Blow up for the $L^2$ critical gKdV equation

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Introduction

We consider the $L^2$ critical (gKdV) equation

\[
\begin{aligned}
\text{(gKdV)} \quad \left\{ \begin{array}{l}
  u_t + (u_{xx} + u^5)_x = 0, \quad (t, x) \in [0, T) \times \mathbb{R}, \\
  u(0, x) = u_0(x), \quad x \in \mathbb{R}.
\end{array} \right.
\end{aligned}
\]

Recall the following important facts:

- The Cauchy problem is locally well-posed in $H^1$ [Kenig-Ponce-Vega, 92] ([Kato, 83])

- Mass and energy conservation

\[
M_0 = \int u^2(t), \quad E_0 = \frac{1}{2} \int u_x^2(t) - \frac{1}{6} \int u^6(t)
\]

- Scaling invariance ($\lambda > 0$)

\[
u^\lambda(t, x) = \frac{1}{\lambda^{1/2}} u \left( \frac{t}{\lambda^3}, \frac{x}{\lambda} \right), \quad \| u^\lambda \|_{L^2} = \| u \|_{L^2}, \quad E(u^\lambda) = \frac{1}{\lambda^2} E(u)
\]
• **Solitons** are special solutions defined by \((\lambda > 0, \, x_0 \in \mathbb{R})\)

\[
R_{\lambda, x_0}^\lambda(t, x) = \frac{1}{\lambda^{1/2}} Q \left( \frac{1}{\lambda} (x - x_0) - \frac{1}{\lambda^3} t \right)
\]

\[
Q(x) = \left( \frac{3}{\cosh^2(2x)} \right)^{1/4}, \quad Q'' - Q + Q^5 = 0, \quad E(Q) = 0
\]

• Global existence for “small” \(L^2\) norm: [Weinstein, 83]

\[
\| u_0 \|_{L^2} < \| Q \|_{L^2} \Rightarrow \text{the solution is global in } H^1
\]

Main questions of this talk:

• Blow up problem for initial data:

\[
u_0 \in H^1, \quad \| Q \|_{L^2} \leq \| u_0 \|_{L^2} \leq \| Q \|_{L^2} + \alpha_0, \quad \alpha_0 \ll 1
\]

• Classification of all possible behaviors for \(\| u_0 - Q \|_{H^1} \ll 1\)
First results on blow up for $L^2$ critical gKdV

[YM-Merle, 00-02]

Assume

$$u_0 \in H^1, \quad \|Q\|_{L^2} \leq \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0, \quad \alpha_0 \ll 1$$

Then:

(i) Blow up in finite or infinite time if $E_0 < 0$.
No information on the blow up regime.

(ii) Assuming blow up, $Q$ is the universal blow up profile.

(iii) Blow up in finite time if $E_0 < 0$ and $\int_{x>1} x^6 u_0^2(x) dx < \infty$.
Moreover, for a sequence $t_n \to T$,

$$\|u_x(t_n)\|_{L^2} \leq \frac{C(u_0)}{T - t_n}$$

(iv) Global existence for minimal mass initial data with decay.
Blow up for $L^2$ critical NLS

\[(\text{NLS}) \begin{cases} 
i \partial_t u + \Delta u + |u|^{\frac{4}{N}} u = 0, & (t, x) \in [0, T) \times \mathbb{R}^N \\ u|_{t=0} = u_0 \end{cases}\]

$$\Delta Q_{\text{NLS}} - Q_{\text{NLS}} + Q_{\text{NLS}}^{1+\frac{4}{N}} = 0, \quad Q_{\text{NLS}} > 0 \text{ even}$$

• [Merle, 93]

The only $H^1$ blow up solution of (NLS) with minimal mass $\|u_0\|_{L^2} = \|Q_{\text{NLS}}\|_{L^2}$ is (up to symmetries)

$$S_{\text{NLS}}(t, x) = \frac{1}{t^{N/2}} e^{-i(\frac{|x|^2}{4t} - \frac{1}{t})} Q_{\text{NLS}} \left( \frac{x}{t} \right)$$

• Existence of unstable nontrivial $\frac{1}{(T-t)}$ blow up solutions.

