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FEM for singular perturbations

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Introduction

What are singular perturbations?

- example:

$$-\varepsilon^2 u'' + u = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0$$

- typical: differential equations with small parameters
- hallmark: in the limit $\varepsilon \to 0$ the equation changes order so that not all boundary conditions can be imposed any more.
- Example:

$$-\varepsilon u'' - u' + u = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0$$

  In the limit $\varepsilon \to 0$, only one boundary condition can be imposed. (In fact, at $x = 1$)
- regular perturbations: equation does not change type in the limit $\varepsilon \to 0$
  example:

$$-u'' + \varepsilon u' + u = f \quad \text{on } \Omega, \qquad u(0) = u(1) = 0$$

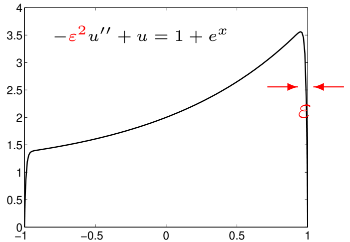
  with limit problem

$$-u'' + u = f \quad \text{on } \Omega, \qquad u(0) = u(1) = 0$$

Typical solution behavior

reaction-diffusion equation

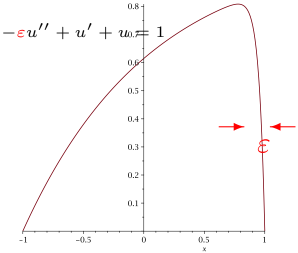
$$-\varepsilon^2 u'' + u = f$$



- limit equation: $u = f$
- layers at both endpoints to ensure b.c.
- length scale ε

convection-reaction-diffusion equation

$$-\varepsilon u'' + u' + u = f$$

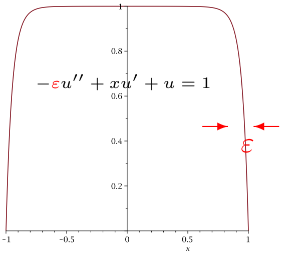


- layer at one endpoint
- length scale ε
- limit equation: in large parts: $u \approx$ solution of $\tilde{u}' + \tilde{u} = f$, $\tilde{u}(-1) = 0$
- layer at $x = 1$ ensures b.c.

turning point problems and the possibility of interior layers

turning point problem I

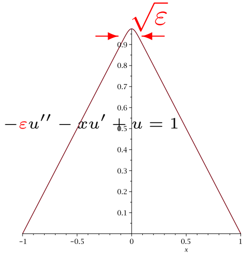
$$-\varepsilon u'' + xu' + u = f$$



- limit equation: $xu' + u = f$
- general solution for $f \equiv 1$: $u = 1 - \frac{c}{x}$
- smoothness requirement $\rightarrow c = 0$
- layers at both endpoints to ensure b.c.
- length scale ε

turning point problem II

$$-\varepsilon u'' - xu' + u = f$$



- limit equation:
  $-x\tilde{u}' + \tilde{u} = f, \quad \tilde{u}(-1) = 0$ on $(-1, 0)$

  $-x\tilde{u}' + \tilde{u} = f, \quad \tilde{u}(1) = 0$ on $(0, 1)$
- interior layer at $x = 0$
- length scale $\sqrt{\varepsilon}$

regularity of 1D reaction-diffusion equations by asymptotic expansions

reaction-diffusion problems: existence and uniqueness

$$\mathcal{L}_\varepsilon u := -\varepsilon^2 u'' + b(x)u = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0, \qquad b \geq b_0 > 0 \tag{1}$$

Lemma (existence and uniqueness)

For each $f \in L^2(\Omega)$, the solution $u_\varepsilon \in H_0^1(\Omega)$ exists and is unique. Moreover,

$$\|u_\varepsilon\|_\varepsilon := \sqrt{\varepsilon^2 \|u_\varepsilon'\|^2_{L^2(\Omega)} + \|u_\varepsilon\|^2_{L^2(\Omega)}} \leq C\|f\|_{L^2(\Omega)}$$

Proof: Lax-Milgram

asymptotic expansions for reaction-diffusion equations

$$\mathcal{L}_\varepsilon u := -\varepsilon^2 u'' + b(x)u = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0, \qquad b \geq b_0 > 0$$

- general technique: matched asymptotic expansions.
  Part of the procedure is to reveal the length scales of the problem
  here: we will always "inject" knowledge of the proper length scale into the ansatz
- here: simpler approach using outer and inner expansion: $u(x) \approx u^{outer}(x) + u^{inner}(x)$
- purpose of u^{outer}: good approximation away from endpoints $x = 0$, $x = 1$
  u^{outer} is (approx.) particular solution
- purpose of u^{inner}: ensures the b.c. since u^{outer} does not satisfy the correct b.c.
- asymptotic expansions aim at "small residual" $\rightsquigarrow$ justification of asymptotic expansions requires a stability result (e.g., Lax-Milgram)

illustration: the case $b \equiv 1$

$$\mathcal{L}_\varepsilon u_\varepsilon := -\varepsilon^2 u_\varepsilon'' + u_\varepsilon = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0,$$

- limit equation: $u = f$ correct with $u^{BL} = Ae^{-x/\varepsilon} + Be^{-(1-x)/\varepsilon}$ such that $(u_0 + u^{BL})(0) = (u_0 + u^{BL})(1) = 0$
- question: is $u_{approx} := u_0 + u^{BL}$ a good approximation?
- the residual $r := u_\varepsilon - u_{approx}$ satisfies

$$\mathcal{L}_\varepsilon r = \mathcal{L}_\varepsilon u_\varepsilon - \mathcal{L}_\varepsilon u_0 = f - (-\varepsilon^2 u_0'' + u_0) = f + \varepsilon^2 u_0'' - f = \varepsilon^2 f'', \qquad r(0) = r(1) = 0$$

- By Lax-Milgram, $\|r\|_\varepsilon = O(\varepsilon^2)$
- question: even better approximations?

illustration: the case $b \equiv 1$

$$\mathcal{L}_\varepsilon u_\varepsilon := -\varepsilon^2 u_\varepsilon'' + u_\varepsilon = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0, \tag{2}$$

- Ansatz for u^{outer}: $u^{outer} \sim \sum_i \varepsilon^i u_i(x)$
- inserting ansatz in (2) yields

$$\sum_i -\varepsilon^{i+2} u_i'' + \varepsilon^i u_i = f = \varepsilon^0 f + \varepsilon^1 \cdot 0 + \varepsilon^2 \cdot 0 + \cdots$$

- equating like powers of ε yields

$$u_0 = f, \qquad u_1 = 0, \qquad u_{i+2} = u_i'', \quad i = 0, 1, \ldots$$

- truncated expansion $u_M^{outer} := \sum_{i=0}^{2M} \varepsilon^i u_i$ satisfies

$$\mathcal{L}_\varepsilon u_M^{outer} - f = -\varepsilon^{2M+2} u_{2M}'' = O(\varepsilon^{2M+2}), \qquad u_M^{outer}(0) = \sum_{i=0}^{2M} \varepsilon^i u_i(0)$$

the case $b \equiv 1$, cont'd

- outer expansion u_M^{outer} does not satisfy the b.c. $\rightsquigarrow$ correct with u_M^{inner}
- Solutions of the homogeneous equation are $u^{left} := e^{-x/\varepsilon}$ and $u^{right} := e^{-(1-x)/\varepsilon}$
- set $u^{inner}(x) \sim \sum_i \varepsilon^i u^{left}(x) u_i(0) + \varepsilon^i u^{right}(x) u_i(1)$
- truncated inner expansion $u_M^{inner}(x) := -\sum_{i=0}^{2M+1} \left(\varepsilon^i u^{left}(x) u_i(0) + \varepsilon^i u^{right}(x) u_i(1)\right)$
- remainder $r_M := u_\varepsilon - (u_M^{outer} + u_M^{inner})$ satisfies:

$$\mathcal{L}_\varepsilon r_M = O(\varepsilon^{2M+2}), \qquad r_M(1) = O(e^{-1/\varepsilon}), \quad r_M(0) = O(e^{-1/\varepsilon})$$

