THE LAGRANGIAN OPERAD

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1. Introduction

1.1. **Basic ideas.** The fundamental idea behind the construction presented in this note is the observation that Lagrangianity is preserved by symplectic reduction [3, Lect. 3]. More precisely, let M be a symplectic manifold, C and L a coisotropic and a Lagrangian submanifold respectively. We denote by M_C the leaf space of the canonical foliation of C. Let $\pi\colon C \to M_C$ be the projection and L_C the image of $L \cap C$ under π. In the best possible situation, M_C is a manifold. Then the restriction of the symplectic form of M to C is basic and induces a symplectic form on M_C. If the intersection of L with C is *clean*,[1] then the image L_C is a Lagrangian immersion of M_C. The immersion will be an imbedding if in addition L intersects leaves of C at most once.

The second fundamental idea is the following: If C is a coisotropic submanifold of a symplectic manifold M and N is a second symplectic manifold, then $C \times N$ is a coisotropic submanifold of $M \times N$, and the reduced manifold $(M \times N)_{(C\times N)}$ is the same as $M_C \times N$.

Given a symplectic manifold N, we denote by $\overline{N}$ the same manifold but with opposite symplectic structure. The final observation is then that the diagonal Δ is Lagrangian, and hence coisotropic, in $N \times \overline{N}$. The reduced manifold $(N \times \overline{N})_\Delta$ is just a point. As a consequence of the previous observation, $\Delta \times M$ is coisotropic in $N \times \overline{N} \times M$, and symplectic reduction yields then M.

2. The Lagrangian operad

Let N be a symplectic manifold and $\overline{N}$ the same manifold with opposite symplectic form. Let $\mathcal{O}_k$, $k \geq 0$, be some subset of the set of Lagrangian submanifolds of $N^k \times \overline{N}$. Given $L_k \in \mathcal{O}_k$ and $L_{i_j} \in \mathcal{O}_{i_j}$, $j = 1, \dots, k$, we may regard $L = L_k \times L_{i_1} \times \cdots \times L_{i_k}$ as a Lagrangian submanifold of $N^k \times \overline{N} \times N^{i_1} \times \overline{N} \times \cdots \times N^{i_k} \times \overline{N}$, which by a permutation of the factors is the same as $X = N^{i_1+\cdots+i_k} \times \overline{N} \times (N \times \overline{N})^k$. More precisely, the $\overline{N}$ left alone is the $\overline{N}$ of $N^k \times \overline{N}$, while the jth factor $N \times \overline{N}$

[1] One says that L and C have a clean intersection if $L \cap C$ is a manifold and $TL \cap TC = T(L \cap C)$. A special case is that of transverse intersection.

is obtained by pairing the jth N in $N^k \times \overline{N}$ with the $\overline{N}$ in $N^{i_j} \times \overline{N}$. Now denote by Δ the diagonal in $N \times \overline{N}$ by Δ^k the multidiagonal in $(N \times \overline{N})^k$, and consider $N^{i_1+\cdots+i_k} \times \overline{N} \times \Delta^k$. This is a coisotropic submanifold of X and symplectic reduction yields $N^{i_1+\cdots+i_k} \times \overline{N}$. If the Lagrangian submanifolds L_k, L_{i_j} are chosen well enough, then L determines a Lagrangian submanifold $L_k\{L_{i_1}, \ldots, L_{i_k}\}$ of $N^{i_1+\cdots+i_k} \times \overline{N}$. (If they are chosen badly, the result might not be a submanifold.)[2]

Assume now that the sets $\mathcal{O}_k$ are chosen in such a way, if possible, that $\forall L_k \in \mathcal{O}_k$ and $\forall L_{i_j} \in \mathcal{O}_{i_j}$, $j = 1, \ldots, k$, the result $L_k\{L_{i_1}, \ldots, L_{i_k}\}$ of the above construction is not only a Lagrangian submanifold of $N^{i_1+\cdots+i_k} \times \overline{N}$ but it is actually an element of $\mathcal{O}_{i_1+\cdots+i_k}$. Then we have a map

$$\mathcal{O}_k \times \mathcal{O}_{i_1} \times \cdots \times \mathcal{O}_{i_k} \to \mathcal{O}_{i_1+\cdots+i_k}.$$

One may immediately verify that the $\mathcal{O}_k$s with these maps form an operad $\mathcal{O}(N)$ which we will call the Lagrangian operad of N. For this operad to have a unit, we will always also require that the diagonal Δ of $N \times \overline{N}$ should be an element of $\mathcal{O}_1$. (It is not difficult to show that $\Delta\{\Delta\} = \Delta$.)

