Global well-posedness and scattering for the mass-critical nonlinear Schrödinger equation

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The mass critical NLS

\[ iu_t + \Delta u = \mu |u|^\frac{4}{d} u, \]

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R}, \ u(x, t) \in \mathbb{C}, \]


*Given* \( u_0 \in L^2(\mathbb{R}^d), \ \mu = +1, \) *the mass critical NLS is globally well-posed and scattering. When* \( \mu = -1, \) *the mass critical NLS is globally well-posed and scattering for* \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}. \) *Q is the unique positive solution to the elliptic problem* \( \Delta Q + Q^{1 + \frac{4}{d}} = Q. \)

\( e^{it} Q \) *solves the mass - critical NLS for* \( \mu = -1. \)
The mass critical NLS

\[
iu_t + \Delta u = \mu |u|^4 u,
\]

\[u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d, t \in \mathbb{R}, \quad u(x, t) \in \mathbb{C},\]


Given \(u_0 \in L^2(\mathbb{R}^d), \mu = +1\), the mass critical NLS is globally well-posed and scattering. When \(\mu = -1\), the mass critical NLS is globally well-posed and scattering for \(\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}\). \(Q\) is the unique positive solution to the elliptic problem \(\Delta Q + Q^{1 + \frac{4}{d}} = Q\).

\(e^{it}Q\) solves the mass-critical NLS for \(\mu = -1\).
Energy Critical Problem

\[ iu_t + \Delta u = \mu |u|^{\frac{4}{d-2}} u, \]
\[ u(0, x) \in H^1. \]

(1)

\( d = 3, \mu = +1, \) radial - Bourgain, nonradial (CKSTT)

\( d = 3, \mu = -1, \) radial - Kenig - Merle

\( d = 4, \mu = +1, \) radial Bourgain, nonradial Ryckman - Visan

\( d = 4, \mu = -1, \) radial - Kenig - Merle

\( d \geq 5, \mu = +1, \) radial Tao, nonradial Visan.

\( d \geq 5, \mu = -1, \) radial Kenig - Merle, nonradial Killip - Visan.
Mass Critical Problem

\[ iu_t + \Delta u = \mu |u|^4 u, \]
\[ u(0, x) \in L^2. \]  \hspace{1cm} (2)

\[ d = 1, \mu = +1, \text{ D.} \]
\[ d = 1, \mu = -1, \text{ D.} \]
\[ d = 2, \mu = +1, \text{ radial Tao, Killip, Visan, nonradial D.} \]
\[ d = 2, \mu = -1, \text{ radial Tao, Killip, Visan, nonradial D.} \]
\[ d \geq 3, \mu = +1, \text{ radial Tao, Visan, Zhang. nonradial D.} \]
\[ d \geq 3, \mu = -1, \text{ radial Killip, Visan, Zhang. nonradial D.} \]
Definition Well-posedness

The mass-critical NLS is said to be globally well-posed if a solution $u(t, x)$ exists for all time,

$$u(t, x) \in C^0_t(\mathbb{R}; L^2(\mathbb{R}^d)) \cap L^\frac{2(d+2)}{d} \left( \mathbb{R}; L^\frac{2(d+2)}{d} (\mathbb{R}^d) \right),$$

a solution depends continuously on $u_0$ in the $L^2(\mathbb{R}^d)$ topology, and the solution is continuous in time. Such a solution is unique.
Definition Scattering

A global solution to the mass critical NLS is said to scatter if there exist \( u_{\pm} \in L^2(\mathbb{R}^d) \) such that

\[
\| u(t, x) - e^{it\Delta} u_+ \|_{L^2(\mathbb{R}^d)} \to 0,
\]

as \( t \to +\infty \) and

\[
\| u(t, x) - e^{it\Delta} u_- \|_{L^2(\mathbb{R}^d)} \to 0
\]

as \( t \to -\infty \). Additionally we say a solution scatters forward in time if it satisfies (3) and backward in time if it satisfies (4).
Theorem

If $u$ solves $iu_t + \Delta u = F$, $(p, q)$, $(\tilde{p}, \tilde{q})$ are admissible pairs,

$$
\|u\|_{L^p_t L^q_x(I\times\mathbb{R}^d)} \lesssim \|u(0)\|_{L^2(\mathbb{R}^d)} + \|F\|_{L^{\tilde{p}'}_t L^{\tilde{q}'}_x(I\times\mathbb{R}^d)}. \quad (5)
$$

A pair $(p, q)$, is called admissible if $\frac{2}{p} = d\left(\frac{1}{2} - \frac{1}{q}\right)$, $p \geq 2$ for $d \geq 3$, $p > 2$ for $d = 2$, $p \geq 4$ for $d = 1$.