Lemma

Let f be smooth. Then, for each fixed $M \in \mathbb{N}_0$ there holds $u_\varepsilon = u_M^{outer} + u_M^{inner} + r_M$ with:

- *u_M^{outer} is smooth and derivatives can be controlled uniformly in ε;*
- *$u_M^{inner} = Ae^{-x/\varepsilon} + Be^{-(1-x)/\varepsilon}$ for some A, B (bounded uniformly in ε);*
- *$\|r_M\|_\varepsilon = O(\varepsilon^{2M+2})$.*

the case of general $b \geq b_0 > 0$

$$\mathcal{L}_\varepsilon u := -\varepsilon^2 u'' + b(x)u = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0, \qquad b \geq b_0 > 0$$

- Ansatz for outer expansion: $u(x) \sim \sum_i \varepsilon^i u_i(x)$
- inserting the Ansatz into differential equation gives[1]

$$\sum_i \varepsilon^i (-u''_{i-2} + b(x) u_i) \stackrel{!}{=} f(x) = \varepsilon^0 f(x) + \varepsilon^1 \cdot 0 + \varepsilon^2 \cdot 0 + \cdots$$

- equating like powers of ε yields the recursion

$$u_0 := \frac{f(x)}{b(x)}, \qquad u_1(x) = 0, \qquad u_{i+2}(x) := \frac{u''_i(x)}{b(x)}, \quad i = 0, 1, \ldots$$

- The truncated series $u_M^{outer} := \sum_{i=0}^{2M} \varepsilon^i u_i(x)$ satisfies

$$\mathcal{L}_\varepsilon u_M^{outer} - f = O(\varepsilon^{2M+2}), \qquad u^{outer}(0) = \sum_{i=0}^{2M} \varepsilon^i u_i(0), \qquad u^{outer}(1) = \sum_{i=0}^{2M} \varepsilon^i u_i(1)$$

[1] here and in the following, we set $u_i \equiv 0$ for $i < 0$

asymptotic expansions, inner expansion II

- ideal requirements on u^{inner}: $\mathcal{L}_\varepsilon u^{inner} = 0$ and $u^{inner}(0) = -u_M^{outer}(0)$, $u^{inner}(1) = -u_M^{outer}(1)$

technique: blow-up at endpoints. Construct the approximation $u^{inner} = u^{inner,l} + u^{inner,r}$, i.e., separately near $x = 0$ and $x = 1$

- near $x = 0$:
  - scaled variables $\widehat{x} = x/\varepsilon$
  - ansatz $u^{inner,l}(\widehat{x}) \sim \sum_i \varepsilon^i \widehat{u}_i(\widehat{x})$
  - rewrite the condition $\mathcal{L}_\varepsilon u^{inner,l} \stackrel{!}{=} 0$ in terms of $\widehat{x}$:

$$\sum_i \varepsilon^i \Big(-\varepsilon^{2-2} \widehat{u}_i''(\widehat{x}) + \underbrace{b(\varepsilon \widehat{x})}_{\stackrel{Taylor}{=} \sum_j \varepsilon^j \widehat{x}^j b_j} \widehat{u}_i(\widehat{x}) \Big) \stackrel{!}{=} 0$$

  - equate like powers of ε to get the recursion

$$-\widehat{u}_i'' + b_0 \widehat{u}_i = -\sum_{j=0}^{i-1} b_{i-j} \widehat{x}^{i-j} \widehat{u}_j \qquad \text{on } (0,\infty)$$

    equipped with the side conditions

$$\widehat{u}_i(0) = -u_i(0), \qquad \widehat{u}_i(\widehat{x}) \to 0 \quad \text{as } \widehat{x} \to \infty$$

asymptotic expansions, inner expansion III

$$-\widehat{u}_i'' + b_0 \widehat{u}_i = -\sum_{j=0}^{i-1} b_{i-j}\widehat{x}^{i-j}\widehat{u}_j \quad \text{on } (0,\infty) \qquad \widehat{u}_i(0) = -u_i(0), \qquad \widehat{u}_i(\widehat{x}) \to 0 \quad \text{as } \widehat{x} \to \infty$$

Lemma

The functions $\widehat{u}_i$ can be computed recursively. For each i, the solution $\widehat{u}_i$ is an entire function of the form $\widehat{u}_i(z) = \pi_i(z)e^{-z/\sqrt{b_0}}$ for some polynomial $\pi_i \in \mathcal{P}_i$ of degree i

- Each solution $\widehat{u}_i$ decays exponentially as $\widehat{x} \to 0$
- The truncated expansion $u_M^{inner,l}(x) := \sum_{i=0}^{2M+1} \varepsilon^i \widehat{u}_i(\widehat{x})$ satisfies
  - $u_M^{inner,l}(0) + u_M^{outer}(0) = 0$ since $u_i(0) = -\widehat{u}_i(0)$ for each i
  - $|u_M^{inner,l}(1)| = O(e^{-c/\varepsilon})$ for some $c > 0$
  - $\mathcal{L}_\varepsilon u_M^{inner,l} = O(\varepsilon^{2M+2})$

asymptotic expansions, inner expansion III

- analogous calculation near $x = 1$ with scaled variable $\widehat{x}^R := (x-1)/\varepsilon$ for $u^{inner,r}$
- $u_M^{inner} := \sum_{i=0}^{2M+1} \varepsilon^i \widehat{u}_i(\widehat{x}) + \sum_{i=0}^{2M+1} \varepsilon^i \widehat{u}_i^r(\widehat{x}^R)$
- obtain $\mathcal{L}_\varepsilon u^{inner} = O(\varepsilon^{2M+2})$ and $|u_M^{outer}(0) + u_M^{inner}(0)| = O(e^{-c/\varepsilon})$

justification of the expansion: Remainder $r_M := u - (u_M^{outer} + u_M^{inner})$ satisfies

$$\mathcal{L}_\varepsilon r_M = O(\varepsilon^{2M+2}), \qquad |r_M(0)| + |r_M(1)| = O(e^{-c/\varepsilon})$$

By Lax-Milgram, we get

$$\|r_M\|_\varepsilon = O(\varepsilon^{2M+2}).$$

Lemma

Let f, b be smooth. For each M one can write $u = u_M^{outer} + u_M^{inner,l} + u^{inner,r} + r_M$ with:

- $u_M^{outer}(x) = \sum_{i=0}^{2M} \varepsilon^i u_i(x)$ *is smooth (uniformly in ε);*
- $u_M^{inner,l}(x) = \sum_{i=0}^{2M+1} \varepsilon^i \widehat{u}_i(x/\varepsilon)$ *is smooth and* $|\frac{d^n}{d\widehat{x}^n} u_M^{inner,l}(\widehat{x})| \leq C_{n,M} e^{-\beta \widehat{x}}$, $\beta > 0$ *suitable;*
- $\|r_M\|_\varepsilon = O(\varepsilon^{2M+2})$.