[Bourgain-Wang, 98], [Krieger-Schlag, 09], [Merle-Raphaël-Szeftel, 11]
“log-log” blow up for (NLS)

- [Landman-Papanicolaou-Sulem-Sulem, 88], etc.
- log-log conjecture

- [Perelman, 01]
  Construction of a large class of log-log blow up solutions close to $Q_{NLS}$.

- [Merle-Raphaël, 03-06]
  (i) Construction of an open set in $H^1$ of log-log blow up solutions close to $Q_{NLS}$ (including all $H^1$ data with $E_0 \leq 0$ close to $Q_{NLS}$)

  $$\|\nabla u_{NLS}(t)\|_{L^2} \sim C^* \sqrt{\frac{\log |\log (T - t)|}{T - t}}$$

  (ii) Quantization of the focused mass at the blow up point $x(T)$:

  $$|u_{NLS}(t)|^2 \rightarrow \|Q_{NLS}\|_{L^2}^2 \delta_{x=x(T)} + |u^*|^2, \quad u^* \in L^2.$$
Statement of new results for critical gKdV

[YM-Merle-Raphaël, 12]

Define ($\alpha_0 \ll 1$)

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{x>1} x^{10} \varepsilon_0^2(x) dx < 1 \right\}$$

THM 1 (Negative or zero energy data close to $Q$)

Let $u_0 \in \mathcal{A}$. If $E(u_0) \leq 0$ and $u(t)$ is not a soliton, then $u(t)$ blows up in finite time $T$ with

$$\|u_x(t)\|_{L^2} \sim \frac{\|Q'\|_{L^2}}{\ell_0(T - t)} \quad \text{for } \ell_0(u_0) > 0$$

$$u(t) - \frac{1}{\lambda^{1/2}(t)} Q \left( \frac{x(t)}{\lambda(t)} \right) \to u^* \quad \text{in } L^2$$

$$\lambda(t) \sim \ell_0(T - t), \quad x(t) \sim \frac{1}{\ell_0^2(T - t)}$$

See [Rodnianski-Sterbenz, 10], [Raphaël-Rodnianski, 12], [Merle-Raphaël-Rodnianski, 11]
THM 2 (Existence and uniqueness of minimal mass blow up sol.)

(i) There exists a solution $S \in C((0, +\infty), H^1)$ with minimal mass

$$\|S(t)\|_{L^2} = \|Q\|_{L^2}$$

such that

$$\|S_x(t)\|_{L^2} \sim \frac{\|Q'\|_{L^2}}{t} \quad \text{as } t \downarrow 0,$$

$$S(t) - \frac{1}{t^{\frac{1}{2}}} Q \left( \cdot + \frac{1}{t} + \bar{c} t \right) \to 0 \quad \text{in } L^2 \quad \text{as } t \downarrow 0,$$

where $\bar{c}$ is a universal constant.

(ii) Let $u(t)$ be a solution with minimal mass which blows up in finite time. Then, $u = S$ up to invariances.
THM 3 (Classification and universality of $S(t)$)

Let $0 < \alpha_0 \ll \alpha^* \ll 1$. Only three scenarios are possible for $u_0 \in A$

**Blow up** $u(t)$ blows up in finite time with blow up rate $\frac{1}{T-t}$.

**Soliton** $u(t)$ is global, bounded and locally converges to a soliton as $t \to +\infty$.

**Exit** there exists $t^* > 0$ such that $u(t)$ exits at $t = t^*$ the $L^2$ neighborhood of size $\alpha^*$ of the family of solitons. Moreover, for some $\tau^*$, $u(t^*)$ is $L^2$ close (related to $\alpha_0$) to $S(\tau^*)$ (up to symmetries).

*Consequence:* Assume that $S(t)$ scatters at $+\infty$. Then, the (Exit) scenario implies scattering.