Theorem

(Cazenave and Weissler) For $\mu = \pm 1$ the mass critical NLS is globally well-posed and scattering if $\|u_0\|_{L^2(\mathbb{R}^d)} < \epsilon(d)$, $\epsilon(d)$ sufficiently small.

The mass critical NLS is locally well-posed on some interval $[-T, T]$, $T(\|u_0\|_{H^1}) > 0$. 
Theorem

(Weinstein) We have the Sobolev embedding

\[ \| f \|_{L_x^2 \mathbb{R}^d}^{d \frac{2(d+2)}{2(d+2)}} \leq \frac{d}{2(d+2)} \frac{\| f \|_{L^2(\mathbb{R}^d)}^{4/d}}{\| Q \|_{L^2(\mathbb{R}^d)}^{4/d}} \| \nabla f \|_{L^2(\mathbb{R}^d)}^2 \]  

Since

\[ E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{\mu d}{2(d+2)} \int |u(t, x)|^{2(d+2)/d} dx, \]

the mass-critical problem is globally well-posed when \( u(0) \in H^1 \), \( \mu = +1 \), or when \( u(0) \in H^1 \), \( \| u(0) \|_{L^2} < \| Q \|_{L^2} \), \( \mu = -1 \).
Theorem

A solution to the mass - critical NLS is globally well - posed and scattering if and only if a solution $u$ to the mass - critical problem satisfies

$$\left\| u \right\|_{L_{t,x}^{\frac{2(d+2)}{d}}((-\infty,\infty)\times\mathbb{R}^d)} < \infty.$$  (7)
Theorem

(Killip, Tao, Visan, Zhang) Theorem 1 is true for $d \geq 2$, $u(0)$ radial.

Define the function

$$A_\mu(m) = \sup \{ \| u \|_{L^2_t L^{2(d+2)}_x (\mathbb{R} \times \mathbb{R}^d)} : \| u(t) \|_{L^2} = m, \}
$$

$u$ solves the mass critical NLS.

Theorem

(Killip, Tao, Visan, Zhang) $A_\mu(m)$ is a continuous function of $m$.

$\{ m : A_\mu(m) = \infty \}$ is a closed set.

$\{ m : A_\mu(m) = \infty \}$ has a minimal element $m_0$. 
**Theorem**

(Killip, Tao, Visan, Zhang) Suppose $m_0 < \infty$ for $\mu = 1$, $m_0 < \| Q \|_{L^2}$ when $\mu = -1$. Then there exist functions $N(t) : [0, \infty) \to (0, \infty)$, $\xi(t), x(t) : [0, \infty) \to \mathbb{R}^d$ such that for all $\eta > 0$ there exists $C(\eta) < \infty$ such that

$$\int_{|x-x(t)| \geq \frac{c(\eta)}{N(t)}} |u(t, x)|^2 dx + \int_{|\xi-\xi(t)| \geq C(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi < \eta,$$  \hspace{1cm} (8)

$N(0) = 1$, $N(t) \leq 1$ for $t \geq 0$, $\xi(0) = x(0) = 0$, $|N'(t)|, |\xi'(t)| \lesssim N(t)^3$, and

$$\int_0^\infty \int |u(t, x)|^{\frac{2(d+2)}{d}} dx dt = +\infty. \hspace{1cm} (9)$$

Also for any compact $I \subset [0, \infty)$

$$\int_I N(t)^2 dt \lesssim \int_I \int |u(t, x)|^{\frac{2(d+2)}{d}} dx dt \lesssim 1 + \int_I N(t)^2 dt. \hspace{1cm} (10)$$
Theorem

(Killip, Tao, Visan, Zhang) $[0, \infty)$ can be divided into intervals of local constancy that satisfy

$$\int_{J_k} \int |u(t, x)|^{\frac{2(d+2)}{d}} \, dx \, dt = 1. \quad (11)$$

On these intervals

$$N(J_k) = \sup_{t \in J_k} N(t) \sim \inf_{t \in J_k} N(t) \sim \int_{J_k} N(t)^3 \, dt. \quad (12)$$

