \in \Lambda^{\mathrm{top}}\Gamma^*$. The particular choice of the prefactors $\frac{\mathrm{i}}{\hbar}$ is done for later convenience. In the physics literature c is called the ghost and $\bar{c}$ the antighost; there are various denominations for λ, among which our favorite one is Lagrange multiplier. From this point of view a ghost is a "point" in $\Pi\Gamma$. However, for all computations what is relevant is the algebra of functions $\Lambda\Gamma^*$ which is generated by linear functions, i.e., elements of Γ^*. If we choose a basis $\{c^i\}$ of Γ^*, then $\Lambda\Gamma^*$ may be identified with the algebra generated by the c^is with relations $c^ic^j = -c^jc^i$, $\forall i,j$. The generators $c^i \in \Gamma^*$ are often referred to as ghost variables. Similarly, one introduces antighost variables $\bar{c}_i \in \Gamma$.

Remark 1.3.1. — The determinant of A might also be obtained in terms of an ordinary Gaussian integral over $\Gamma \times \Gamma^*$ at the price however of putting A^{-1} in the exponential. In the finite-dimensional case, this would not make much of a difference. In the infinite-dimensional generalization, however, we may use the Feynman-diagrams techniques only when dealing with local functions, see 1.2.3.1. If the action of the group is local, A will be a differential operator, so $\langle \bar{c}, \mathsf{A}c \rangle$ will be a local function (unlike the quadratic function defined in terms of the Green function A^{-1}).

We may now write the expectation value of an invariant function g w.r.t. to a given invariant action function S as

$$\langle g \rangle = \frac{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S}\, g\,\mu}{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S}\,\mu} = \frac{\int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar}S_F}\, g\,\widetilde{\mu}}{\int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar}S}\,\widetilde{\mu}} =: \langle g \rangle_F, \tag{1.3.1}$$

with

$$\widetilde{\mathcal{M}} = \mathcal{M}' \times \Pi\Gamma \times \Gamma^* \times \Pi\Gamma^*,$$
$$S_F = S + \langle \lambda, F \rangle - \langle \bar{c}, \mathsf{A}c \rangle,$$

and $\widetilde{\mu} = \mu\,\omega$. The function S_F is usually called the gauge-fixed action function.

1.3.2. The BRST method. — By construction it is clear that the r.h.s. of (1.3.1) is independent of F. We recall that our assumptions so far were rather restrictive: namely, we need a measure-preserving action on $\mathcal{M}$ of a compact Lie group $\mathcal{G}$, and we have to assume that the principal bundle $\mathcal{M} \to \mathcal{M}/\mathcal{G}$ is trivial. On the other hand, in order to define

$$\langle g \rangle_F = \frac{\int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar}S_F}\, g\,\widetilde{\mu}}{\int_{\widetilde{\mathcal{M}}} e^{\frac{i}{\hbar}S}\,\widetilde{\mu}}, \tag{1.3.2}$$

we just need the infinitesimal action

$$\begin{array}{rccc} X\colon & \Gamma & \to & \mathfrak{X}(\mathcal{M}) \\ & \gamma & \mapsto & X_\gamma \end{array}$$

of a Lie algebra Γ on $\mathcal{M}$. In this case A is simply given by

$$\mathsf{A}\gamma = \mathrm{L}_{X_\gamma} F, \qquad \gamma \in \Gamma. \tag{1.3.3}$$

We also want to relax the condition that $F^{-1}(0)$ defines a section and simply require that $\mathsf{A}(x)$ should be nondegenerate $\forall x \in \mathcal{M}'$. Moreover, we obviously require that the integrals are well-defined and that the denominator of (1.3.2) does not vanish. We denote by $\mathcal{F} \subset C^\infty(\mathcal{M}', \Gamma)$ the space of allowed gauge-fixing functions.

As $\langle g \rangle_F$ is well-defined[(4)] in this case (and susceptible to an infinite-dimensional generalization along the lines discussed in Sect. 1.2), it makes sense to take it into consideration in more general instances. However, as the gauge-fixing function is

[(4)] Observe that if the group is not compact, an invariant function is not a test function. It is then understood that it is replaced by a test function that in a neighborhood of the zeros of F coincides with the given function. To avoid cumbersome notation, we will never explicitly change the function, so we will write, e.g., $\int_{\mathbb{R}} \delta_0(x)\,dx = 1$, without mentioning that the constant function 1 is replaced by a test function which is one in a neighborhood of zero.

chosen arbitrarily, we want conditions for $\langle\, g\,\rangle_F$ to be, at least locally, independent of F.

Definition 1.3.2. — A locally constant function on $\mathcal{F}$ is called gauge-fixing independent.

We have the

Theorem 1.3.3. — *Let $X\colon \Gamma \to \mathcal{M}$ be an infinitesimal action of the Lie algebra Γ on the manifold $\mathcal{M}$. If S and g are invariant functions and*

$$\operatorname{div} X_\gamma + \operatorname{Tr}\operatorname{ad}_\gamma = 0, \qquad \forall \gamma \in \Gamma, \tag{1.3.4}$$

then $\langle\, g\,\rangle_F$ is gauge-fixing independent.

Here ad denotes the adjoint representation of Γ and $\operatorname{Tr}\operatorname{ad}_\gamma$ is regarded as a constant function on $\mathcal{M}$. The condition in particular says that the divergence of X is a constant function. In the particular case when the Lie algebra is unimodular (i.e., $\operatorname{Tr}\operatorname{ad}_\gamma = 0$ $\forall\gamma$), the condition says that the infinitesimal action must be measure preserving. The case discussed in 1.3.1 is covered by the Theorem as the Lie algebra of a compact Lie group is unimodular.

1.3.2.1. The BRST operator and a proof of Theorem 1.3.3. — The infinitesimal action of Γ on $\mathcal{M}$ makes $C^\infty(\mathcal{M})$ into a Γ-module. As a consequence, we may consider the Lie algebra complex $\Lambda\Gamma^* \otimes C^\infty(\mathcal{M})$. The Lie-algebra differential δ is in particular a derivation on $\Lambda\Gamma^*$ acting on $C^\infty(\mathcal{M})$ by vector fields. As such it is a derivation on $\Lambda\Gamma^* \otimes C^\infty(\mathcal{M})$ and defines a vector field[(5)] on $\mathcal{M} \times \Pi\Gamma$. If we choose a basis $\{e_i\}$ of Γ and denote by $\{f_{ij}^k\}$ the corresponding structure constants, then the algebra of functions on $\Pi\Gamma$ may be identified with the graded commutative algebra with odd generators c^i (the ghost variables) and

$$\delta c^i = -\frac{1}{2} f_{jk}^i c^j c^k.$$

On functions $f \in C^\infty(M)$, δ acts by

$$\delta f = c^i \,\mathrm{L}_{X_{e_i}} f.$$

We may extend the vector field δ to the whole $\widetilde{\mathcal{M}}$ by adding to it a vector field on $\Gamma^* \times \Pi\Gamma^*$. Using the dual basis $\{e^i\}$ of Γ^*, the algebra of functions may be identified with the graded commutative algebra with odd generators $\bar{c}_i$ and even generators λ_i.