Classification results for NLKG, NLW [Nakanishi-Schlag, 10], [Krieger-Nakanishi-Schlag, 10] ([Duyckaerts-Kenig-Merle, 06-09])

Stable manifold: [Krieger-Schlag, 05], [Beceanu, 07]
Blow up rates for initial data with slow decay $u_0 \not\in \mathcal{A}$

THM 4 (Unstable blow up rates)

There exist blow up solutions with the following blow up rates:

(i) Blow up in finite time: for any $\nu > \frac{11}{13}$,

$$\|u_x(t)\|_{L^2} \sim t^{-\nu} \quad \text{as} \quad t \to 0^+.$$  

(ii) Blow up in infinite time:

$$\|u_x(t)\|_{L^2} \sim e^t \quad \text{as} \quad t \to +\infty.$$  

For any $\nu > 0$,

$$\|u_x(t)\|_{L^2} \sim t^\nu \quad \text{as} \quad t \to +\infty.$$  

Moreover, such solutions can be taken arbitrarily close to solitons.

See [Krieger-Schlag-Tataru, 08], [Bejenaru-Tataru, 09], [Donninger-Krieger, 12], [Perelman, 12]
Formal derivation of the dynamics in $\mathcal{A}$

$$u(t, x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} Q_{b(t)} \left( \frac{x - x(t)}{\lambda(t)} \right), \quad Q_b = Q + bP$$

$$u_t = -\frac{\lambda_t}{\lambda} (\Lambda Q_b)^\lambda - \frac{x_t}{\lambda} (Q'_b)^\lambda + b_t P^\lambda, \quad \Lambda Q_b = \frac{1}{2} Q_b + y(Q_b)_y,$$

$$\Rightarrow -\lambda^2 \lambda_t \Lambda Q_b + (Q''_b - \lambda^2 x_t Q_b + Q^5_b)' + \lambda^3 b_t P = 0$$

Fix $\lambda^2 x_t = 1$ and $-\lambda^2 \lambda_t = b$. At first order in $b$,

$$b \Lambda Q + b (LP)' + \lambda^3 b_t P + O(b^2) = 0$$

where $LP = -P'' + P - 5Q^4P$. We fix

$$(LP)' = -\Lambda Q \quad \text{and} \quad \lambda^3 b_t = -2b^2$$
Combining the equations of $\lambda_t$ and $b_t$, one gets

$$\frac{d}{dt} \left( \frac{b}{\lambda^2} \right) = \frac{1}{\lambda^2} \left( b_t - 2 \frac{\lambda_t}{\lambda} b \right) = 0$$

and

$$-\lambda_t = \frac{b}{\lambda^2} = \ell_0 \quad \text{(scaling law)}$$

Three scenarios:

- $\ell_0 > 0$:
  \[
  \lambda_t = -\ell_0 < 0 \implies \text{blow up and } \lambda(t) = \ell_0(T - t)
  \]
  Example: $E_0 < 0$ but also $E_0 = 0$ (rigidity argument)

- $\ell_0 = 0$:
  \[
  \lambda(t) = \text{Cte} \implies \text{soliton}
  \]

- $\ell_0 < 0$:
  \[
  \lambda_t = -\ell_0 > 0 \implies \text{defocusing and then (Exit)}
  \]
Full ansatz - control of the remainder term

We decompose the solution $u(t, x)$ as

$$u(t, x) = \frac{1}{\lambda^{1/2}(t)} Q_b(t) \left( \frac{x - x(t)}{\lambda(t)} \right) + \frac{1}{\lambda^{1/2}(t)} \varepsilon \left( t, \frac{x - x(t)}{\lambda(t)} \right)$$

where $(b, \lambda, x)$ are adjusted to obtain orthogonality conditions on $\varepsilon$.