Remark: Possibly after modifying $C(\eta)$ by a constant we can transform

$$N(t) \mapsto \alpha(t) N(t), \quad (13)$$

for some $\epsilon > 0$,

$$\epsilon < \alpha(t) < \frac{1}{\epsilon}. \quad (14)$$
Study two situations separately.

\[
\int_0^\infty N(t)^3 \, dt < \infty
\]  

(15)

This is called the rapid frequency cascade.

\[
\int_0^\infty N(t)^3 \, dt = +\infty.
\]  

(16)

This is called the quasi-soliton.
Long time Strichartz estimates

**Theorem**

(D’) (2009) Suppose $d > 2$ and $u(t, x)$ is a minimal mass blowup solution in the form of slide 10. Suppose $\int J N(t)^3 dt = K$, $J$ is a compact subset of $[0, \infty)$. Then

$$\| u|_{\xi - \xi(t)}> M \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbb{R}^d)} \lesssim \left( \frac{K}{M} \right)^{1/2} + \sigma_J(\frac{M}{2}),$$

(17)

where $\sigma_J(\frac{M}{2})$ is a frequency envelope that majorizes

$$\inf_{t \in J} \| u|_{\xi - \xi(t)}> \frac{M}{2} \|_{L_x^2(\mathbb{R}^d)}.$$

(18)

A more technical version of this theorem was proved for dimensions $d = 1, 2$ by (D’ 2010). In this case we use a norm constructed from the spaces of $U^2_{\Delta}$ of (Hadac, Herr, and Koch), (Koch and Tataru).
Rapid Frequency Cascade

Suppose $\int_0^\infty N(t)^3 \, dt = K < \infty$. For $0 \leq s < 1 + \frac{4}{d}$,

$$\|u\|_{L^\infty_t \dot{H}^s([0,\infty) \times \mathbb{R}^d)} \lesssim K^s. \quad (19)$$

This implies $E(t) \equiv 0$, which is a contradiction.
Quasi - soliton

Suppose $\int_{t} N(t)^3 dt = K$.

In the defocusing case use the interaction Morawetz estimates of (Colliander, Keel, Staffilani, Takaoka, Tao) ($d = 3$), (Tao, Visan, Zhang) ($d \geq 4$), (Colliander, Grillakis, Tzirakis) ($d = 1, 2$), (Planchon, Vega) ($d \geq 1$).
\[ N(t)^3 \lesssim \frac{d^2}{dt^2} \int |u(t, x)|^2 |x - y||u(t, y)|^2 dxdy. \] (20)

\[ \frac{d}{dt} \int |u(t, x)|^2 |x - y||u(t, y)|^2 dxdy \]

\[ = 2 \int |u(t, y)|^2 \text{Im}[\bar{u}(t, x)\partial_j u(t, x)] \frac{(x - y)_j}{|x - y|} dxdy. \] (21)

Use the long time Strichartz estimates to estimate the error arising from truncating \( u \) to frequencies \( \leq CK \). Have \( K \lesssim o(K) \). Contradicts \( K \nearrow \infty \) as \( T \to \infty \).
In the focusing case have to construct an interaction Morawetz estimate. In the defocusing case the potential is \( \frac{(x-y)_j}{|x-y|} \).

The error estimates apply equally well in the focusing and defocusing case for a potential \( a_j(t, x-y) \) such that

1. \( |a_j(t, x)| \lesssim 1 \),
2. \( |\nabla a_j(t, x)| \lesssim \frac{1}{|x|} \) for \( d \geq 2 \), \( \|\nabla a(t, x)\|_{L^1(\mathbb{R})} \lesssim 1 \),
3. \( a_j(t, x) = -a_j(t, -x) \),
4. \( \|\partial_t a_j(t, x)\|_{L^1(\mathbb{R}^2)} \lesssim 1 \).
Construction of the interaction Morawetz estimate.

1. \( N(t) \equiv 1, \ u \ \text{even}, \ d = 1. \)

2. \( N(t) \equiv 1, \ d = 1. \)

3. \( N(t) \equiv 1. \)

4. \( \int_0^{+\infty} N(t)^3 \, dt = +\infty. \)
Construction of the interaction Morawetz estimate.