(5) According to 1.2.1.3.1, a vector field on the superspace $\Pi\Gamma$ is an element of $\Lambda\Gamma^* \otimes \Gamma$. If Γ is a Lie algebra, the commutator may be regarded as an element of $\Lambda^2\Gamma^* \otimes \Gamma$ and as such it is a vector field on $\Pi\Gamma$. Vector fields on $\mathcal{M} \times \Pi\Gamma$ may be identified with elements of $\Lambda\Gamma^* \otimes \Gamma \otimes C^\infty(M) \oplus \Lambda\Gamma^* \otimes \mathfrak{X}(M)$. The commutator tensor the constant function 1 is an element of $\Lambda^2\Gamma^* \otimes \Gamma \otimes C^\infty(M)$ while the infinitesimal action of Γ on $\mathcal{M}$ is an element of $\Gamma^* \otimes \mathfrak{X}(M)$. As such they define vector fields on $\mathcal{M} \times \Pi\Gamma$ and δ is their sum.

On them we define[6]

$$\delta \bar{c}_i = \lambda_i, \qquad \delta\lambda_i = 0. \tag{1.3.5}$$

As a consequence of this definition, we immediately get the

Lemma 1.3.4. — *δ is a differential (i.e., an odd derivation that squares to zero) on $\Lambda\Gamma^* \otimes C^\infty(\mathcal{M}) \otimes \Lambda\Gamma \otimes S\Gamma$. It has degree one with respect to the grading $\deg(\alpha \otimes f \otimes \beta \otimes \gamma) := \deg(\alpha) - \deg(\beta)$.*

A cohomological vector field is by definition a differential on the algebra of functions. We may restate the Lemma by saying that δ is a cohomological vector field (of degree one) on $\widetilde{\mathcal{M}}$. In the physics literature, it is often referred to as the BRST operator. A function on $\mathcal{M}$ may also be considered as a function on $\widetilde{\mathcal{M}}$. By definition of δ, a function f is invariant iff $\delta f = 0$.

Exercise 1.3.5. — Show that

$$\operatorname{div} \delta = \operatorname{div} X_c + \operatorname{Tr} \operatorname{ad}_c .$$

Hint: Use Exercise 1.2.2.

Now observe that a gauge-fixing function $F\colon \mathcal{M} \to \Gamma$ may be regarded as an element of $C^\infty(\mathcal{M}) \otimes \Gamma$. By the inclusions

$$C^\infty(\mathcal{M}) \otimes \Gamma \hookrightarrow C^\infty(\mathcal{M}) \otimes \Lambda\Gamma \hookrightarrow \Lambda\Gamma^* \otimes C^\infty(\mathcal{M}) \otimes \Lambda\Gamma \otimes S\Gamma,$$

we may then associate to F a function Ψ_F on $\widetilde{\mathcal{M}}$. With the same notations as above, we have $\Psi_F = \bar{c}_i F^i$. The odd function Ψ_F is called the gauge-fixing fermion. A very simple computation shows that

$$S_F = S + \delta\Psi_F.$$

We now assume that the action function S is invariant; i.e., $\delta S = 0$.

Lemma 1.3.6. — *Let g be a function on $\widetilde{\mathcal{M}}$. If $\delta g = 0$ and δ is divergence-free, then*

$$I_F := \int_{\widetilde{\mathcal{M}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_F}\, g\, \widetilde{\mu},$$

is gauge-fixing independent.

The Lemma together with the result of Exercise 1.3.5 proves Theorem 1.3.3 immediately.

---

[6] More abstractly, observe that a polynomial vector field on $\Gamma^* \times \Pi\Gamma^*$ is an element of $\Lambda\Gamma \otimes S\Gamma \otimes (\Gamma^* \oplus \Gamma^*)$. The identity operator may be regarded as an element of $\Gamma \otimes \Gamma^*$. Using the inclusion map

$$\begin{array}{rccc} \iota\colon & \Gamma \otimes \Gamma^* & \to & \Lambda\Gamma \otimes S\Gamma \otimes (\Gamma^* \oplus \Gamma^*) \\ & a \otimes b & \mapsto & 1 \otimes a \otimes (b \oplus 0) \end{array}$$

we may the regard it as a vector field on $\Gamma^* \times \Pi\Gamma^*$. Observe that this vector field corresponds to the de Rham differential on $\Pi\Gamma^*$.

Proof. — Let F_t be a curve of gauge-fixing functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{F_t} = \frac{\mathrm{i}}{\hbar} \int_{\widetilde{\mathcal{M}}} \delta \left(\frac{\mathrm{d}}{\mathrm{d}t} \Psi_{F_t} \right) \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_F} \, g \, \widetilde{\mu} = \frac{\mathrm{i}}{\hbar} \int_{\widetilde{\mathcal{M}}} \delta \left(\frac{\mathrm{d}}{\mathrm{d}t} \Psi_{F_t} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_F} \, g \right) \widetilde{\mu} = \\ = \frac{\mathrm{i}}{\hbar} \int_{\widetilde{\mathcal{M}}} \operatorname{div} \delta \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_F} \, g \, \widetilde{\mu} = 0.$$

□

A particular case of a δ-closed function is a δ-exact function. These functions are irrelevant as for computing expectation values. In fact,

$$\int_{\widetilde{\mathcal{M}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_F} \, \delta h \, \widetilde{\mu} = \int_{\widetilde{\mathcal{M}}} \delta \left(\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_F} \, h \right) \widetilde{\mu} = 0$$

if S is invariant and δ is divergence free. We may then extend Theorem 1.3.3 to the

Theorem 1.3.7. — *Let $X \colon \Gamma \to \mathcal{M}$ be an infinitesimal action of the Lie algebra Γ on the manifold $\mathcal{M}$. If the action function S is invariant and the BRST operator δ is divergence-free (i.e., if condition* (1.3.4) *is satisfied), then*

1. $\langle\, g \,\rangle_F$ *is gauge-fixing independent for* $\forall g \in \ker \delta$*;*
2. $\langle\, g \,\rangle_F = 0$ $\forall g \in \operatorname{im} \delta$*.*

Thus the expectation value defines a linear function on the δ-cohomology.

Observe that 2. produces identities relating expectation values of different quantities. Such identities are called Ward identities and usually have nontrivial content.

1.3.2.2. Examples. — We discuss two very simple examples.

1.3.2.2.1. Translations. — Let $\mathcal{M} = \Gamma = \mathbb{R}$. Let Γ act by infinitesimal translations. Denoting by x a coordinate on $\mathcal{M}$, we then have $\delta x = c$ and $\delta c = 0$. We assume S and g to be constant. Then

$$\langle\, g \,\rangle_F = \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S} g \int \delta_0(F(x)) F'(x)}{\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S} \int \delta_0(F(x)) F'(x)} = g$$

if the denominator does not vanish. Obviously $\langle\, g \,\rangle_F$ is gauge-fixing independent.