A trivial choice of Lagrangian operad always exists: simply set $\mathcal{O}_1 = \{\Delta\}$, and $\mathcal{O}_k = \emptyset$, $k \neq 1$.

If we are given a Lagrangian submanifold Δ_2 of $N^2 \times \overline{N}$, we may then consider the Lagrangian operad $\mathcal{O}(N; \Delta_2)$ generated by Δ_1 and Δ_2.

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[2] The construction may also be explained as follows. Take first $L_k \in \mathcal{O}_k$ and $L_j \in \mathcal{O}_j$ and choose $i \in \{1, \ldots, k\}$. Then $L_k \times L_j$ is Lagrangian in $N^k \times \overline{N} \times N^j \times \overline{N}$ which by permuting the factors is the same as $N^{k+j-1} \times \overline{N} \times N \times \overline{N}$. Here the last $\overline{N}$ comes from $N^j \times \overline{N}$, while the last N is the ith factor in $N^k \times \overline{N}$; the j factors N in $N^j \times \overline{N}$ are inserted in place of the removed factor N in $N^k \times \overline{N}$. In other words, we use the permutation

$$(1, \ldots\ldots, k+j+2) \mapsto (1, \ldots, i-1, k+2, \ldots, k+j+1, i+1, \ldots, k, k+1, i, k+j+2).$$

Consider the coisotropic submanifold $N^{k+j-1} \times \overline{N} \times \Delta$ in $N^{k+j-1} \times \overline{N} \times N \times \overline{N}$ and perform the reduction. Denote by $L_k \circ_i L_j$ the element of $\mathcal{O}_{k+j-1}$ obtained this way. If we now have k elements $L_{i_1}, \ldots, L_{i_k}$, we may define $L_k\{L_{i_1}, \ldots, L_{i_k}\}$ as

$$(\ldots((L_k \circ L_{i_k}) \circ_{k-1} L_{i_{k-1}}) \circ_{k-2} \ldots) \circ_1 L_{i_1}.$$

Vice versa, *if the diagonal Δ of $N \times \overline{N}$ is an element of $\mathcal{O}_1$*, one may recover the $\circ$ operation from $L_k\{L_{i_1}, \ldots, L_{i_k}\}$ by setting

$$L_k \circ_i L_j = L_k\{\Delta, \ldots, \Delta, \underbrace{L_j}_{i\text{th}}, \Delta, \ldots, \Delta\}.$$

Each $\mathcal{O}_k$, $k > 0$, will now be a finite, nonempty set.[3] We may enlarge this operad by allowing in $\mathcal{O}_k$ a small (or formal) neighborhhood of each of its elements in the space of Lagrangian submanifolds.

A special situation arises if we are given in addition a Lagrangian submanifold Δ_0 of $\overline{N}$. In this case, we may consider the Lagrangian operad $\mathcal{O}(N; \Delta_0, \Delta_2)$ generated by Δ_0, Δ_1 and Δ_2 and then enlarge it as described above.

If N is a symplectic groupoid [1, 2], then we are given for free elements Δ_0 and Δ_1: viz., the identity section and the graph of the product, respectively. It turns out that $\mathcal{O}(N)(\Delta_0, \Delta_2)$ has exactly one element in each degree. (The inverse does not play a role here as its graph is a Lagrangian submanifold of N^2 or of $\overline{N}^2$.)

A particular case of symplectic groupoid is $N = T^*M$ with its canonical symplectic structure and product defined by $(x, p) \cdot (x, p') = (x, p + p')$. In this case, $\mathcal{O}_k = \{\Delta_k\}$, with $\Delta_0 = \{(x, 0) \in \overline{T^*M}\}$, $\Delta_1 = \{((x, p), (x, p)) \in T^*M \times \overline{T^*M}\}$, and in general

$$\Delta_k = \{((x, p_1), (x, p_2), \ldots, (x, p_k), (x, p_1 + \cdots + p_k)) \in T^*M^k \times \overline{T^*M}\}.$$

We will call this operad the cotangent Lagrangian operad of M. In the next Sections we will work out properties of the enlargement of this operad to "formal neighborhoods" of the Δ_k's in the case $M = \mathbb{R}^n$.