1. \( N(t) \equiv 1, \ u \text{ even}, \ d = 1. \)

Take \( \psi(0) = 0, \ \psi'(x) = \phi(x), \ \phi \text{ even}, \ \phi \equiv 1 \text{ on } [-R, R], \ \phi \text{ supported on } [-2R, 2R]. \)

\[
\partial_t \text{Im}(\bar{u} \partial_j u) = -2 \partial_k \text{Re}(\partial_k \bar{u} \partial_j u) + \frac{1}{2} \partial_j \partial_k^2(|u|^2) + \frac{2}{d + 2} \partial_j(|u|^{\frac{2(d+2)}{d}}). 
\tag{22}
\]

\[
M(t) = \int \psi(x) \text{Im}[\bar{u} \partial_j u](t, x) dx.
\tag{23}
\]

\[
\dot{M}(t) = 2 \int \phi(x)[|u_x|^2 - \frac{1}{3}|u|^6] dx - \frac{1}{2} \int \phi''(x)|u(t, x)|^2 dx \\
\geq C(\|u\|_{L^2}) \|u_x\|^2_{L^2} - o_R(1) \gtrsim N(t)^2 = N(t)^3.
\tag{24}
\]
\( \mathcal{N}(t) \equiv 1, \; d = 1. \)

Let \( \psi(0) = 0, \; \psi'(x) = \phi(x) \),

\[
\phi(x - y) = \frac{1}{R} \int \chi(x - s)\chi(y - s)ds,
\]

(25)

\( \chi \equiv 1 \) on \([-R, R] \), \( \chi \equiv 0 \) on \(|x| \geq R + 1 \).

\[
M(t) = \int \psi(x - y)|u(t, y)|^2 \Im[\bar{u}\partial_x u](t, x)dx\,dy. \tag{26}
\]

\[
\dot{M}(t) = 2 \int \psi'(x - y)|u(t, y)|^2\left[|u_x|^2 - \frac{1}{3}|u|^6\right]dx
\]

\[
-2 \int \psi'(x - y)\Im(\bar{u}\partial_x u)\Im(\bar{u}\partial_x u)dx\,dy
\]

\[
-\frac{1}{2} \int \psi'''(x - y)|u(t, y)|^2|u(t, x)|^2dx\,dy. \tag{27}
\]
If we could get rid of $-2 \int \psi'(x - y) \text{Im}(\bar{u} \partial_x u) \text{Im}(\bar{u} \partial_x u) \, dx \, dy$ we could proceed as in case 1. For any $\xi(s)$,

\[
\frac{1}{R} \int \chi(x - s) \chi(y - s) |u(t, y)|^2 |u_x|^2 \, dx \, dy
\]

\[
- \frac{1}{R} \int \chi(x - s) \chi(y - s) \text{Im}(\bar{u} \partial_y u) \text{Im}(\bar{u} \partial_x u) \, dx \, dy
\]

\[
= \frac{1}{R} \int \chi(x - s) \chi(y - s) |u(t, y)|^2 |\partial_x (e^{-ix \cdot \xi(s)} u)|^2 \, dx \, dy
\]

\[
- \frac{1}{R} \int \chi(x - s) \chi(y - s) \text{Im}(e^{ix \cdot \xi(s)} \bar{u} \partial_y (e^{-ix \cdot \xi(s)} u)) \times \text{Im}(e^{ix \cdot \xi(s)} \bar{u} \partial_x (e^{-ix \cdot \xi(s)} u)) \, dx \, dy.
\]

Therefore,

\[
\dot{M}(t) \geq C(\|u\|_{L^2}) \|u\|_6^6 - o_R(1) \geq N(t)^2 = N(t)^3.
\]
3. \( N(t) \equiv 1 \). In this case

\[
M(t) = \int S_{d-1} \int \int \psi((x - y)\omega)|u(t, y)|^2 \text{Im}(\bar{u}\partial_\omega u)(t, x)dx dy d\omega. \tag{31}
\]

4. \( \int_0^\infty N(t)^3 dt = \infty \).

Here we take

\[
M(t) = \int S_{d-1} \int \int \psi((x - y)\tilde{N}(t))|u(t, y)|^2 \text{Im}(\bar{u}\partial_\omega u)(t, x)dx dy d\omega.
\tag{32}
\]

\( \tilde{N}(t) \leq N(t), \tilde{N}(t) \sim N(t), \)

\[
\frac{|\tilde{N}'(t)|}{\tilde{N}(t)^3} \leq \frac{|N'(t)|}{N(t)^3}. \tag{33}
\]

\[
\int |\tilde{N}'(t)| dt \leq \delta(\|u\|_{L^2}) \int N(t)^3 dt. \tag{34}
\]