We treat similarly the case of rotation-invariant functions on $\mathcal{M} = S^1$. A section here is just a point. We take $\mathcal{M}'$ to be a neighborhood of this point. After identifying $\mathcal{M}'$ with $\mathbb{R}$, we proceed as above with F any function with a single nondegenerate zero corresponding to the image of the section.

1.3.2.2.2. Plane rotations. — Assume $\mathcal{M} = \mathbb{R}^2 \setminus \{0\}$ and $\Gamma = \mathfrak{so}(2)$ acting by infinitesimal rotations. Calling x and y the coordinates on $\mathbb{R}^2$, we have

$$\delta x = yc, \quad \delta y = -xc, \quad \delta c = 0.$$

Let $\mathcal{M}' = \{x > 0\}$. A possible choice for F is the function $F(x, y) = y$. Then

$$S_F(x, y, c, \lambda, \bar{c}) = S(x, y) + \lambda y + \bar{c} x c,$$

where S is the given rotation-invariant action function. Then

$$\langle\, g \,\rangle_F = \frac{\int_0^{+\infty} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(x,0)}\, g(x,0)\, x \,\mathrm{d}x}{\int_0^{+\infty} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(x,0)}\, x \,\mathrm{d}x},$$

which is equal to the expected

$$\frac{\int_{\mathcal{M}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(x,y)}\, g(x,y)\, \mathrm{d}x\mathrm{d}y}{\int_{\mathcal{M}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(x,y)}\, \mathrm{d}x\mathrm{d}y}$$

for a rotation-invariant function g.

1.3.3. Infinite dimensions. — In the infinite dimensional case, we consider (1.3.2) whenever it makes sense as a perturbative expansion. We assume then to have an infinite-dimensional manifold of fields $\mathcal{M}$, an infinite-dimensional Lie algebra Γ acting freely on $\mathcal{M}$, and a local action function S. The space $\widetilde{\mathcal{M}}$ and the BRST operator δ will be defined exactly as above, and δ will still be a cohomological vector field on $\widetilde{\mathcal{M}}$.

A gauge-fixing function F will be allowed if the corresponding A is nondegenerate and the critical point of the action function at a zero of F is also nondegenerate. Then, for suitable functions g, we will be able to define $\langle\, g \,\rangle_F$ as a perturbative expansion in terms of Feynman diagrams. An observable in this new context is a δ-cohomology class g for which $\langle\, g \,\rangle_F$ is well-defined.

Theorem 1.3.7 will hold whenever $\operatorname{div}\delta = 0$. The divergence of δ is however not defined a priori and has to be understood in terms of expectation values (as we discussed, e.g., in 1.2.2.3). The usual pragmatical way to proceed, however, is to assume Theorem 1.3.7 to hold and to derive from it properties of the expectation values. Once they are properly defined in terms of Feynman diagrams, one checks if the identities hold and calls any deviation an anomaly.

Observe that if our aims are of mathematical nature, "Theorem" 1.3.7 then provides a source for a lot of interesting conjectures (which in most cases, fortunately, turn out to be true).

1.3.3.1. The trivial Poisson sigma model on the plane. — We want to consider a two-dimensional generalization of the example discussed in 1.2.2, which will be also at the basis for the study of the Poisson sigma model in the following Sections. Let ξ and η be a zero-form and a one-form on the plane respectively. We assume them to vanish at infinity sufficiently fast (e.g., as Schwarz functions) so that we may define the action function

$$S := \int_\Sigma \eta\, \mathrm{d}\xi, \tag{1.3.6}$$

with $\Sigma = \mathbb{R}^2$. Our manifold of fields is then $\mathcal{M} = \Omega^0_0(\mathbb{R}^2) \oplus \Omega^1_0(\mathbb{R}^2)$. On this space we have an action of the abelian Lie algebra $\Gamma = \Omega^0_0(\mathbb{R}^2)$, given by the monomorphism $\iota \circ \mathrm{d}$,

$$\Gamma \overset{\mathrm{d}}{\hookrightarrow} \Omega^1_0(\mathbb{R}^2) \overset{\iota}{\hookrightarrow} \mathcal{M}$$

with ι the inclusion, and the action function is clearly invariant. The BRST differential is then given on coordinates by

$$\delta\xi = 0, \qquad \delta\eta = \mathrm{d}c.$$

To define a gauge-fixing function, we choose a Riemannian metric on $\mathbb{R}^2$. Denoting by $*$ the induce Hodge-star operator, we have the pairing $(\alpha, \beta) = \int_{\mathbb{R}^2} (*\alpha)\,\beta$ of forms on $\mathbb{R}^2$. Denoting by $\mathrm{d}^* = *\mathrm{d}*$ the formal adjoint of the de Rham differential, we choose the gauge-fixing function $F(\xi, \eta) = \mathrm{d}^*\eta$. Different choices of metrics give gauge-fixing functions connected by paths. It turns out that the corresponding operator A, see (1.3.3), is the Laplace operator on zero-forms which is invertible for the given conditions at infinity. Moreover, there is a unique critical point (i.e., a solution of $\mathrm{d}\xi = \mathrm{d}\eta = 0$) satisfying the gauge-fixing condition $\mathrm{d}^*\eta = 0$: viz., $\xi = \eta = 0$. Thus, the proposed gauge fixing is allowed.

By integration, we identify Γ^* with $\Omega^2(\mathbb{R}^2)$. Then the corresponding gauge-fixing fermion is $\Psi_F = \int_{\mathbb{R}^2} \bar{c}\,\mathrm{d}^*\eta$. This leads to the gauge-fixed action function

$$S_F = \int_{\mathbb{R}^2} \eta\,\mathrm{d}\xi + \lambda\,\mathrm{d}^*\eta - \bar{c}\,\mathrm{d}^*\mathrm{d}c. \tag{1.3.7}$$

Using the pairing of forms, we may rewrite it as

$$S_F = \frac{1}{2}(\phi, \mathsf{M}\phi) - (*\bar{c}, \Delta c),$$

where $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ is the Laplace operator,

$$\phi = \begin{pmatrix} \xi \\ \eta \\ \lambda \end{pmatrix} \in \Omega^0_0(\mathbb{R}^2) \oplus \Omega^1_0(\mathbb{R}^2) \oplus \Omega^2(\mathbb{R}^2),$$

and

$$\mathsf{M} = \begin{pmatrix} 0 & *\mathrm{d} & 0 \\ *\mathrm{d} & 0 & \mathrm{d}* \\ 0 & \mathrm{d}* & 0 \end{pmatrix}.$$

The propagators between ϕ and c or $\bar{c}$ are clearly zero. By (1.2.6) we get

$$\langle\, *\bar{c}(z)\, c(w)\,\rangle_0 = -\mathrm{i}\hbar G_0(w, z),$$

where G_0 is the Green function of the Laplace operator acting on functions. By (1.2.4), in order to get the propagator between two fields ϕ, we have to invert the symmetric operator M. We use a little trick as in [**2**], and compute first its square

$$\mathsf{M}^2 = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}.$$

Then we observe that $M^{-1} = M\,M^{-2}$; thus,

$$M^{-1} = \begin{pmatrix} 0 & *\mathrm{d}\Delta^{-1} & 0 \\ *\mathrm{d}\Delta^{-1} & 0 & \mathrm{d}*\Delta^{-1} \\ 0 & \mathrm{d}*\Delta^{-1} & 0 \end{pmatrix}$$