3. The local Lagrangian operad

Let us consider the the cotangent Lagrangian operad of $M = \mathbb{R}^n$. Since $N = T^*\mathbb{R}^n$ is a trivial bundle, beside the projection $\pi \colon N \to \mathbb{R}^n$ we also have the projection onto the fiber $\varpi \colon N \to (\mathbb{R}^n)^*$. Setting $B_k = ((\mathbb{R}^n)^*)^k \times \mathbb{R}^n$, we have the projections

$$\begin{array}{cccc} \varpi \times \ldots \varpi \times \pi \colon & N^k \times \overline{N} & \to & B_k \\ & ((x_1, p^1), \ldots, (x_k, p^k), (x, p)) & \mapsto & (p^1, \ldots, p^k, x). \end{array}$$

Observe moreover that $N^k \times \overline{N}$ can be identified with T^*B_k and that functions on B_k determine Lagrangian sections (so potential elements of a Lagrangian operad of N). In particular, the elements Δ_k of the cotangent Lagrangian operads of $\mathbb{R}^n$ are determined by the following generating functions

$$S_0(x) = 0,$$

$$S_k(p^1, \ldots, p^k, x) = x \cdot \sum_{i=1}^{k} p^i, \qquad k > 0,$$

[3] E.g., $\mathcal{O}_3$ will contain $\Delta_2\{\Delta_2, \Delta_1\}$ and $\Delta_2\{\Delta_1, \Delta_2\}$, which in some cases may coincide.

Here the dot denotes the canonical pairing between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$.

We want now to enlarge this operad by allowing formal neighborhoods of the Lagrangian sections Δ_k. We consider two possible ways to do it which we will denote by $\mathcal{O}^\epsilon$ and $\mathcal{O}^p$. In the former case we set

$$\mathcal{O}^\epsilon_k = \{S_k + \epsilon f : f \in C^\infty(B_k)[[\epsilon]], f(0) = 0\},$$

while in the latter the definition is

$$\mathcal{O}^p_k = \{S_k + f : f \in C^\infty(\mathbb{R}^n)[[p^1, \dots, p^k]],\ \deg_p f \geq 2\}.$$

When it is not necessary to distinguish between the two cases, we will simply write $\mathcal{O}$. Each of these enlargements of the cotangent Lagrangian operad of $\mathbb{R}^n$ will be referred to as the local Lagrangian operad in n dimensions.

Observe that a generating function $F \in \mathcal{O}_k$ determines the Lagrangian submanifold

$$\left(\left(\frac{\partial F}{\partial p^1}, p^1\right), \dots, \left(\frac{\partial F}{\partial p^k}, p^k\right), \left(x, \frac{\partial F}{\partial x}\right)\right).$$

We now want now to give an explicit expression for the generating function $F\{G_1, \dots, G_k\}$ corresponding to the Lagrangian section obtained by composing the Lagrangian sections of F, $G_1, \dots, G_k$.

Theorem 3.1. *The operad composition in $\mathcal{O}^p$ is given by $F\{G_1, \dots, G_k\} = \Phi \in \mathcal{O}^p_{l_1+\dots+l_k}$ with*

$$\Phi(\pi^{11}, \dots, \pi^{1l_1}, \pi^{21}, \dots, \pi^{2l_2}, \dots\dots, \pi^{k1}, \dots, \pi^{kl_k}, x) = \tag{3.1}$$
$$= \left[F(p^1, \dots, p^k, x) + \sum_{i=1}^k \left(G_i(\pi^{i1}, \dots, \pi^{il_i}, y_i) - p^i \cdot y_i\right)\right]\Bigg|_{\substack{y_i = \frac{\partial F}{\partial p^i} \\ p^i = \frac{\partial G_i}{\partial y_i}}\ \forall i}.$$

In $\mathcal{O}^\epsilon$ the composition is given by $F\{G_1, \dots, G_k\} = \Phi - \Phi(0) \in \mathcal{O}^\epsilon_{l_1+\dots+l_k}$.