FEM for reaction-diffusion equations

$$\mathcal{L}_\varepsilon u := -\varepsilon^2 u'' + b(x)u = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0, \qquad b \geq b_0 > 0$$

weak formulation

Find $u_\varepsilon \in H_0^1(\Omega)$ s.t.

$$a_\varepsilon(u_\varepsilon, v) := \int_\Omega \varepsilon^2 u_\varepsilon' v' + b(x) u_\varepsilon v \, dx = \ell(v) := \int_\Omega f(x) v \, dx \qquad \forall v \in H_0^1(\Omega)$$

abstract FEM

Given closed $V_N \subset H_0^1(\Omega)$ find $u_N \in V_N$ s.t.

$$a_\varepsilon(u_N, v) = \ell(v) \qquad \forall v \in V_N \tag{3}$$

Theorem (Céa Lemma/quasioptimality)

There is a unique solution u_N of (3) and there is $C > 0$ depending only on Ω, b s.t.

$$\|u_\varepsilon - u_N\|_\varepsilon \leq C \inf_{v \in V_N} \|u_\varepsilon - v\|_\varepsilon$$

classical FEM

- $\mathcal{T} = \{K_i\}_{i=0}^{n-1}$ = mesh with elements $K_i = (x_i, x_{i+1})$ and nodes $0 = x_0 < x_1 < \cdots < x_n = 1$
- $S^{p,1}(\mathcal{T}) := \{v \in H^1(\Omega) \,|\, v|_{K_i} \in \mathcal{P}_p \quad \forall K_i \in \mathcal{T}\}$ = space of piecewise polyn. of deg. p
- $V_N := S_0^{p,1}(\mathcal{T}) := S^{p,1}(\mathcal{T}) \cap H_0^1(\Omega)$

how to choose $\mathcal{T}$?

- $u_\varepsilon = u_M^{outer} + u_M^{inner} + r_M$
- idea: design $\mathcal{T}$ such that u_M^{outer} and u_M^{inner} can be approximated well
- u_M^{outer} is smooth (uniformly in ε) $\Longrightarrow$ (refinements of) uniform meshes are OK
- u_M^{inner} behaves near $x = 0$ like $e^{-x/\varepsilon}$ $\Longrightarrow$ refine mesh near $x = 0$ (and analogously near $x = 1$)

Shishkin mesh

Shishkin mesh

Given a transition parameter $\tau > 0$ the Shishkin mesh $\mathcal{T}_N^S$ is given by the piecewise uniform mesh with N nodes each in $[0, \tau]$, $[\tau, 1-\tau]$, and $[1-\tau, 1]$.

Lemma

Let f be smooth. If $\tau = \min\{\lambda \varepsilon \log N, 1/2\}$ for sufficiently large $\lambda > 0$ then the piecewise linear interpolant $I_h u_M^{inner}$ satisfies

$$\|u_M^{inner} - I_h u_M^{inner}\|_{L^2(\Omega)} + \sqrt{\varepsilon}\|(u_M^{inner} - I_h u_M^{inner})'\|_{L^2(\Omega)} \leq C N^{-1} \log^{3/2} N$$

Proof: see blackboard; note that the factor $\sqrt{\varepsilon}$ is a stronger result than the energy norm with factor ε

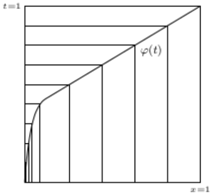
Corollary

Let f be smooth. For $\lambda > 0$ sufficiently large, the FEM based on $S^{1,1}(\mathcal{T}_N^S) \cap H_0^1(\Omega)$ yields

$$\|u_\varepsilon - u_N\|_\varepsilon \leq C N^{-1} \log^{3/2} N$$

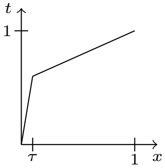
Proof: see blackboard

mesh grading functions

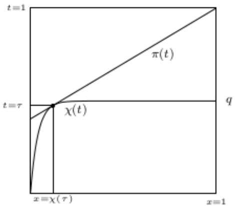


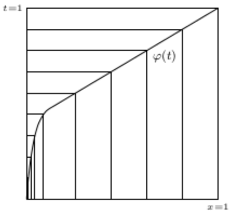
mesh grading function $\varphi : [0,1] \to [0,1]$

- mesh points $x_i = \varphi(t_i)$, where t_i are uniformly distrib. on $[0,1]$
- $h_i \approx \varphi'(t_i) N^{-1}$, $N =$ number of nodes
- uniform mesh where φ is affine
- equidistributing interpolation error could suggest good choices of φ
- example: $\varphi(t) = t \to$ uniform mesh
- example: Shishkin mesh: $\varphi =$ piecewise affine



Bakhvalov mesh





- "typical" sol. behavior near endpoint $x = 0$: $1 - e^{-\beta x/\varepsilon}$
- idea: near $x = 0$, want
$$q(1 - e^{-\beta x_i/(\sigma\varepsilon)}) = t_i,$$
for $q \in (0, 1)$, σ user chosen parameters.
- $q \approx$ proportion of mesh points in the layer; σ controls grading in the layer
- away from $x = 0$: uniform mesh

- $\varphi(t) = \begin{cases} \chi(t) := -\frac{\sigma\varepsilon}{\beta} \ln \frac{q-t}{q} & t \in [0, \tau], \\ \pi(t) := \chi(\tau) + \chi'(\tau)(t - \tau) & \text{otherwise} \end{cases}$
- note: $\varphi \in C^1$ by construction
- τ such that $\varphi(1) = 1$. ($\Longrightarrow$ can compute $\tau \approx q - \frac{\sigma\varepsilon}{\beta}$; $\qquad \chi(\tau) \approx \frac{\sigma\varepsilon}{\beta} \ln \frac{\beta q}{\sigma\varepsilon}$)

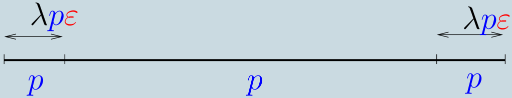
Remark: convergence results for singular perturbation problems without logarithmic factors, but mesh construction more complicated

high order methods

- Shishkin meshes and fixed order methods yield only algebraic convergence
- question: is exponential convergence possible?
- answer: use $V_N = S^{p,1}(\mathcal{T}(\varepsilon,p)) \cap H_0^1(\Omega)$ with $\mathcal{T}(\varepsilon,p)$ given by the nodes

$$0, \quad \tau, \quad 1-\tau, \quad 1, \qquad \text{with } \tau = \lambda p \varepsilon \text{ and } \lambda \text{ sufficiently small and let } p \to \infty$$

spectral boundary layer mesh $\mathcal{T}(\varepsilon,p)$ (Schwab & Suri '96)

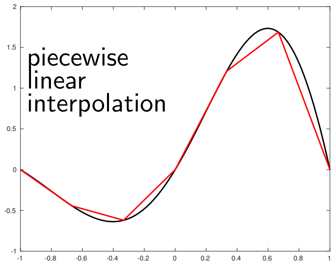


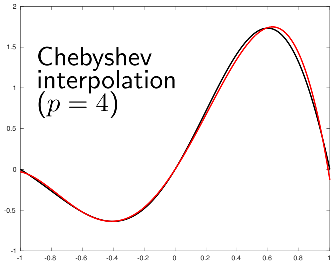
ingredients of the proof of exponential convergence:

- analyticity (with control of constants) of the decomposition $u_\varepsilon = u_M^{outer} + u_M^{inner} + r_M$
- polynomial approximation results for analytic functions

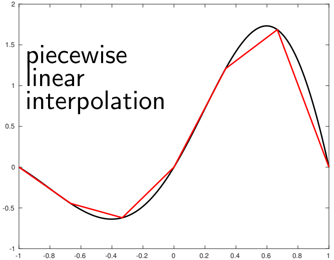
interlude: high order methods and the geometric mesh

interlude: low order vs. high order methods





interlude: low order vs. high order methods



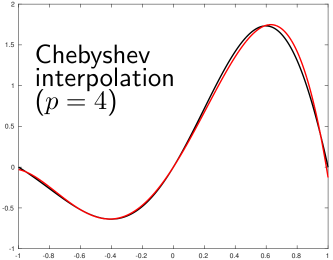
error:

$$\|f - If\|_{L^\infty} \leq \frac{1}{8}h^2\|f''\|_{L^\infty}$$

$\Longrightarrow$

algebraic convergence

$$\|f - If\|_{L^\infty} \leq CN^{-2}$$



error:

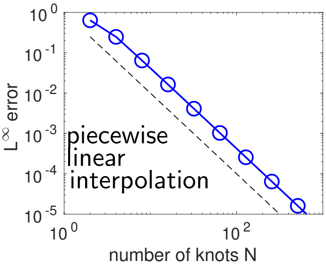
$$\|f - I_p^{Cheb}f\|_{L^\infty} \leq \frac{2^{-p}}{(p+1)!}\|f^{(p+1)}\|_{L^\infty}$$

$\Longrightarrow$

exponential convergence possible:

$$\|f - I_p^{Cheb}f\|_{L^\infty} \leq Ce^{-bp}$$

interlude: low order vs. high order methods



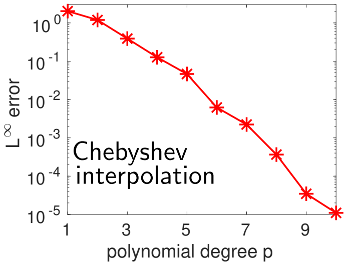
error:

$$\|f - If\|_{L^\infty} \leq \frac{1}{8} h^2 \|f''\|_{L^\infty}$$

$\Longrightarrow$

algebraic convergence

$$\|f - If\|_{L^\infty} \leq C N^{-2}$$



error:

$$\|f - I_p^{Cheb} f\|_{L^\infty} \leq \frac{2^{-p}}{(p+1)!} \|f^{(p+1)}\|_{L^\infty}$$

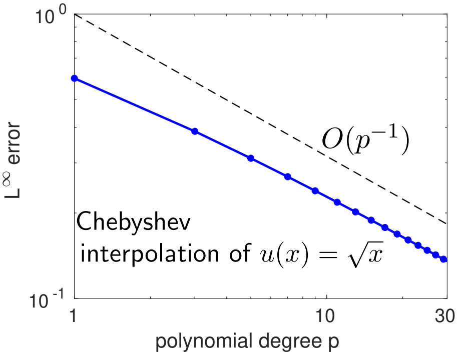
$\Longrightarrow$

exponential convergence possible:

$$\|f - I_p^{Cheb} f\|_{L^\infty} \leq C e^{-bp}$$

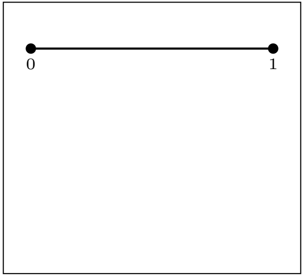
example of failure: $u(x) = x^\alpha$ on $(0,1)$, $\alpha \in (0,1)$

observe: $u(x) = x^\alpha$ is not smooth at $x = 0$



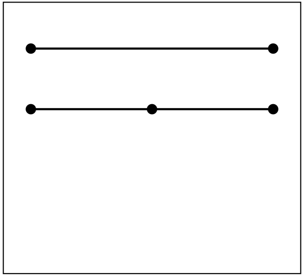
for exponential convergence: use piecewise polynomial approximation on geometric mesh

geometric mesh



Scherer 1981, Babuška & Guo (1986), Schwab, p and hp FEM (1998)

geometric mesh



Scherer 1981,   Babuška & Guo (1986),   Schwab, p and hp FEM (1998)

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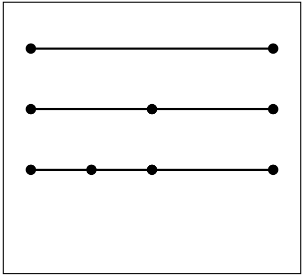
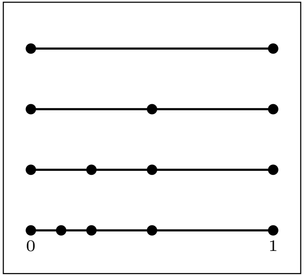


Scherer 1981, Babuška & Guo (1986), Schwab, p and hp FEM (1998)

geometric mesh



Scherer 1981, Babuška & Guo (1986), Schwab, p and hp FEM (1998)

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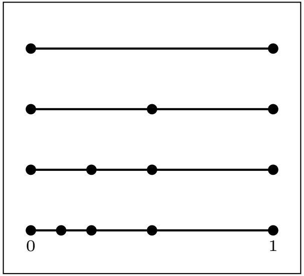

geometric mesh $\mathcal{T}_{geo}$ with

- L layers and
- grading factor $\sigma \in (0,1)$:

$$x_0 = 0, \quad x_1 = \sigma^L, \quad x_2 = \sigma^{L-1}, \quad \ldots, \quad x_L = 1$$

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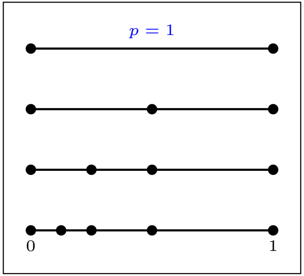

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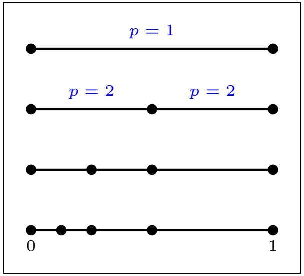

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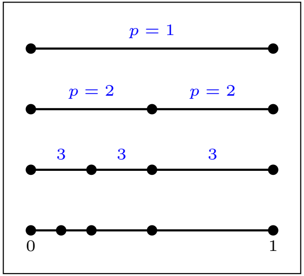

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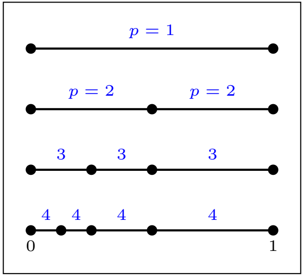

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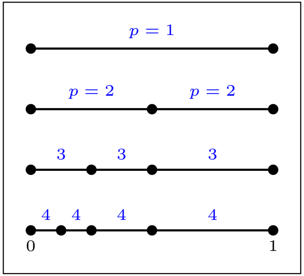

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$$x_0 = 0, \quad x_1 = \sigma^L, \quad x_2 = \sigma^{L-1}, \quad \ldots, \quad x_L = 1$$

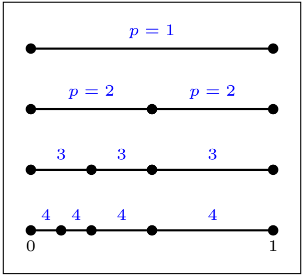
Theorem

On a geometric mesh with p layers:

$$\|x^\alpha - I^{Cheb}_{pw,p} x^\alpha\|_{L^\infty} \leq Ce^{-bp},$$
$$N = p(p+1)$$

Scherer 1981, Babuška & Guo (1986), Schwab, p and hp FEM (1998)

geometric mesh



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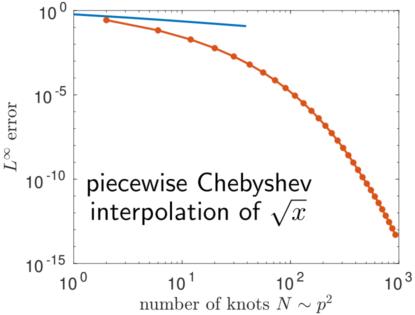


Scherer 1981, Babuška & Guo (1986), Schwab, p and hp FEM (1998)

regularity and exponential convergence for 1D reaction-diffusion equation

analytic regularity by decompositions: $b \equiv 1$

$$\mathcal{L}_\varepsilon u_\varepsilon := -\varepsilon^2 u_\varepsilon'' + u_\varepsilon = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0, \tag{4}$$

- $u_M^{outer} = \sum_{i=0}^{2M} \varepsilon^i u_i(x)$ with

$$u_0 = f, \quad u_1 = 0, \quad u_{i+2} = u_i'', \quad i \geq 0$$

- $\Longrightarrow u_{2i}(x) = f^{(2i)}(x)$ and $u_{2i+1}(x) = 0$.
- $u_M^{inner} = \sum_{i=0}^{2M+1} \varepsilon^i u_i(0) e^{-x/\varepsilon} + \sum_{i=0}^{2M+1} \varepsilon^i e^{-(1-x)/\varepsilon} =: u_M^{inner,l} + u_M^{inner,r}$
- $\mathcal{L}_\varepsilon u_M^{outer} - f = -\varepsilon^{2M+2} u_{2M+2}$

question: what is a good M?