In particular,

$$\langle\, \xi(z)\, \eta(w)\, \rangle_0 = \mathrm{i}\hbar *_z \mathrm{d}_z G_1(z,w) = \mathrm{i}\hbar *_w \mathrm{d}_w G_0(w,z),$$

where G_1 is the Green function of the Laplace operator acting on one-forms. For further convenience we introduce the "superfields"

$$\widetilde{\xi} = \xi - \mathrm{d}^* \bar{c}, \tag{1.3.8a}$$

$$\widetilde{\eta} = c + \eta, \tag{1.3.8b}$$

and compute the "superpropagator"

$$\mathrm{i}\hbar\theta(z,w) := \left\langle\, \widetilde{\xi}(z)\, \widetilde{\eta}(w)\, \right\rangle_0 = \langle\, \xi(z)\, \eta(w)\, \rangle_0 - \langle\, \mathrm{d}^* \bar{c}(z)\, c(w)\, \rangle_0 = \tag{1.3.9}$$
$$= \mathrm{i}\hbar(*_z \mathrm{d}_z + *_w \mathrm{d}_z) G_0(w,z) \in \Omega^1(C_2(\mathbb{R}^2)),$$

where $C_2(\mathbb{R}^2)$ denotes the configuration space of two points on $\mathbb{R}^2$.

Lemma 1.3.8. — *If we choose the Euclidean metric, then*

$$\theta = \frac{\mathrm{d}\phi_E}{2\pi},$$

where d *is the differential on* $C_2(\mathbb{R}^2)$ *and* $\phi_E(z,w)$ *is the Euclidean angle between a fixed reference line and the line passing through* z *and* w.

Proof. — The Green function for the Euclidean Laplace operator in two dimensions is

$$G_0(z,w) = \frac{1}{2\pi} \log|z-w|,$$

where $|\ |$ is the Euclidean norm. It is easier to work with complex coordinates, in which we have

$$G_0(z,w) = \frac{1}{4\pi} \log(z-w)(\bar{z}-\bar{w}).$$

Then

$$\mathrm{d}_z G_0(z,w) = \frac{1}{4\pi} \left(\frac{\mathrm{d}z}{z-w} + \frac{\mathrm{d}\bar{z}}{\bar{z}-\bar{w}} \right).$$

The Euclidean Hodge operator in complex coordinates acts as follows: $*\mathrm{d}z = -\mathrm{i}\mathrm{d}z$, $*\mathrm{d}\bar{z} = \mathrm{i}\mathrm{d}\bar{z}$. Hence,

$$*_z \mathrm{d}_z G_0(z,w) = \frac{1}{4\pi\mathrm{i}} \left(\frac{\mathrm{d}z}{z-w} - \frac{\mathrm{d}\bar{z}}{\bar{z}-\bar{w}} \right).$$

Analogously we get

$$*_w \mathrm{d}_w G_0(z,w) = \frac{1}{4\pi\mathrm{i}} \left(\frac{\mathrm{d}w}{w-z} - \frac{\mathrm{d}\bar{w}}{\bar{w}-\bar{z}} \right).$$

Summing up, we obtain

$$\theta = \frac{1}{4\pi\mathrm{i}} \left(\frac{\mathrm{d}z - \mathrm{d}w}{z-w} - \frac{\mathrm{d}\bar{z} - \mathrm{d}\bar{w}}{\bar{z}-\bar{w}} \right) = \frac{1}{4\pi\mathrm{i}} \mathrm{d} \log \frac{z-w}{\bar{z}-\bar{w}}.$$

On the other hand, $z - w = |w - z|e^{i\phi}$ yields

$$\phi = \frac{1}{2i} \log \frac{z - w}{\bar{z} - \bar{w}},$$

which completes the proof. □

The cohomology class of θ is then the generator of $H^1(C_2(\mathbb{R}^2); \mathbb{Z})$. It is not difficult to see that other choices of metric yield the same class.[(7)]

1.3.3.1.1. Expectation values. — Any function of ξ is BRST-invariant; e.g., we may consider the evaluation of ξ at some point u. A function of $\int_\gamma \eta$ is also invariant for any closed curve γ. Thus, the expectation value

$$\left\langle \xi(u) \int_\gamma \eta \right\rangle_0 = i\hbar W_\gamma(u), \qquad u \notin \operatorname{im} \gamma,$$

is independent of the gauge-fixing. As we also have $i\hbar W_\gamma(u) = \left\langle \widetilde{\xi}(u) \int_\gamma \widetilde{\eta} \right\rangle_0$, by the previous considerations we see immediately that $W_\gamma(u)$ is the winding number of γ around u. Observe that this number is invariant under deformations of the curve γ or displacement of u, an indication that this theory is also topological. We may formally prove this invariance by using the same techniques as in 1.2.2.4. For example, let us deform γ to γ'. Denoting by σ a two-chain whose boundary is $\gamma - \gamma'$, we get

$$W_\gamma(u) - W_{\gamma'}(u) = \left\langle \xi(u) \int_\sigma d\eta \right\rangle_0 .$$

If we introduce the sequence of divergence-free vector fields $X_r(\xi, \eta) = \lambda_r \oplus 0$, where $\{\lambda_r\}$ is a sequence of functions that converges almost everywhere to the characteristic function of the image of σ, we get

$$W_\gamma(u) - W_{\gamma'}(u) = \lim_{r\to\infty} \langle\, \xi(u)\, X_r(S)\, \rangle_0 = i\hbar \lim_{r\to\infty} \langle\, X_r(\xi(u))\, \rangle_0 = 0,$$

under the assumption that u does not belong to σ.

---

(7) A simple observation is that $*_w\theta(z, w)$ is the Green function of the operator $P = *d\Delta^{-1}*$ which is a parametrix for the de Rham differential on forms that vanish at infinity; viz.,

$$dP + Pd = \mathrm{id}.$$

The convolution relating P to θ may be written as

$$P\alpha = -\pi_{2*}(\theta\, \pi_1^*\alpha), \qquad \alpha \in \Omega_0^*(\mathbb{R}^2),$$

with π_1 and π_2 the two projections to $\mathbb{R}^2$. Then

$$dP\alpha + Pd\alpha = -\pi_{2*}(d\theta\, \pi_1^*\alpha) + \pi_*^\partial(\theta)\, \alpha,$$

where $\pi_*^\partial(\theta)(w)$ denotes the integral of θ along a limiting small circle around w. Since P is a parametrix and α is arbitrary, we see that in general θ is closed and has integral one along the generator of $H_1(C_2(\mathbb{R}^2); \mathbb{Z})$.