Proof. The product of the Lagrangian submanifolds determined by F and Gs is a Lagrangian section of $N^k \times \overline{N} \times N^{l_1} \times \overline{N} \times \dots \times N^{l_k} \times \overline{N}$ over $B^k \times B^{l_1} \times \dots \times B^{l_k}$. If we denote a point in the base manifold by $(p^1, \dots, p^k, x; \pi^{11}, \dots, \pi^{1l_1}, y_1; \dots\dots; \pi^{k1}, \dots, \pi^{kl_k}, y_k)$, then the

Lagrangian section is given by

$$\begin{aligned}(3.2)\quad \Bigg(&\left(\frac{\partial F}{\partial p^1},p^1\right),\dots,\left(\frac{\partial F}{\partial p^k},p^k\right),\left(x,\frac{\partial F}{\partial x}\right),\\ &\left(\frac{\partial G_1}{\partial \pi^{11}},\pi^{11}\right),\dots,\left(\frac{\partial G_1}{\partial \pi^{1l_1}},\pi^{1l_1}\right),\left(y_1,\frac{\partial G_1}{\partial y_1}\right),\\ &\dots\dots\\ &\left(\frac{\partial G_k}{\partial \pi^{k1}},\pi^{k1}\right),\dots,\left(\frac{\partial G_k}{\partial \pi^{kl_k}},\pi^{kl_k}\right),\left(y_k,\frac{\partial G_k}{\partial y_k}\right)\Bigg).\end{aligned}$$

Restriction to the diagonal imposes then the equalities:

$$\left(\frac{\partial F}{\partial p^i},p^i\right)=\left(y_i,\frac{\partial G_i}{\partial y_i}\right), \tag{3.3}$$

$i=1,\dots,k$. Observe that these equations may be solved both in $\mathcal{O}^\epsilon$ and in $\mathcal{O}^p$; the unique solution has the form

$$y_i(\pi^{11},\dots\dots,\pi^{kl_k},x)=x+\cdots, \tag{3.4a}$$

$$p^i(\pi^{11},\dots\dots,\pi^{kl_k},x)=\sum_{r=1}^{l_i}\pi^{ir}+\cdots. \tag{3.4b}$$

Projecting down to $N^{l_1+\dots l_k}\times\overline{N}$, we get finally a Lagrangian section over $B^{l_1+\dots l_k}$. If we denote by $(\pi^{11},\dots\dots,\pi^{kl_k},x)$ a point in the base space, a point in the section is given by

$$\left(\left(\frac{\partial G_1}{\partial \pi^{11}},\pi^{11}\right),\dots\dots,\left(\frac{\partial G_k}{\partial \pi^{kl_k}},\pi^{kl_k}\right),\left(x,\frac{\partial F}{\partial x}\right)\right)$$

evaluated at the solution of (3.3). We now have to check that Φ generates this Lagrangian section. But this is true since, as can be easily verified,

$$\frac{\partial \Phi}{\partial \pi^{il_i}}=\frac{\partial G_i}{\partial \pi^{il_i}}\Bigg|_{\substack{y_i=\frac{\partial F}{\partial p^i}\\ p^i=\frac{\partial G_i}{\partial y_i}}\ \forall i},\qquad \frac{\partial \Phi}{\partial x}=\frac{\partial F}{\partial x}\Bigg|_{\substack{y_i=\frac{\partial F}{\partial p^i}\\ p^i=\frac{\partial G_i}{\partial y_i}}\ \forall i}.$$

Finally, using (3.4), one sees that $\Phi=S_{l_1+\dots+l_k}+\cdots$. One can also easily verify that the dots actually correspond to what we ask for in $\mathcal{O}^\epsilon$ or in $\mathcal{O}^p$. In the former case, it may be checked by particular examples that $\Phi(0)$ may not vanish; so, in order to get an element of $\mathcal{O}^\epsilon$ one has to subtract the constant $\Phi(0)\in\epsilon\mathbb{R}[[\epsilon]]$. $\square$

Remark 3.2 (Saddle point formula). Formula (3.1) for Φ can be interpreted in terms of saddle point evaluation for $\hbar \to 0$ of the following integral:

$$\int_{((\mathbb{R}^n)^*\times\mathbb{R}^n)^k} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left[F(p^1,\dots,p^k,x)+\sum_{i=1}^k\left(G_i(\pi^{i1},\dots,\pi^{il_i},y_i)-p^i\cdot y_i\right)\right]} \prod_{i=1}^k \frac{\mathrm{d}^n p^i\,\mathrm{d}^n y_i}{(2\pi\hbar)^n} = \\ = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Phi(\pi^{11},\dots,\pi^{1l_1},\pi^{21},\dots,\pi^{2l_2},\dots\dots,\pi^{k1},\dots,\pi^{kl_k},x)}\,(C+O(\hbar)),$$

with $C = 1 + O(\epsilon)$ for the case of $\mathcal{O}^\epsilon$ and $C = 1 + O(\pi)$ for the case of $\mathcal{O}^p$.