- idea: choose M (in dependence on ε) such that residual $\|\mathcal{L}_\varepsilon u_M^{outer} - f\|$ is minimized

analytic regularity of the decomposition

Lemma

Let $f \in C^\infty(\Omega)$ satisfy

$$\|f^{(n)}\|_{L^\infty(\Omega)} \leq C_f \gamma_f^n n! \qquad \forall n \in \mathbb{N}_0$$

Then the choice $M \sim \varepsilon^{-1}$ leads to a decomposition $u_\varepsilon = u_M^{outer} + u_M^{inner,l} + u_M^{inner,r} + r_M$ with

$$\|(u_M^{outer})^{(n)}\|_{L^\infty(\Omega)} \leq C_u \gamma_u^n n! \qquad \forall n \in \mathbb{N}_0$$

$$|(u_M^{inner,l})^{(n)}(x)| \leq C\varepsilon^{-n} e^{-x/\varepsilon} \qquad \forall n \in \mathbb{N}_0$$

$$\|r_M^{(n)}\|_{L^\infty(\Omega)} \leq e^{-c/\varepsilon}, \qquad n \in \{0,1,2\}.$$

Proof: see blackboard

- Motivation for choice of M:

$$\|\mathcal{L}_\varepsilon u_M^{outer} - f\|_{L^\infty(\Omega)} \leq \varepsilon^{2M+2}\|f^{(2M+2)}\|_{L^\infty(\Omega)} \leq C_f(\varepsilon\gamma_f)^{2M+2}(2M+2)! \leq C_f(\varepsilon\gamma_f(2M+2))^{2M+2}$$

- choose M such that $(2M+2)\varepsilon\gamma_f \approx 1/2$ so as to get exponential convergence in M

the case of non-constant b

$$\mathcal{L}_\varepsilon u_\varepsilon := -\varepsilon^2 u_\varepsilon'' + b(x) u_\varepsilon = f \quad \text{on } \Omega = (0,1), \qquad u(0) = u(1) = 0, \tag{5}$$

Lemma (Melenk '97)

Let f, $b \in C^\infty(\Omega)$ *satisfy*

$$\|f^{(n)}\|_{L^\infty(\Omega)} \le C_f \gamma_f^n n! \qquad \forall n \in \mathbb{N}_0, \qquad \|b^{(n)}\|_{L^\infty(\Omega)} \le C_b \gamma_b^n n! \qquad \forall n \in \mathbb{N}_0,$$

and $b \ge b_0 > 0$.
Then one can select $M \sim 1/\varepsilon$ *such that* $u_\varepsilon = u_M^{outer} + u_M^{inner,l} + u_M^{inner,r} + r_M$ *with*

$$\begin{aligned} \|(u_M^{outer})^{(n)}\|_{L^\infty(\Omega)} &\le C_u \gamma_u^n n! \qquad \forall n \in \mathbb{N}_0 \\ |(u_M^{inner,l})^{(n)}(x)| &\le C \gamma^n e^{-cx/\varepsilon} \max\{n, \varepsilon\}^{-n} \qquad \forall n \in \mathbb{N}_0, \\ \|r_M^{(n)}\|_{L^\infty(\Omega)} &\le e^{-b/\varepsilon}, \qquad n \in \{0,1,2\}. \end{aligned}$$

Proof: induction argument to control u_i of outer expansion and $\widehat{u}_i$ of inner expansion

Polynomial approximation on the reference element $\widehat{K} = (-1,1)$

Theorem

Let $\widehat{u} \in C^\infty(\widehat{K})$ *satisfy*

$$\|\widehat{u}^{(n)}\|_{L^2(\widehat{K})} \le C_u \gamma_u^n n! \qquad \forall n \in \mathbb{N}_0$$

There there are constants C, $c > 0$ *(depending only on* γ_u*) and a polynomials* $\pi_p \in \mathcal{P}_p$ *with* $(\widehat{u} - \pi_p)(\pm 1) = 0$ *such that*

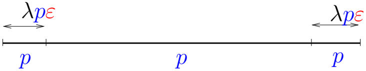
$$\|\widehat{u} - \pi_p\|_{H^1(\widehat{K})} \le C C_u e^{-cp}$$

Proof: see blackboard

1. construct $\pi_p(x) := \widehat{u}(-1) + \int_{-1}^x \Pi_{p-1}^{L^2} \widehat{u}'(t)\,dt$
2. use $\displaystyle\int_{-1}^1 (1-x^2)^k L_i^{(k)}(x) L_j^{(k)}(x)\,dx = \delta_{ij} \frac{2}{2i+1} \frac{(i+k)!}{(i-k)!}$
3. conclude from (2) for Legendre expansion $v = \sum_{i=0}^\infty v_i L_i(x)$ that $\displaystyle\int_{-1}^1 (1-x^2)^k |v^{(k)}(x)|^2\,dx = \sum_{i=k}^\infty \frac{2}{2i+1} |v_i|^2 \frac{(i+k)!}{(i-k)!}$
4. conclude from (3) exponential decay of the coefficients b_i of $\widehat{u}' = \sum_i b_i L_i$ via the choice $k = \lambda i$ for sufficiently small λ

approximation on $\mathcal{T}(\varepsilon, p)$

$\mathcal{T}(\varepsilon, p)$:



Theorem (Melenk '97)

Let f, $b \in C^\infty(\Omega)$ *satisfy*

$$\|f^{(n)}\|_{L^\infty(\Omega)} \leq C_f \gamma_f^n n! \qquad \forall n \in \mathbb{N}_0, \qquad \|b^{(n)}\|_{L^\infty(\Omega)} \leq C_b \gamma_b^n n! \qquad \forall n \in \mathbb{N}_0,$$

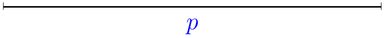
and $b \geq b_0 > 0$. *Then there are* $\lambda_0 > 0$, C, $\beta > 0$ *(depending only on* f *and* b*) such that for* $\lambda \in (0, \lambda_0]$

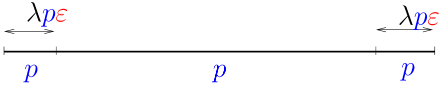
$$\inf_{v \in S_0^{p,1}(\mathcal{T}(\varepsilon,p))} \|u_\varepsilon - v\|_\varepsilon \leq C \lambda^{-1/2} e^{-\beta \lambda p}$$

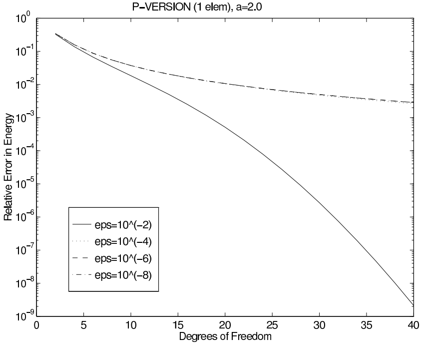
Proof: key is the approximation of the boundary layer part u_M^{inner}.