1.3.3.2. The trivial Poisson sigma model on the upper half plane. — We now consider the action function in (1.3.6) with Σ the upper half plane $H = \mathbb{R}\times\mathbb{R}^+$. As a boundary condition,, we impose that the one-form η vanishes when restricted to the boundary $\partial H = \mathbb{R}\times\{0\}$. The Lie algebra that acts on the space of fields consists now of zero-forms on H vanishing on ∂H (more generally we might consider zero-forms that are constant on ∂H, but then the action would not be free). The BRST complex may be defined exactly as above, and we may choose the same gauge-fixing function. We define the superpropagator as in (1.3.9), but we denote it by ϑ instead of θ to avoid confusion. Then we have the following generalization of Lemma 1.3.8:

Lemma 1.3.9. — *If we choose the Euclidean metric, then*

$$\vartheta = \frac{\mathrm{d}\phi_h}{2\pi},$$

where d *is the differential on* $C_2(H)$ *and* $\phi_h(z,w)$ *is the angle between the vertical line through* w *and the geodesic joining* w *to* z *in the hyperbolic Poincaré metric.*

Proof. — We use the classical method of the images. The Green function G_0^H of the Laplace operator on H is the restriction to H of the Green function of the Laplace operator on $\mathbb{R}^2$ plus a harmonic function such that the sum satisfies the boundary conditions. In complex coordinates, we need $G_0^H(w,z) = 0$ if w is real. This is achieved by

$$G_0^H(w,z) = G_0(w,z) - G_0(\bar{w},z).$$

Then

$$\vartheta(z,w) = \theta(z,w) - \theta(z,\bar{w}).$$

Since the hyperbolic angle is given by

$$\phi_h(z,w) = \frac{1}{2\mathrm{i}}\log\frac{(z-w)(\bar{z}-w)}{(\bar{z}-\bar{w})(z-\bar{w})},$$

this completes the proof. □

Observe that ϑ is the generator of $H^1(C_2(H), H\times\partial H;\mathbb{Z})$.

1.3.3.3. Generalizations. — The simplest generalization of the trivial Poisson sigma model described in 1.3.3.1 and 1.3.3.2 consists in allowing more fields. Namely, we take a collection of n zero-forms ξ^i and of n one-forms η_i, $i = 1,\dots,n$, and consider the action function $\int_\Sigma \eta_i\,\mathrm{d}\xi^i$. We may also think of ξ and η as forms taking values in $\mathbb{R}^n$. The Lie algebra Γ of symmetries will now consist of the direct sum of n copies of the previous one; in other words it will be the abelian Lie algebra of $\mathbb{R}^n$-valued zero-forms. Denoting by c_i, $i = 1,\dots,n$, the generators of the algebra of functions of $\Pi\Gamma$, we may write the BRST operator as $\delta\xi^i = 0$, $\delta\eta_i = c_i$. If we choose the gauge-fixing function to be $F_i(X,\eta) = \mathrm{d}^*\eta_i$, the computation is exactly analogous to the one given above. In particular, we may introduce again "superfields" $\widetilde{\xi^i} = \xi^i - \mathrm{d}^*\bar{c}^i$ and $\widetilde{\eta}_i = c_i + \eta_i$ (where the c^is are the generators of the algebra of functions of $\Pi\Gamma$)

and compute the "superpropagator"

$$\left\langle \,(\widetilde{\xi}^i(z)\,\widetilde{\eta}_i(w)\,\right\rangle_0 = \begin{cases} \mathrm{i}\hbar\delta^i_j\theta(z,w) & \text{on the plane,} \\ \mathrm{i}\hbar\delta^i_j\vartheta(z,w) & \text{on the upper half plane.} \end{cases} \tag{1.3.10}$$

The next generalization consists in dropping the assumption that the zero-form field vanishes at infinity. More precisely, we denote by X^i a collection of maps to $\mathbb{R}^n$ with no conditions on the boundary or at infinity and consider the action function

$$S = \int_\Sigma \eta_i \,\mathrm{d}X^i, \tag{1.3.11}$$

where Σ is the plane or the upper half plane and the η_is are one-forms vanishing on the boundary and at infinity. Critical points are pairs of constant maps together with closed one-forms. They are now degenerate also after modding out by the action of the abelian Lie algebra of zero-forms. However, the degeneracy is now of a very simple type as it is parametrized by the finite-dimensional manifold $\mathbb{R}^n$—the value of the constant map. As discussed in 1.2.3.1, it is enough to choose a measure on $\mathbb{R}^n$ and impose Fubini's theorem. We choose a delta-measure picked at a point $x \in \mathbb{R}^n$ and require the X to map infinity to x. If we write $X^i = x^i + \xi^i$, then ξ^i vanishes at infinity and we are reduced to the previous situation.

The last generalization is to replace $\mathbb{R}^n$ by a manifold M. We think of X^i as a local coordinate expression of a map $X\colon \Sigma \to M$. For the action function to be covariant, we should assume $\eta_i(u)$, $u \in \Sigma$, to be the local coordinate expression of a one-form on Σ taking values in the cotangent space of M at $X(u)$. Namely, we assume $\eta \in \Gamma(\Sigma, T^*\Sigma \otimes X^*T^*M)$. The manifold of fields $\mathcal{M}$ may then be identified with the manifold of bundle maps $T\Sigma \to T^*M$ and the action function may be written in an invariant way as

$$S = \int_\Sigma \langle\, \eta\,,\, \mathrm{d}X\,\rangle\,,$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing between the tangent and the cotangent bundle of M, and $\mathrm{d}X$ is the differential of the map X regarded as a section of $T^*\Sigma \otimes X^*TM$. If we now require X to map infinity to a given point $x \in M$, we may expand around the critical solution by setting $X = x + \xi$, $\xi\colon \Sigma \to T_xM$ and by regarding η as a one-form taking values in T^*_xM. Choosing coordinates we may identify T_xM with $\mathbb{R}^n$ ($n = \dim M$) and reduce to the previous case.

Finally, we may allow Σ to be any two-manifold. The above discussion will change drastically if Σ is not simply connected, as the space of solutions modulo symmetries, with $X = x$, will now be parametrized by $H^1(\Sigma, T^*_xM)$ and one has to choose a measure on this vector space as well.

1.4. The Poisson sigma model

We want now to discuss deformations of the trivial Poisson sigma model described in 1.3.3.1, 1.3.3.2 and 1.3.3.3. This Section is included only to provide a motivation for the action function that describes the deformation quantization of affine Poisson structures, see 1.5, and may be safely skipped.

Our starting point is the trivial Poisson sigma model (1.3.11). We want to deform its action function without introducing extra structure on Σ (the plane or the upper half plane). This means that the terms we are allowed to add must be two-forms on Σ built in terms of the fields X^i and η_i; viz., they must be linear combinations of terms $\alpha^{ij}(X)\eta_i\eta_j$, $\beta^i_j(X)\eta_i\,\mathrm{d}X^j$ and $\gamma_{ij}(X)\mathrm{d}X^i\,\mathrm{d}X^j$ (we do not consider a term $\phi^i(X)\mathrm{d}\eta_i$ as integration by parts reduces it to a term of the second type). The second and third terms may however be absorbed by a redefinition of η adding to it terms linear in η and $\mathrm{d}X$. Thus, modulo field redefinitions, the most general deformation of the action function has the form

$$S = \int_\Sigma \left(\eta_i\,\mathrm{d}X^i + \frac{1}{2}\epsilon\alpha^{ij}(X)\eta_i\eta_j\right) + O(\epsilon^2)),$$

where ϵ is the deformation parameter and α^{ij} is assumed to be skew-symmetric. The aim of this Section is to show that it makes sense to consider only those deformations in which the α^{ij}s are the components of a Poisson bivector field and that the BRST method is available only if the Poisson structure is affine.