The function $\varepsilon(s, y)$ and $(b, \lambda, x)$ are governed by

$$\varepsilon_s - (L\varepsilon)_y = \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda Q + \left( \frac{x_s}{\lambda} - 1 \right) Q' + \frac{\lambda_s}{\lambda} \Lambda \varepsilon + O(b^2 + |b_s| + |\varepsilon|^2)$$

and $\int \varepsilon Q = \int \varepsilon \Lambda Q = \int \varepsilon y \Lambda Q = 0$ ($s$ is the rescaled time $\frac{ds}{dt} = \frac{1}{\lambda^3}$)

The uniform control of some norm of $\varepsilon$ is a fundamental point in all the regimes to justify the dynamics of the parameters.
Tools for a simplified linear model (with orthogonality)

\[ \varepsilon_s - (L\varepsilon)_y = \alpha(s)Q + \beta(s)Q' \]

- Energy conservation at the level of \( \varepsilon \):

\[ \forall s, \quad (L\varepsilon(s), \varepsilon(s)) = \text{Cte} \]

- Monotonicity argument: for \( A \gg 1 \),

\[ \frac{d}{ds} \int_{y > -A} (\varepsilon^2 + \varepsilon^2 - 5Q^4\varepsilon^2)(s, y)dy \leq e^{-\frac{A}{10}} \|\varepsilon(s)\|_{H^1}^2 \]

- Viriel type argument (under orthogonality conditions):

\[ -\frac{d}{ds} \int y\varepsilon^2 = H(\varepsilon, \varepsilon) \geq \mu_0 \|\varepsilon(s)\|_{H^1}^2 \]
Main estimate on $\varepsilon$

Definition of a Liapunov functional for $\varepsilon(s)$

$$\mathcal{F}(s) \sim \int \left[ \varepsilon_y^2 \psi_1 + \varepsilon^2 \psi_2 - 5Q^4 \varepsilon^2 \psi_1 \right] (s, y) dy$$

where

- $\psi_1(y) = 0$ for $y < -A$, $\psi_1(y) = 1$ for $y > -\frac{1}{2}A$,
- $\psi_2(y) = 0$ for $y < -A$, $\psi_2(y) = 1 + y$ for $y > -\frac{1}{2}A$.

$\mathcal{F}(t)$ is a mixed energy monotonicity and Viriel quantity

PROP. Under a suitable assumption on space decay of $\varepsilon(s, y)$ on the right (which requires decay on the initial data), it holds

$$\frac{d}{ds} \left( \frac{\mathcal{F}}{\lambda^2} \right) + \frac{\|\varepsilon\|^2_{H^1_{\text{loc}}}}{\lambda^2} \lesssim \frac{b^4}{\lambda^2}$$

The blue term is reminiscent of the “Kato smoothing effect”. The term $\frac{b^4}{\lambda^2}$ is due to the equation of $Q_b$ (order $b$ only).
Full estimates

- Control of $\frac{b}{\lambda^2}$

$$\left| \frac{b(t_2)}{\lambda^2(t_2)} - \frac{b(t_1)}{\lambda^2(t_1)} \right| \lesssim \frac{b^2(t_1)}{\lambda^2(t_1)} + \frac{b^2(t_2)}{\lambda^2(t_2)} + \frac{\mathcal{F}(t_1)}{\lambda^2(t_1)}$$

- Equation of $\lambda$

$$\left| \lambda^2 \lambda_t + b \right| \lesssim \|\varepsilon(t)\|_{H^1_{\text{loc}}}^2 + |b|^2$$

- Control of $\varepsilon$

$$\frac{\mathcal{F}(t_2)}{\lambda^2(t_2)} + \int_{t_1}^{t_2} \frac{\|\varepsilon(t)\|_{H^1_{\text{loc}}}^2}{\lambda^5} dt \lesssim \frac{\mathcal{F}(t_1)}{\lambda^2(t_1)} + \frac{b^3(t_1)}{\lambda^2(t_1)} + \frac{b^3(t_2)}{\lambda^2(t_2)}$$
Analysis of the (Exit) case

Definition of the $L^2$ (Exit) time ($\alpha^*$ small but fixed) :

$$t^* = \sup\{0 < t < T, \text{ such that } \forall t' \in [0, t], \ u(t) \in T_{\alpha^*}\}$$

where $T_{\alpha^*}$ is an $L^2$ tube around the family of solitons:

$$T_{\alpha^*} = \left\{ u \in H^1 \text{ with } \inf_{\lambda_0 > 0, x_0 \in \mathbb{R}} \left\| u - \frac{1}{\lambda_0^2} Q \left( \frac{\cdot - x_0}{\lambda_0} \right) \right\|_{L^2} < \alpha^* \right\}$$