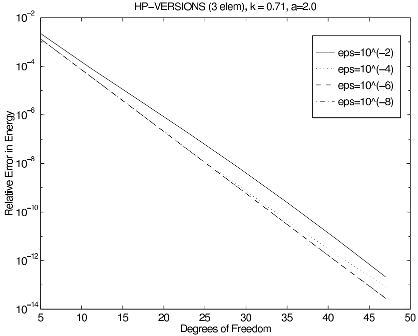
pure p-version versus 3-element mesh

$$-\varepsilon^2 u'' + u = (2 - x^2)^{-1} \qquad \text{on } (-1,1), \qquad u(\pm 1) = 0$$





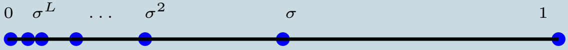




geometric meshes

geometric mesh with L layers

- geometric mesh $\mathcal{T}^L_{geo,\sigma}$ with L layers



characterizing feature

- element at $x = 0$: size σ^L
- all other elements:

$$\frac{\operatorname{diam} K}{\operatorname{dist}(K, 0)} = \text{const}$$

- geometric meshes are cornerstone of high order methods to resolve (algebraic) singularities
- can also resolve boundary layers if L is s.t. scale resolution condition $\sigma^L \approx \varepsilon$ is satisfied

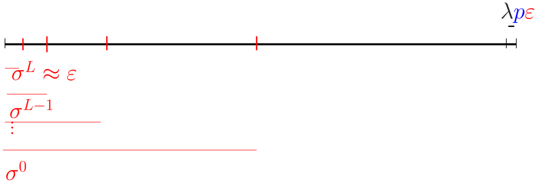
Exercise: Show that if L is such that $\sigma^L \approx \varepsilon$, then

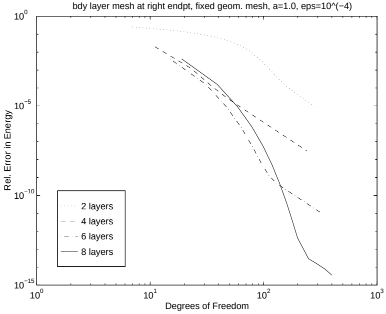
$$\inf_{v \in S_0^{p,1}(\mathcal{T}^L_{geo,\sigma})} \|u_\varepsilon - v\|_\varepsilon \leq C e^{-\beta p}$$

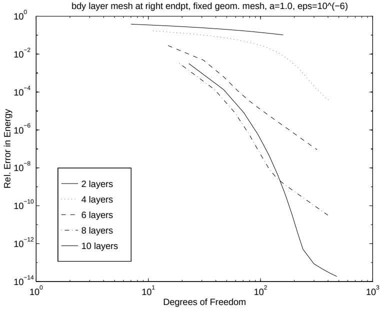
for some $\beta > 0$. The problem size is $N = \dim S_0^{p,1}(\mathcal{T}^L_{geo,\sigma}) \sim pL \sim p|\log \varepsilon|$

resolving layers and singular rhs by geometric mesh

$$-\varepsilon^2 u'' + u = (1+x)^{-0.45} \quad \text{on } (-1,1), \qquad u(\pm 1) = 0$$







asymptotics of the reaction-convection-diffusion equation

$$-\varepsilon u_\varepsilon'' - b(x) u_\varepsilon' + a(x) u_\varepsilon = f \quad \text{on } \Omega = (0,1), \quad u_\varepsilon(0) = u_\varepsilon(1) = 0. \tag{6}$$

- $b \geq b_0 > 0$
- consider the constant coefficient case:
  - fundamental system for homogeneous equation: $e^{\lambda_1 x}$, $e^{\lambda_2 x}$ with

$$\lambda_1 = \frac{b + \sqrt{b^2 + 4a\varepsilon}}{-2\varepsilon} \approx -\frac{b}{\varepsilon}, \qquad \lambda_2 = \frac{b - \sqrt{b^2 + 4a\varepsilon}}{-2\varepsilon} \approx \frac{a}{b}$$

  - expect layer at $x = 0$ of length scale $O(\varepsilon)$

asymptotic expansion for reaction-convection-diffusion equation

- asymptotic expansion: ansatz $u^{outer} \sim \sum_i \varepsilon^i u_i(x)$ yields

$$\sum_i \varepsilon^i(-u''_{i-1} - bu'_i + au_i) \stackrel{!}{=} f$$

- equating coefficients yields recursion

$$\begin{aligned} -bu'_0 + au_0 &= f \quad \text{on } \Omega, && u_0(1) = 0, && \\ -bu'_i + au_i &= u''_{i-1} \quad \text{on } \Omega, && u_i(1) = 0, && i \geq 1 \end{aligned}$$

- expansion $\sum_i \varepsilon^i u_i$ satisfies the b.c. at $x = 1$.
- the boundary layer at $x = 0$ is obtained by the inner expansion $u^{inner}(x) \sim \sum_i \varepsilon^i \widehat{u}_i(x/\varepsilon)$
- using, e.g., the maximum principle and a suitable barrier function, one can show that the remainder r_M satisfies $\|r_M\|_{L^\infty(\Omega)} = O(\varepsilon^{M+1})$

Melenk 1997

an example of a system with multiple scales

an example of a system

$$\mathcal{L}_{\varepsilon,\mu}\mathbf{U} := -\mathbf{E}_{\varepsilon,\mu}\mathbf{U}'' + \mathbf{A}(x)\mathbf{U} = \mathbf{F} \quad \text{on } \Omega = (0,1), \qquad \mathbf{U}(0) = \mathbf{U}(1) = 0,$$

$$\mathbf{E}_{\varepsilon,\mu} := \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \mu^2 \end{pmatrix}, \qquad \mathbf{A} \text{ pointwise SPD}, \qquad \mathbf{A}(x) \geq c_0 > 0, \qquad 0 < \varepsilon \leq \mu \leq 1$$

- existence and uniqueness by Lax-Milgram
- FEM discretization quasi-optimal by Céa Lemma
- solution structure: boundary layers for small ε and/or μ of length scales $O(\varepsilon)$ and $O(\mu)$
- layer structure depends on the scale separation of the scales ε, μ, 1, i.e., the ratios

$$\frac{\mu}{1} \qquad \text{and} \qquad \frac{\varepsilon}{\mu}$$

4 cases:

(I) no scale separation: neither $\mu/1$ nor ε/μ is small

(II) 3 scales: $\mu/1$ is small and ε/μ is small

(III) 2 scales: $\mu/1$ is small and ε/μ is *not* small

(IV) 2 scales: $\mu/1$ is *not* small and ε/μ is small

Melenk, Xenophontos, Oberbroeckling '13

asymptotic expansions for the 3-scale case

- layers on scales $O(\mu)$, $O(\varepsilon)$ $\to$ stretched variables $\widetilde{x} := x/\mu$ and $\widehat{x} := x/\varepsilon$ near $x = 0$ (and corresponding ones $\widetilde{x}^R$, $\widehat{x}^R$ at $x = 1$)
- formal ansatz

$$\mathbf{U}(x) \sim \sum_{i,j} \left(\frac{\mu}{1}\right)^i \left(\frac{\varepsilon}{\mu}\right)^j \left[\mathbf{U}_{ij}(x) + \widetilde{\mathbf{U}}_{ij}(\widetilde{x}) + \widehat{\mathbf{U}}_{ij}(\widehat{x}) + \widetilde{\mathbf{U}}^R_{ij}(\widetilde{x}^R) + \widehat{\mathbf{U}}^R_{ij}(\widehat{x}^R)\right]$$