Exercise 3.3. Let $F(p^1,\dots,p^k,x) = S_k(p^1,\dots,p^k,x)+\epsilon\sum_i p^i\cdot a_i \in \mathcal{O}^\epsilon_k$ and $G_i(\pi^{i1},\dots,\pi^{il_i},y_i) = S_{l_i}(\pi^{i1},\dots,\pi^{il_i},y_i)+\epsilon b^i\cdot y_i \in \mathcal{O}^\epsilon_{l_i}$ with $a_i\in\mathbb{R}^n$ and $b^i\in(\mathbb{R}^n)^*$, $i=1,\dots,k$. Show that

$$\Phi(\pi^{11},\dots,\pi^{1l_1},\pi^{21},\dots,\pi^{2l_2},\dots\dots,\pi^{k1},\dots,\pi^{kl_k},x) = \\ = S_{l_1+\dots+l_k}(\pi^{11},\dots,\pi^{1l_1},\pi^{21},\dots,\pi^{2l_2},\dots\dots,\pi^{k1},\dots,\pi^{kl_k},x)+ \\ +\epsilon\left(\sum_i b^i\cdot x + \sum_{ir}\pi^{ir}\cdot a_i\right)+\epsilon^2\sum_i b^i\cdot a_i.$$

This shows that $\Phi(0)$ may not vanish.

Exercise 3.4. Given $S_{l_i}+W_i\in\mathcal{O}_{l_i}$, $i=1,\dots,k$, show that

$$S_k\{S_{l_1}+W_1,\dots,S_{l_k}+W_k\}(\pi^{11},\dots\dots,\pi^{kl_k},x) = \\ = S_{\sum_i l_i}+\sum_{i=1}^k W_i(\pi^{i1},\dots,\pi^{il_i},x). \tag{3.5}$$

3.1. **Simplified notation.** As our functions are always of the form S plus something, it is useful to get rid of S in the notation. We then define $V\boldsymbol{\{}W_1,\dots,W_k\boldsymbol{\}}$ by

$$(S_k+V)\{S_{l_1}+W_1,\dots,S_{l_k}+W_k\} = S_{l_1+\dots+l_k}+V\boldsymbol{\{}W_1,\dots,W_k\boldsymbol{\}}$$

in $\mathcal{O}^p$, and

$$(S_k+\epsilon V)\{S_{l_1}+\epsilon W_1,\dots,S_{l_k}+\epsilon W_k\} = S_{l_1+\dots+l_k}+\epsilon V\boldsymbol{\{}W_1,\dots,W_k\boldsymbol{\}}$$

in $\mathcal{O}^\epsilon$. Let us denote by 0_k the zero function on B_k. Using the simplified notation, we may rewrite (3.5) as follows:

$$0_k\boldsymbol{\{}W_1,\dots,W_k\boldsymbol{\}} = W_1\cup\dots\cup W_k \tag{3.6}$$

with

$$(W_1 \cup \cdots \cup W_k)(\pi^{11}, \ldots\ldots, \pi^{kl_k}, x) = \sum_{i=1}^{k} W_i(\pi^{i1}, \ldots, \pi^{il_i}, x).$$

Notice that this definition of the cup product is independent of x, which can then be considered a parameter.

Exercise 3.5 (The quadratic operad)**.** Let $S_k + \widetilde{V} \in \mathcal{O}^p_k$, $S_{l_i} + \widetilde{W}_i \in \mathcal{O}^p_{l_i}$, $i = 1, \ldots, k$. Denote by V and W_i the quadratic parts in the momenta of $\widetilde{V}$ and $\widetilde{W}_i$. Then show that the quadratic part of $\widetilde{V}\{\widetilde{W}_1, \ldots, \widetilde{W}_k\}$ is given by $V\{W_1, \ldots, W_k\}_0$ with

$$\begin{aligned}(3.7)\quad & V\{W_1, \ldots, W_k\}_0(\pi^{11}, \ldots\ldots, \pi^{kl_k}, x) = \\ &\qquad = V(\sum_{j=1}^{l_1} \pi^{1j}, \ldots, \sum_{j=1}^{l_k} \pi^{kj}, x) + \sum_{i=1}^{k} W_i(\pi^{i1}, \ldots, \pi^{il_i}, x).\end{aligned}$$

This defines an operad structure on quadratic functions. Namely, if we forget the parameter x and set

$$\mathcal{Q}_k(\mathbb{R}^n) = \{\text{quadratic functions on } (\mathbb{R}^n)^k\},$$

then $\mathcal{Q}$ is an operad with composition defined as above.