We want first to deform the symmetries accordingly (without adding extra structure on Σ). We do it already in the BRST framework. We recall that in the trivial case the BRST operator acted by $\delta X^i = 0$, $\delta\eta_i = \mathrm{d}c_i$ and $\delta c_i = 0$, with $c \in \Pi\Gamma$ and $\Gamma = \Omega^0_0(\Sigma, \mathbb{R}^n)$. We want to deform the trivial δ so that $\delta S = O(\epsilon^2)$ for the new S.[8]

Lemma 1.4.1. — *Modulo field redefinitions, there is a unique BRST operator deforming the trivial one such that $\delta S = O(\epsilon^2)$ and $\delta^2 = O(\epsilon^2) + R$ with R vanishing at critical points. It acts by*

$$\begin{aligned}
\delta X^i &= -\epsilon\alpha^{ij}(X)c_j + O(\epsilon^2),\\
\delta\eta_i &= \mathrm{d}c_i + \epsilon\partial_i\alpha^{jk}(X)\eta_j c_k + O(\epsilon^2),\\
\delta c_i &= -\frac{1}{2}\epsilon\partial_i\alpha^{jk}(X)c_j c_k + O(\epsilon^2).
\end{aligned}$$

Moreover, R vanishes on the whole $\mathcal{M}\times\Pi\Gamma$ if α is at most linear.

Proof. — Recalling that δ applied to X or η must be linear in the ghosts c, the most general deformation of δ (without adding extra structure on Σ) is of the form

$$\begin{aligned}
\delta X^i &= \epsilon v^{ij}(X)c_j + O(\epsilon^2),\\
\delta\eta_i &= \mathrm{d}c_i + \epsilon(a_i^{jk}(X)\eta_j c_k + b_i^j(X)\mathrm{d}c_j + d_{ij}^k(X)\mathrm{d}X^j c_k) + O(\epsilon^2),
\end{aligned}$$

for some functions v^{ij}, a_i^{jk}, b_i^j and d_{ij}^k on $\mathbb{R}^n$. Thus,

$$\begin{aligned}
\delta S = \epsilon\int_\Sigma \big((a_i^{jk}(X)\eta_j c_k + b_i^j(X)\mathrm{d}c_j + d_{ij}^k(X)\mathrm{d}X^j c_k)\,\mathrm{d}X^i +{}&\\
+\,\eta_i(\mathrm{d}X^r\partial_r v^{ij}(X)c_j + v^{ij}(X)\,\mathrm{d}c_j) + \alpha^{ij}(X)\,\mathrm{d}c_i\,\eta_j\big) + O(\epsilon^2).&
\end{aligned}$$

[8] We consider only the restriction of δ to $\mathcal{M}\times\Pi\Gamma$ as its restriction to $\Gamma^*\times\Pi\Gamma^*$ needs no deformation.

As the identity $\delta S = O(\epsilon^2)$ must hold for any η, we get the following two equations

$$a_i^{jk}(X)c_k\mathrm{d}X^i + \mathrm{d}X^r\partial_r v^{jk}(X)c_k + v^{jk}(X)\,\mathrm{d}c_k + \alpha^{kj}(X)\,\mathrm{d}c_k = 0, \tag{1.4.1a}$$

$$\int_\Sigma (b_i^j(X)\mathrm{d}c_j + d_{ij}^k(X)\mathrm{d}X^j c_k)\,\mathrm{d}X^i = 0. \tag{1.4.1b}$$

In particular, choosing X to be a constant map (with value x), we deduce from the first equation

$$v^{jk}(x)\,\mathrm{d}c_k + \alpha^{kj}(x)\,\mathrm{d}c_k = 0,$$

and as this has to hold for any c, we finally have

$$\alpha^{jk} = v^{jk}.$$

Inserting this into (1.4.1a) yields

$$a_i^{jk}(X)c_k\mathrm{d}X^i - \mathrm{d}X^i\partial_i\alpha^{jk}(X)c_k = 0,$$

and as this also has to hold for all c and X, we get

$$a_i^{jk} = \partial_i\alpha^{jk}.$$

Integrating by parts, (1.4.1b) yields

$$\int_\Sigma (-\mathrm{d}X^r\partial_r b_i^j(X)c_j + d_{ij}^k(X)\mathrm{d}X^j c_k)\,\mathrm{d}X^i = 0,$$

and as this has to hold for all X and c, we finally get

$$d_{ij}^k = \partial_j b_i^k.$$

Thus, we have proved that

$$\begin{aligned}\delta X^i &= -\epsilon\alpha^{ij}(X)c_j + O(\epsilon^2),\\ \delta\eta_i &= \mathrm{d}c_i + \epsilon(\partial_i\alpha^{jk}(X)\eta_j c_k + \mathrm{d}(b_i^j(X)c_j)) + O(\epsilon^2),\end{aligned}$$

which, after the redefinition $c_i \mapsto c_i - \epsilon b_i^j(X)c_j + O(\epsilon^2)$, yields the the first two equations in the Lemma. As for the last equation, we recall that the BRST operator on c must be quadratic in c, so its general form is

$$\delta c_i = \frac{1}{2}\epsilon f_i^{jk}(X)c_j c_k + O(\epsilon^2).$$

To determine the "structure" functions f_i^{jk}, we compute δ^2. Observe that $\delta^2 X^i = \delta^2 c_i = 0(\epsilon^2)$ automatically. On the other hand,

$$\begin{aligned}\delta^2\eta_i = \epsilon\left(\frac{1}{2}\mathrm{d}(f_i^{jk}(X)c_j c_k) + \partial_i\alpha^{jk}(X)\mathrm{d}c_j c_k\right) + 0(\epsilon^2) = \\ = \epsilon\left((f_i^{jk}(X) + \partial_i\alpha^{jk}(X))\mathrm{d}c_j c_k + \frac{1}{2}\mathrm{d}X^r\partial_r f_i^{jk}(X)c_j c_k\right) + 0(\epsilon^2).\end{aligned}$$

At a critical point (where $\mathrm{d}X^i = O(\epsilon)$) the third summand of the last equation vanishes. Thus, $\delta^2 = O(\epsilon^2)$ at critical points implies

$$f_i^{jk} = -\partial_i\alpha^{jk},$$

which proves the last equation in the Lemma. Observe that

$$\delta^2\eta_i = -\frac{1}{2}\epsilon \mathrm{d}X^r\partial_r\partial_i\alpha^{jk}(X)c_jc_k,$$

which is identically zero (not only at critical points) if α is at most linear. □

We want now to extend deformations beyond the first order in ϵ. Even without knowing the following terms, we already have the following

Lemma 1.4.2. — *$\delta^2 = O(\epsilon^3)$ at critical points only if α is Poisson.*