- write

$$\mathbf{A}(x) \overset{\text{Taylor}}{=} \sum_k \mathbf{A}_k x^k = \sum_k \mathbf{A}_k \mu^k \widetilde{x}^k = \sum_k \mathbf{A}_k \mu^k \left(\frac{\varepsilon}{\mu}\right)^k \widehat{x}^k$$

- write the operator $\mathcal{L}_{\varepsilon,\mu}$ as:

on the $\widetilde{x}$-scale: $$-\mu^{-2}\mathbf{E}_{\varepsilon,\mu}\partial^2_{\widetilde{x}}\mathbf{U}(\widetilde{x}) + \sum_k \mu^k \mathbf{A}_k \widetilde{x}^k \mathbf{U}(\widetilde{x}),$$

on the $\widehat{x}$-scale: $$-\varepsilon^{-2}\mathbf{E}_{\varepsilon,\mu}\partial^2_{\widehat{x}}\mathbf{U}(\widehat{x}) + \sum_k \mu^k \left(\frac{\varepsilon}{\mu}\right)^k \mathbf{A}_k \widehat{x}^k \mathbf{U}(\widehat{x}).$$

asymptotic expansions for the 3-scale case, II

viewing the variables x, $\widetilde{x}$, $\widehat{x}$ as independent variables and inserting the ansatz into the differential equation yields

$$O(1)\text{-scale:} \qquad \sum_{i,j} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \left[-\mathbf{E}_{\varepsilon,\mu}\mathbf{U}''_{ij}(x) + \mathbf{A}(x)\mathbf{U}_{ij}(x)\right] = \mathbf{F}(x),$$

$$O(\mu)\text{-scale:} \qquad \sum_{i,j} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \left[-\mu^{-2}\mathbf{E}_{\varepsilon,\mu}\widetilde{\mathbf{U}}''_{ij}(\widetilde{x}) + \sum_k \mu^k \mathbf{A}_k \widetilde{x}^k \widetilde{\mathbf{U}}_{ij}(\widetilde{x})\right] = 0$$

$$O(\varepsilon)\text{-scale:} \qquad \sum_{i,j} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \left[-\varepsilon^{-2}\mathbf{E}_{\varepsilon,\mu}\widehat{\mathbf{U}}''_{ij}(\widehat{x}) + \sum_k \varepsilon^k \mathbf{A}_k \widehat{x}^k \widehat{\mathbf{U}}_{ij}(\widehat{x})\right] = 0$$

and analogous equations for $\widetilde{\mathbf{U}}^R(\widetilde{x}^R)$ and $\widehat{\mathbf{U}}^R(\widehat{x}^R)$
now one equates like powers of μ and ε/μ!

asymptotic expansions for the 3-scale case, III

write

$$\mathbf{U}_{ij}(x) = \begin{pmatrix} u_{ij}(x) \\ v_{ij}(x) \end{pmatrix}, \qquad \widetilde{\mathbf{U}}_{ij}(\widetilde{x}) = \begin{pmatrix} \widetilde{u}_{ij}(\widetilde{x}) \\ \widetilde{v}_{ij}(\widetilde{x}) \end{pmatrix}, \qquad \widehat{\mathbf{U}}_{ij}(\widehat{x}) = \begin{pmatrix} \widehat{u}_{ij}(\widehat{x}) \\ \widehat{v}_{ij}(\widehat{x}) \end{pmatrix},$$

and arrive at

$$-\begin{pmatrix} u''_{i-2,j-2} \\ v''_{i-2,j} \end{pmatrix} + \mathbf{A}(x)\mathbf{U}_{ij}(x) = \mathbf{F}(x)\delta_{(i,j),(0,0)}$$

$$-\begin{pmatrix} \widetilde{u}''_{i,j-2} \\ \widetilde{v}''_{i,j} \end{pmatrix} + \sum_{k=0}^{i} \mathbf{A}_k \widetilde{x}^k \widetilde{\mathbf{U}}_{i-k,j}(\widetilde{x}) = 0$$

$$-\begin{pmatrix} \widehat{u}''_{i,j} \\ \widehat{v}''_{i,j+2} \end{pmatrix} + \sum_{k=0}^{\min\{i,j\}} \mathbf{A}_k \widehat{x}^k \widehat{\mathbf{U}}_{i-k,j-k}(\widehat{x}) = 0$$

This recursion is complemented with the following side conditions:

$$\mathbf{U}_{ij}(0) + \widetilde{\mathbf{U}}_{ij}(0) + \widehat{\mathbf{U}}_{ij}(0) = 0, \qquad \text{and decay conditions for } \widetilde{\mathbf{U}},\ \widehat{\mathbf{U}} \text{ at } \infty$$

asymptotic expansions for the 3-scale case, IV

$$-\begin{pmatrix} u''_{i-2,j-2} \\ v''_{i-2,j} \end{pmatrix} + \mathbf{A}(x)\mathbf{U}_{ij}(x) = \mathbf{F}(x)\delta_{(i,j),(0,0)}$$

recursion for the $\mathbf{U}_{ij}$:

$$\mathbf{U}_{0,0}(x) = \mathbf{A}^{-1}(x)\mathbf{F}(x),$$

$$\mathbf{U}_{i,j}(x) = \mathbf{A}^{-1}(x)\begin{pmatrix} u''_{i-2,j-2} \\ v''_{i-2,j} \end{pmatrix}, \qquad (i,j) \neq (0,0)$$

asymptotic expansions for the 3-scale case, V

$$-\begin{pmatrix} \widetilde{u}''_{i,j-2} \\ \widetilde{v}''_{i,j} \end{pmatrix} + \underbrace{\sum_{k=0}^{i} \mathbf{A}_k \widetilde{x}^k \widetilde{\mathbf{U}}_{i-k,j}(\widetilde{x})}_{=\mathbf{A}_0 \mathbf{U}_{ij} \text{ for } \mathbf{A} = \text{const}} = 0 \tag{7}$$

- simplify notation by assuming $\mathbf{A} \equiv \mathbf{A}_0$ (i.e., $\mathbf{A}_k = 0$ for $k \geq 1$)
- study the case $j = 0$. Then the first equation of (7) is an algebraic equation:

$$\mathbf{A}_{11}\widetilde{u}_{i,0} + \mathbf{A}_{12}\widetilde{v}_{i,0} = 0 \tag{8}$$

- solve for $\widetilde{u}_{i,0}$ and insert into the second equation of (7):

$$-\widetilde{v}''_{i,0} + \frac{\mathbf{A}_{11}\mathbf{A}_{22} - \mathbf{A}_{12}\mathbf{A}_{21}}{\mathbf{A}_{11}}\widetilde{v}_{i,0} = 0 \tag{9}$$

- complement (9) with the side conditions

$$\widetilde{v}_{i,0}(0) = -v_{i,0}(0), \qquad \widetilde{v}_{i,0}(\widetilde{x}) \to 0 \quad \text{for } \widetilde{x} \to \infty$$

- finally solve for $\widetilde{u}_{i,0}$ with (8).

asymptotic expansions for the 3-scale case, V

$$-\begin{pmatrix} \widehat{u}''_{i,j} \\ \widehat{v}''_{i,j+2} \end{pmatrix} + \underbrace{\sum_{k=0}^{\min\{i,j\}} \mathbf{A}_k \widehat{x}^k \widehat{\mathbf{U}}_{i-k,j-k}(x)}_{=\mathbf{A}_0\widehat{\mathbf{U}}} = 0 \tag{10}$$