Let us work out in details the quadratic operad for $n = 1$. In this case $\mathcal{Q}_k = \{\text{symmetric } k \times k \text{ matrices}\}$. Given $V \in \mathcal{Q}_k$ and $W_i \in \mathcal{Q}_{l_i}$, $i = 1, \ldots, k$, the composition $V\{W_1, \ldots, W_k\}_0 \in cQ_{l_1+\cdots+l_k}$ is defined as the matrix

$$\begin{aligned}&(V\{W_1, \ldots, W_k\}_0)_{ir_i, js_j} = V_{ij} + \delta_{ij}(W_i)_{r_i s_i}, \\ &\qquad\qquad i, j = 1, \ldots, k,\ r_i = 1, \ldots, l_i,\ s_j = 1, \ldots, l_j.\end{aligned}$$

This clearly defines an operad structure on $\mathcal{Q}$. The formula simplifies when $V = 0_k$, the $k \times k$ zero matrix: $0_k\{W_1, \ldots, W_k\}_0 = W_1 \oplus \cdots \oplus W_k$.

Exercise 3.6. Let $S_k + \epsilon\widetilde{V} \in \mathcal{O}^\epsilon_k$, $S_{l_i} + \epsilon\widetilde{W}_i \in \mathcal{O}^\epsilon_{l_i}$, $i = 1, \ldots, k$. Write $\widetilde{V} = V + \epsilon v + O(\epsilon^2)$ and $\widetilde{W}_i = W_i + \epsilon w_i + O(\epsilon^2)$. Then show that

$$\begin{aligned}&\widetilde{V}\{\widetilde{W}_1, \ldots, \widetilde{W}_k\} = V\{W_1, \ldots, W_k\}_0 + \\ &\qquad + \epsilon(v\{w_1, \ldots, w_k\}_0 + V\{W_1, \ldots, W_k\}_1) + O(\epsilon^2),\end{aligned}$$

with $\{\ \}_0$ defined in (3.7) and $V\{W_1,\dots,W_k\}_1 = V\{W_1,\dots,W_k\}_1^0 - V\{W_1,\dots,W_k\}_1^0(0)$ with

$$V\{W_1,\dots,W_k\}_1^0(\pi^{11},\dots\dots,\pi^{kl_k},x) = \\ = \sum_{i=1}^k \frac{\partial V}{\partial p^i}(\sum_{j=1}^{l_1}\pi^{1j},\dots,\sum_{j=1}^{l_k}\pi^{kj},x)\,\frac{\partial W_i}{\partial x}(\pi^{i1},\dots,\pi^{il_i},x).$$

Remark 3.7. The last exercise in particular shows that functions on products of $\mathbb{R}^n$s form an operad with the composition law given in (3.7). Quadratic functions form the suboperad described in Exercise 3.5.

3.2. **Products.** Let us fix $S_2 + V \in \mathcal{O}_2$. Then we define $W_1 *_V W_2$ as $V\{W_1, W_2\}$. If $V = 0$, by (3.6) we simply have

$$W_1 *_0 W_2 = W_1 \cup W_2.$$

So the product $*_0$ is associative and commutative. It may be proved, by checking the identity $(0_1 *_V 0_1) *_V 0_1 = 0_1 *_V (0_1 *_V 0_1)$, that $S_2 + V$ is the generating function of a symplectic groupoid structure on $T^*\mathbb{R}^k$ if $*_V$ is associative. The converse is conjectured.