Proof. — We have

$$\delta^2X^i = -\epsilon^2\left(\alpha^{rk}(X)c_k\partial_r\alpha^{ij}(X)c_j + \frac{1}{2}\alpha^{ij}(X)\partial_j\alpha^{rj}c_rc_j\right) + O(\epsilon^3).$$

As this has to hold for all c, we get the Jacobi identity for α. □

It is now possible to prove that, under the assumption that α is Poisson, this deformation is not only infinitesimal. Namely, we have the

Theorem 1.4.3. — *Given a Poisson bivector field α, the odd vector field*

$$\begin{aligned}\delta X^i &= -\epsilon\alpha^{ij}(X)c_j,\\ \delta\eta_i &= \mathrm{d}c_i + \epsilon\partial_i\alpha^{jk}(X)\eta_jc_k,\\ \delta c_i &= -\frac{1}{2}\epsilon\partial_i\alpha^{jk}(X)c_jc_k,\end{aligned}$$

is cohomological for α at most linear or at critical points $\forall\epsilon$. Moreover,

$$S := \int_\Sigma\left(\eta_i\,\mathrm{d}X^i + \frac{1}{2}\epsilon\alpha^{ij}(X)\eta_i\eta_j\right)$$

is δ-closed $\forall\epsilon$.

For a proof see [**6**], [**8**] or [**5**] (or do the simple computation as an exercise). The geometrical meaning of this Theorem is that there is a distribution of vector fields on $\mathcal{M}$ under which the action function is invariant. In general, this distribution is involutive only on the submanifold of critical points of S. It is involutive on the whole of $\mathcal{M}$ if α is at most linear and in this case it can be regarded as the free, infinitesimal action of a Lie algebra (see 1.5 for more details in this case).

We remark that the action function S may be generalized to the case when one wants to consider a Poisson manifold (M,α) which is not an $\mathbb{R}^n$. To do so, one regards X as a map $\Sigma \to M$ and, for a given map X, η is taken to be a section of $T^*\Sigma\otimes X^*T^*M$. Denoting by $\langle\ ,\ \rangle$ the canonical pairing between the tangent and the cotangent bundle of M and by $\alpha^\#$ the bundle map $T^*M \to TM$ induced by the Poisson bivector field α, we may write

$$S = \int_\Sigma\left(\langle\,\eta\,,\,\mathrm{d}X\,\rangle + \frac{1}{2}\epsilon\,\langle\,\eta\,,\,\alpha^\#(X)\eta\,\rangle\right).$$

1.4.1. Observables. — In the case when Σ is the upper half plane, c has to vanish on the boundary. This implies that $\delta X(u) = 0$ for $u \in \partial\Sigma$. As a consequence,

$$\mathcal{O}_{f_1,\dots,f_k;u_1,\dots,u_k} := f_1(X(u_1)) \cdots f_k(X(u_k)),$$
$$f_1, \dots, f_k \in C^\infty(\mathbb{R}^n),\ u_1, \dots, u_k \in \partial\Sigma \cong \mathbb{R},\ u_1 < \cdots < u_k,$$

are observables (i.e., δ-closed functions). It is shown in [**5**] that, with the gauge-fixing $\mathrm{d}^*\eta = 0$ for the Euclidean metric on Σ, one has

$$\langle\, \mathcal{O}_{f_1,\dots,f_k;u_1,\dots,u_k} \,\rangle(x) = f_1 \star \cdots \star f_k(x),$$

where $\langle\ \ \rangle(x)$ denotes the expectation value for $X(\infty) = x$ (and expanding only around the trivial critical solution $X = x$, $\eta = 0$) while $\star$ is Kontsevich's star product for the given Poisson structure. In 1.5 we will derive this result in the case when α is at most linear so that the BRST method is available.

1.5. Deformation quantization of affine Poisson structures

An affine Poisson structure on a vector space $\mathbb{R}^n$ is the datum of Poisson bivector field α that is at most linear. The linear part of α gives the dual of $\mathbb{R}^n$ a Lie algebra structure while the constant part is a 2-cocycle in the Lie algebra cohomology with trivial coefficients. From now on we will denote by $\mathfrak{g}$ this Lie algebra. The fields of the Poisson sigma model for an affine Poisson structure are then a map $X\colon \Sigma \to \mathfrak{g}^*$ and a one-form η on Σ taking values in $\mathfrak{g}$. As $\mathfrak{g}$ is a Lie algebra, we may regard η as a connection one-form for a trivial principal bundle P over Σ (with structure group any Lie group whose Lie algebra is $\mathfrak{g}$). For definiteness, we will fix Σ to be the upper half plane and we will require η to vanish at infinity and on the boundary. The action function reads

$$S = \int_\Sigma \left(\eta_i \,\mathrm{d}X^i + \frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j \right),$$

where

$$\alpha^{ij}(x) = \chi^{ij} + x^k f_k^{ij}$$

is the given affine Poisson structure on $\mathfrak{g}^*$. Integrating by parts, we may also rewrite it as

$$S = \int_\Sigma \left(\langle\, X \,,\, F_\eta \,\rangle + \frac{1}{2}\chi(\eta,\eta) \right),$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ while

$$(F_\eta)_i = \mathrm{d}\eta_i + \frac{1}{2} f_i^{jk}\eta_j\eta_k$$

is the curvature two-form of the connection one-form η. In accordance with Sect. 1.4, there is a Lie algebra Γ—which as a vector space consists of functions $\Sigma \to \mathfrak{g}$ vanishing at infinity and on the boundary—that acts on the manifold of fields $\mathcal{M}$ and leaves the action function invariant. The BRST operator on $\mathcal{M} \times \Pi\Gamma$ has the form displayed in Theorem 1.4.3 (with $\epsilon = 1$). Geometrically then we may regard Γ as the Lie algebra of infinitesimal gauge transformations of the principal bundle P; the field η actually transforms as a connection one-form, while X transform as a section of the coadjoint bundle in case $\chi = 0$. For $\chi \neq 0$, we may regard $X \oplus 1$ as a section of the coadjoint bundle for the Lie algebra $\widehat{\mathfrak{g}} \simeq \mathfrak{g} \oplus \mathbb{R}$ obtained by central extension of $\mathfrak{g}$ through χ. The BRST operator on $\Pi\Gamma^* \times \Gamma^*$ has the usual form displayed in (1.3.5).