New and general approach to:

1. Construct the minimal mass solution $S$
2. Prove universality of the (Exit) case and a “no-return lemma” based on the properties of $S$
Existence of a minimal mass solution

Choose a sequence of well-prepared initial data, for example:

\[ u_n(0) = Q_{b_n(0)}, \quad b_n(0) = -\frac{1}{n}, \quad \|u_n(0)\|_{L^2} - \|Q\|_{L^2} \sim -\frac{c}{n}, \quad \varepsilon_n(0) = 0 \]

(Blowup) and (Soliton) are not possible \( \Rightarrow \) (Exit) regime

\[ u_n(t, x) = \frac{1}{\lambda_n^2(t)} (Q_{b_n(t)} + \varepsilon_n) \left( t, \frac{x - x_n(t)}{\lambda_n(t)} \right) \]

\[ (\lambda_n)_t \sim -b_n(0), \quad \lambda_n(t) \sim 1 - b_n(0)t, \quad b_n(t) = b_n(0)\lambda_n^2(t). \]

At the (Exit) time \( t_n^* \): \( b_n(t_n^*) = -\alpha^*, \quad \lambda_n^2(t_n^*) \sim \frac{b_n(t_n^*)}{b_n(0)} \sim n\alpha^* \)

(defocalisation)
Renormalize the solution at $t_n^*$:

$$v_n(\tau, x) = \lambda_n^{\frac{1}{2}}(t_n^*)u_n(t_n^* + \tau \lambda_n^3(t_n^*), \lambda_n(t_n^*)x + x_n(t_n^*)).$$

$$v_n(\tau, x) = \frac{1}{\lambda_n^{\frac{1}{2}}(\tau)} \left( Q_{b v_n} + \varepsilon_{v_n} \right) \left( \tau, \frac{x - x_{v_n}(\tau)}{\lambda_{v_n}(\tau)} \right)$$

$$\lambda_{v_n}(\tau) \sim \frac{1}{\lambda_n(t_n^*)} \left[ 1 - b_n(0)(t_n^* + \tau \lambda_n^3(t_n^*)) \right]$$

$$\sim \frac{1}{\lambda_n(t_n^*)} \left[ \lambda_n(t_n^*) - \tau b_n(0) \lambda_n^3(t_n^*) \right] = 1 - \tau b_n(t_n^*) = 1 + \tau \alpha^*.$$

Mass, energy conservation and $\varepsilon_n(0) = 0 \Rightarrow \sup_{\tau} \|\varepsilon_{v_n}\|_{H^1} \leq \delta(\alpha^*)$.

Extract a weak limit $v_n(0) \rightharpoonup v(0)$ in $H^1$ weak such that the corresponding solution $v(\tau)$ blows up backwards at $\tau^* \sim -\frac{1}{\alpha^*}$.

Moreover, $\|v(0)\|_{L^2} \leq \|Q\|_{L^2}$ by weak limit and blow up yields

$$\|v(0)\|_{L^2} = \|Q\|_{L^2}.$$
Description of the general (Exit) scenario

PROP Let \((u_n(0))\) be a sequence in \(H^1\) satisfying:

1. \(u_n(0) \in \mathcal{A}\);
2. \(\|u_n(0) - Q\|_{H^1} \leq \frac{1}{n}\);
3. the solution \(u_n\) satisfies the (Exit) scenario

Then, there exists \(\tau^* = \tau^*(\alpha^*)\) such that

\[
\frac{1}{n} \lambda_n(t^*_n) u_n(t^*_n, \lambda_n(t^*_n) \cdot + x_n(t^*_n)) \to \frac{1}{S(\tau^*)} S(\tau^*, \lambda_S(\tau^*) \cdot + x_S(\tau^*))
\]

in \(L^2\) as \(n \to +\infty\).

The idea of the proof is similar as before, except that the \(H^1\) bound is lost for general (not well-prepared) initial data.

The uniqueness of \(S\) is decisive.