3.3. **Compositions.** Using the distinguished element S_1 (function 0_1), we may define the ith composition of two elements F and G (functions V and W) by the following rule:

$$F \circ_i G = F\{S_1,\dots,S_1,\underbrace{G}_{i\text{th}},S_1,\dots,S_1\},$$
$$V \circ_i W = V\{0_1,\dots,0_1,\underbrace{W}_{i\text{th}},0_1,\dots,0_1\}.$$

Formula (3.1) may be simplified as follows: $F \circ_i G = \Psi$ (or $\Psi - \Psi(0)$ in $\mathcal{O}^\epsilon$ with

$$\Psi(p^1,\dots,p^{i-1},(\pi^1,\dots,\pi^l),p^{i+1},\dots,p^k,x) = \\ = \left(F(p^1,\dots,p^k,x) + G(\pi^1,\dots,\pi^l,y) - p^i\cdot y\right)\Big|_{\substack{y=\frac{\partial F}{\partial p^i}\\ p^i=\frac{\partial G}{\partial y}}}\cdot$$

This can also be given by a saddle-point approximation:

$$\int_{(\mathbb{R}^n)^*\times\mathbb{R}^n} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(F(p^1,\dots,p^k,x)+G(\pi^1,\dots,\pi^l,y)-p^i\cdot y\right)}\,\frac{\mathrm{d}^n p^i\,\mathrm{d}^n y}{(2\pi\hbar)^n} = \\ = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Psi(p^1,\dots,p^{i-1},(\pi^1,\dots,\pi^l),p^{i+1},\dots,p^k,x)}\,(C + O(\hbar)),$$

with $C = 1 + O(\epsilon)$ or $C = 1 + O(\pi)$. Observe that with this notation, $S_2 + V$ is the generating function of a symplectic groupoid structure on $T^*\mathbb{R}^n$ iff

$$V \circ_1 V = V \circ_2 V.$$

3.3.1. *Multiplication and commutator.* As is done in linear operads (but ours is not!), one may be tempted to define a multiplication $\mathcal{O}^k \times \mathcal{O}^l \to \mathcal{O}^{k+l-1}$ by

$$F \circ G := \sum_{i=1}^{k} (-1)^{(i-1)(l-1)} F \circ_i G,$$

and a commutator

$$[\, F\, ,\, G\,] := F \circ G - (-1)^{(k-1)(l-1)}\, G \circ F.$$

Using the simplified notation, we also define

$$V \circ W := \sum_{i=1}^{k} (-1)^{(i-1)(l-1)} V \circ_i W,$$

and

$$[\![\, V\, , W\,]\!] := V \circ W - (-1)^{(k-1)(l-1)}\, W \circ W.$$

In the case of linear operads, the multiplication is not associative but the corresponding commutator is still a Lie bracket. In our case, the multiplication is in general not well-defined as $F \circ G$ might not start with the required term S_{k+l-1}. Actually, the multiplication is well-defined iff k is odd and l is even. The commutator is instead well-defined iff either $k = 2$ and l is odd or k is odd and $l = 0$. We may however keep using the multiplication and the commutator in general, keeping in mind that the result is a function that does not necessarily belong to $\mathcal{O}$. The notational advantage is that, e.g., $S_2 + V$ is the generating function of a symplectic groupoid structure on $T^*\mathbb{R}^n$ iff V satisfies the "Maurer–Cartan" equation

$$[\![\, V\, , V\,]\!] = 0.$$

Exercise 3.8 (The coboundary operator)**.** Show that $[\![\, 0_2 \,,W \,]\!] = \delta W$ with

$$\begin{aligned}(3.8)\quad \delta W(p^1,\dots,p^{l+1},x) &= W(p^1,\dots,p^l,x)+\\ &+\sum_{i=1}^{l}(-1)^{l+i}\, W(p^1,\dots,p^{i-1},p^i+p^{i+1},p^{i+2},\dots,p^{l+1},x)+\\ &\qquad + (-1)^{l-1}\, W(p^2,\dots,p^{l+1},x).\end{aligned}$$

Observe that $\delta^2=0$.

Exercise 3.9. Let V and W be quadratic functions in the momenta in k and l arguments respectively. Denote by $[\![\, V \,,W \,]\!]_0$ the quadratic part of $[\![\, V \,,W \,]\!]$. Show that

$$\begin{aligned}&[\![\, V \,,W \,]\!]_0(p^1,\dots,p^{k+l-1},x) =\\ &=\sum_{i=1}^{k}(-1)^{(i-1)(l-1)}\left[V(p^1,\dots,p^{i-1},\sum_{j=1}^{i+l-1}p^j,p^{i+l},\dots,p^{k+l-1},x)+W(p^i,\dots,p^{i+l-1},x)\right]+\\ &-\sum_{i=1}^{l}(-1)^{(i+l)(k-1)}\left[V(p^i,\dots,p^{i+k-1},x)+W(p^1,\dots,p^{i-1},\sum_{j=1}^{i+k-1}p^j,p^{i+k},\dots,p^{k+l-1},x)\right].\end{aligned}$$

Is this a Lie bracket?