1.5.1. Gauge-fixing and Feynman diagrams. — As in 1.3.3.1 and 1.3.3.2, we choose a metric on Σ and define the gauge-fixing function $F(X,\eta) = \mathrm{d}^*\eta$. The gauge-fixing fermion is then $\Psi_F = \int_\Sigma \langle\, \bar{c}\,,\, \mathrm{d}^*\eta\,\rangle$ and the gauge-fixed action function reads

$$S_F = \int_\Sigma \left(\eta_i \,\mathrm{d}X^i + \frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j + \lambda^i \,\mathrm{d}^*\eta_i - \bar{c}^k \,\mathrm{d}^*(\mathrm{d}c_k + \partial_k\alpha^{ij}(X)\eta_i c_j) \right).$$

We now fix the value of X at infinity to x. We write $X = x + \xi$, where the field ξ has to vanish at infinity. We observe that S_F has the form given in (1.2.13) on page 22 (with y the collection of ξ, $\bar{c}$, λ, and z the collection of η and c). Thus, we write $S_F = S_0 + S_1$, with

$$S_0 = \int_\Sigma \left(\eta_i \,\mathrm{d}X^i + \lambda^i \,\mathrm{d}^*\eta_i - \bar{c}^k \,\mathrm{d}^*\,\mathrm{d}c_k\right),$$
$$S_1 = \int_\Sigma \left(\frac{1}{2}\alpha^{ij}(x+\xi)\eta_i\eta_j - \bar{c}^k \,\mathrm{d}^*(\partial_k\alpha^{ij}(x+\xi)\eta_i c_j) \right),$$

and regard S_1 as a perturbation of S_0. The unperturbed action function S_0 consists of $\dim\mathfrak{g}$ copies of the action function (1.3.7) which we have studied before. The perturbation S_1 may be rewritten, integrating by parts, as

$$S_1 = \int_\Sigma \left(\frac{1}{2}\alpha^{ij}(x+\xi)\eta_i\eta_j - \mathrm{d}^*\bar{c}^k \,\partial_k\alpha^{ij}(x+\xi)\eta_i c_j \right).$$

Introducing "superfields" $\widetilde{\xi}$ and $\widetilde{\eta}$ as in (1.3.8), we have finally

$$S_1 = \int_\Sigma \frac{1}{2}\alpha^{ij}(x+\widetilde{\xi})\widetilde{\eta}_i\widetilde{\eta}_j,$$

where integration on Σ is understood to select the two-form component. This shows that, as long as also the observables under consideration may be written as functions of the superfields, expectation values are computed in terms of the superpropagators (1.3.10) only. If we denote the superpropagator graphically as an arrow from $\widetilde{\eta}$ to $\widetilde{\xi}$, the perturbation S_1 is represented by the two vertices in figure 7, with the bivalent vertex corresponding to $\alpha^{ij}(x) = \chi^{ij} + x^k f_k^{ij}$ and the trivalent vertex corresponding to the structure constants.

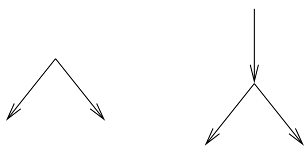

FIGURE 7. The two vertices.

We now consider the observables $\mathcal{O}_{f_1,\dots,f_k;u_1,\dots,u_k}$ introduced in (1.4.1). As evaluation at a point, i.e., integration along a zero-cycle, is understood to select the zero-form component of a differential form, we may write

$$\mathcal{O}_{f_1,\dots,f_k;u_1,\dots,u_k} = f_1(x+\widetilde{\xi}(u_1))\cdots f_k(x+\widetilde{\xi}(u_k)),$$
$$f_1,\dots,f_k \in C^\infty(\mathbb{R}^n),\ u_1,\dots,u_k \in \partial\Sigma \cong \mathbb{R},\ u_1 < \cdots < u_k.$$

As a consequence, in computing the expectation value $\langle\ \mathcal{O}_{f_1,\dots,f_k;u_1,\dots,u_k}\ \rangle(x)$, we only need the superpropagator. The corresponding Feynman diagrams then have three kinds of vertices:

1. bivalent vertices in the upper half plane corresponding to $\alpha^{ij}(x)$;
2. trivalent vertices in the upper half plane corresponding to f_k^{ij};
3. l-valent vertices, $l \geq 0$ with only incoming arrows at one of the boundary points u_i corresponding to the lth derivative of f_i.

As observed in 1.2.3.1, the normal ordering prescription excludes all graphs containing a tadpole (i.e., an edge whose head and tail are the same vertex). The combinatorics prevents automatically vacuum subgraphs. See figure 8 for examples.

FIGURE 8. An allowed graph and a non-allowed graph (tadpole).

In the case $k = 2$ and considering the gauge-fixing $\mathrm{d}^*\eta = 0$ w.r.t. the Euclidean metric on the upper half plane, i.e., with the superpropagator determined by the one-form ϑ in Lemma 1.3.9, we get

$$\langle\ \mathcal{O}_{f,g;0,1}\ \rangle(x) = f \star g(x),$$

where $\star$ denotes Kontsevich's star product for the given affine Poisson structure. We leave the (now) easy proof of this fact as an exercise.

1.5.1.1. Independence from the evaluation points. — Reasoning as in 1.2.2.4, we give a formal proof of the independence of the expectation values of $\mathcal{O}_{f_1,\dots,f_k;u_1,\dots,u_k}$ from the points $u_1,\dots,u_k$. The first observation is that

$$f(X(v)) - f(X(u)) = \int_u^v \mathrm{d}X^i\partial_i f(X) = \int_u^v (\mathrm{d}X^i + \mathrm{d}^*\lambda^i)\partial_i f(X) - \delta\Phi,$$

with $\Phi = \int_u^v \mathrm{d}^*\bar{c}^i\partial_i f(X)$. Let ω_r be a sequence of one-forms on Σ vanishing on the boundary and at infinity that converges to the measure concentrated on the interval $(u,v) \in \Sigma$.[9] Let $Y_{f,r}$ be the local vector field on $\widetilde{\mathcal{M}}$ corresponding to the infinitesimal

[9] Denoting by (a,b), $b \geq 0$, the coordinates on the upper half plane Σ, a possible choice for this sequence is

$$\omega_r(a,b) = rb\,\mathrm{e}^{-\frac{rb^2}{2}}\,\chi_r(a)\,\mathrm{d}a,$$

displacement of η_i by $\omega_r \partial_i f(X)$. Then

$$f(X(v)) - f(X(u)) = \lim_{r\to\infty} Y_{f,r}(S_F) - \delta\Phi.$$

If $\mathcal{O}$ is a BRST-observable depending on fields outside the closed interval $[u, v]$, we have

$$\langle\, (f(X(v)) - f(X(u)))\, \mathcal{O} \,\rangle = \mathrm{i}\hbar \lim_{r\to\infty} \langle\, Y_{f,r}(\mathcal{O}) \,\rangle - \langle\, \delta(\Phi\mathcal{O}) \,\rangle = 0.$$

1.5.1.2. Associativity. — The independence from the evaluation points shows in particular that

$$\lim_{v\to u^+} \langle\, \mathcal{O}_{f,g,h;u,v,w} \,\rangle_0(x) = \lim_{v\to w^-} \langle\, \mathcal{O}_{f,g,h;u,v,w} \,\rangle_0(x).$$

Intuitively the l.h.s. corresponds to evaluating first the expectation value of $\mathcal{O}_{f,g;u,v}$, placing the result at u and finally computing the expectation value of $\mathcal{O}_{\langle\, \mathcal{O}_{f,g;u,v} \,\rangle_0, h; w}$. The result is then $(f\star g)\star h$. Repeating the computation on the r.h.s. we get $f\star(g\star h)$.

This rather formal argument explains why one should expect the star product defined by the Poisson sigma model to be associative.

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where χ_r is a sequence of smooth, compactly supported functions converging to the characteristic function of the interval (u, v) almost everywhere.

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