Exercise 3.10. Let $S_k+\epsilon\widetilde{V}\in\mathcal{O}_k^\epsilon$, $S_l+\epsilon\widetilde{W}\in\mathcal{O}_l^\epsilon$. Write $\widetilde{V}=V+\epsilon v+O(\epsilon^2)$ and $\widetilde{W}=W+\epsilon w+O(\epsilon^2)$. Then show that

$$\begin{aligned}[\![\, V \,,W \,]\!] &= (-1)^{(k-1)(l-1)}\,\delta V+\delta W+\\ &\qquad + \epsilon([\![\, V \,,W \,]\!]_1+(-1)^{(k-1)(l-1)}\,\delta v+\delta w)+O(\epsilon^2),\end{aligned}$$

with δ defined in (3.8) and

$$\begin{aligned}(3.9)\quad &[\![\, V \,,W \,]\!]_1(p^1,\dots,p^{k+l-1},x) :=\\ &=\sum_{i=1}^{k}(-1)^{(i-1)(l-1)}\,\frac{\partial V}{\partial p^i}(p^1,\dots,p^{i-1},\sum_{j=1}^{i+l-1}p^j,p^{i+l},\dots,p^{k+l-1},x)\frac{\partial W}{\partial x}(p^i,\dots,p^{i+l-1},x)+\\ &-\sum_{i=1}^{l}(-1)^{(i+l)(k-1)}\,\frac{\partial w}{\partial p^i}(p^1,\dots,p^{i-1},\sum_{j=1}^{i+k-1}p^j,p^{i+k},\dots,p^{k+l-1},x)\frac{\partial V}{\partial x}(p^i,\dots,p^{i+k-1},x),\end{aligned}$$

assuming that V and W are at least linear in the momenta.

Exercise 3.11 ("Formality")**.** Let ξ be a multivector field of degree k. Let u_0 be defined by

$$u_0(\xi)(p^1,\dots,p^k,x) := \left\langle\, \xi\,,\, p^1\wedge\cdots\wedge p^k \,\right\rangle(x) = \xi^{\mu_1\dots\mu_k}(x)p^1_{\mu_1}\dots p^k_{\mu_k}.$$

Then show that $\delta u_0(\xi) = 0$ and $[\![u_0(\xi) \,, u_0(\eta)]\!]_1 = u_0([\xi \,, \eta])$, with δ defined in (3.8) and $[\![\;\, , \;\,]\!]_1$ defined in (3.9).

4. Comments

4.1. **Kontsevich's formula and the local Lagrangian operad.** For $p \in (\mathbb{R}^n)^*$, let $u_p \in C^\infty(\mathbb{R}^n)$ be the function $u_p(x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p \cdot x}$. Let $\xi_1, \dots, \xi_k$ be multivector fields of degrees $r_1, \dots, r_k$, and let U denote Kontsevich's formality map. Then

$$U(\hbar^{r_1 - 1}\xi_1, \dots, \hbar^{r_k - 1}\xi_k)\,(u_{p_1} \otimes \cdots \otimes u_{p_s}) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} u(\xi_1, \dots, \xi_k; s) + O(1)},$$

with $u(\xi_1, \dots, \xi_k; s) \in \mathcal{O}_s$. (Actually, $\mathcal{O}_s^\epsilon$ is the multivector fields are actually in $\epsilon\Gamma(\wedge T^*M)[[\epsilon]]$ and in $\mathcal{O}_s^p$ if we assume $r_i > 1\ \forall i$.) Does this map u define some sort of semiclassical formality?

4.2. **Trees.** To compose elements of the local Lagrangian operad as described in Thm. 3.1, one has to solve certain algebraic equations. As we work with formal power series (in ϵ or in p), these equations may always be solved, starting as in (3.4). The iterative procedure may be described in terms of trees with rules very similar to those appearing in the formal or Runge–Kutta integration of vector fields (the same trees of formal diffeomorphisms appear in Connes–Kreimer description of renormalization). In view of subsection 4.1, these trees should also be related to connected Feynman diagrams of the Poisson sigma model. (I thank Giovanni Felder and his student Benoît Dherin for pointing out the contents of this subsection.)

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