- set $\widehat{v}_{i,0} = \widehat{v}_{i,1} = 0$
- then first equation of (10) yields

$$-\widehat{u}''_{i,0} + \mathbf{A}_{11}\widehat{u}_{i,0} = 0$$

$$\widehat{u}_{i,0}(0) = -u_{i,0} \qquad \widehat{u}_{i,0}(\widehat{x}) \to 0 \qquad \text{as } \widehat{x} \to \infty$$

- solve second equation of (10) for $\widehat{v}_{i,2}$:

$$\widehat{v}_{i,2}(\widehat{x}) = \int_{\widehat{x}}^{\infty} \int_{t}^{\infty} \mathbf{A}_{21}\widehat{u}_{i,0}(\tau)\, d\tau\, dt$$

- so far, we have obtained $\widetilde{\mathbf{U}}_{i,0}$, $\widehat{\mathbf{U}}_{i,0}$. The functions $\widetilde{\mathbf{U}}_{i,j}$, $\widehat{\mathbf{U}}_{i,j}$ for $j > 0$ are obtained recursively

asymptotic expansions for the 3-scale case, V

Theorem (Melenk, Xenophontos, Oberbroeckling '13)

Let $\mathbf{A}$, $\mathbf{F}$ *be analytic. Then,* $\mathbf{U}$ *can be written as*

$$\mathbf{U} = \mathbf{U}_M(x) + \widetilde{\mathbf{U}}_M(\widetilde{x}) + \widehat{\mathbf{U}}_M(\widehat{x}) + \widetilde{\mathbf{U}}_M^R(\widetilde{x}^R) + \widehat{\mathbf{U}}_M^R(\widehat{x}^R) + \mathbf{R}_M$$

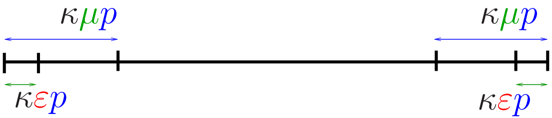
where

$$\begin{aligned}
\|D_x^n \mathbf{U}_M\|_{L^\infty(\Omega)} &\leq C\gamma^n n! & \forall n \in \mathbb{N}_0\\
|D_{\widetilde{x}}^n \widetilde{\mathbf{U}}_M(\widetilde{x})| &\leq C\gamma^n e^{-b\widetilde{x}} & \forall n \in \mathbb{N}_0\\
|D_{\widehat{x}}^n \widehat{\mathbf{U}}_M(\widehat{x})| &\leq C\gamma^n e^{-b\widehat{x}} & \forall n \in \mathbb{N}_0\\
\|\mathbf{R}_M\|_{L^\infty(\Omega)} &\leq Ce^{-b/\mu} + Ce^{-b\mu/\varepsilon}
\end{aligned}$$

Proof:

- structurally similar to the scalar case
- optimize expansion order M

exponential convergence for multiscale problems



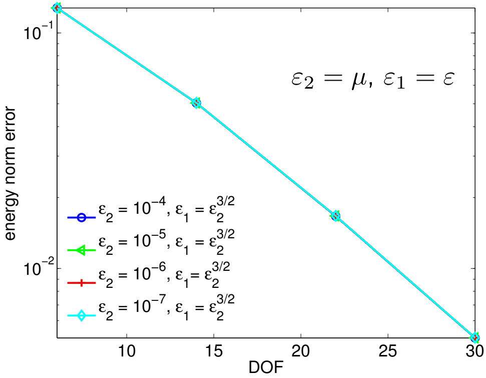
Theorem (Melenk, Xenophontos, Oberbroeckling '13)

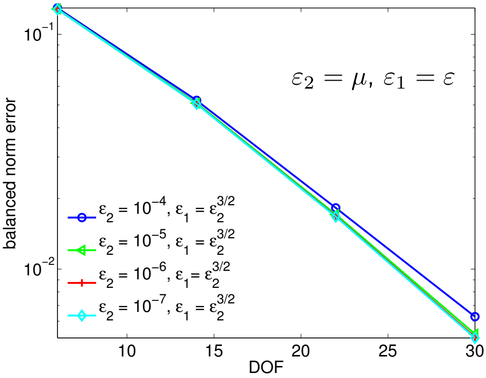
The FEM approximation $\mathbf{U}_N \in S_0^{p,1}(\Delta_{\varepsilon,\mu,p})$ *satisfies*

$$\|\mathbf{U} - \mathbf{U}_N\|_{\mathbf{E}}^2 \leq Ce^{-bp}$$

Approximation in balanced norms: system case

$$\mathbf{A} = \begin{pmatrix} 2\,(x+1)^2 & -\,(1+x^2) \\ -2\cos(\pi x/4) & 2.2e^{1-x} \end{pmatrix}, \quad \mathbf{F}(x) = \frac{1}{1/2+x}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$





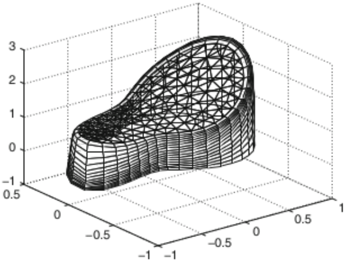
energy norm:
$\|\mathbf{E}^{1/2}\mathbf{U}'\|_{L^2}^2 + \|\mathbf{U}\|_{L^2}^2$

balanced norm:
$\|\mathbf{E}^{1/4}\mathbf{U}'\|_{L^2}^2 + \|\mathbf{U}\|_{L^2}^2$

regularity and hp-FEM for the 2D reaction-diffusion equation

2D reaction diffusion equation

$$-\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon = f \quad \text{on } \Omega, \qquad\qquad u_\varepsilon = g \quad \text{on } \partial\Omega \tag{11}$$



observations:

- boundary layer at $\partial\Omega$: rapid changes in normal direction, smooth solution variation in tangential direction (smooth data)
- $\rightarrow$ appropriate mesh design: long thin elements

the case of smooth $\partial\Omega$

$$\mathcal{L}_\varepsilon u_\varepsilon := -\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon = f \quad \text{on } \Omega, \qquad\qquad u_\varepsilon = g \quad \text{on } \partial\Omega$$

- (outer expansion) make the formal ansatz $u_\varepsilon \sim \sum_i \varepsilon^i u_i(x,y)$, which leads to

$$u^{outer}(x,y) \sim \sum_i \varepsilon^{2i} \Delta^i f = f + \varepsilon^2 \Delta f + \varepsilon^4 \Delta^2 f + \cdots$$

- (boundary fitted coordinates): Let $\mathbf{X} : \mathbb{T}_L \to \partial\Omega$ be a (smooth, periodic) parametrization of $\partial\Omega$ and $\mathbf{n}(\theta)$ be the outer normal vector at $\mathbf{X}(\theta)$. Set:

$$\psi : (0,\rho_0) \times \mathbb{T}_L \to \mathbb{R}^2, \qquad (\rho,\theta) \mapsto \psi(\rho,\theta) := \mathbf{X}(\theta) - \rho \mathbf{n}(\theta)$$

- fact: for ρ_0 sufficiently small, ψ is smoothly invertible, and its range is a half-tubular neighborhood of $\partial\Omega$

the case of smooth $\partial\Omega$ for $-\varepsilon^2\Delta u + u = f$, $u|_{\partial\Omega} = g$

- $\kappa(\theta)$ = curvature of $\partial\Omega$ at $\mathbf{X}(\theta)$ and

$$\sigma(\theta) := \frac{1}{1-\rho\kappa(\theta)}$$

- in fitted coordinates, we have $\Delta u(\rho,\theta) = \partial_\rho^2 u - \kappa(\theta)\sigma(\rho,\theta)\partial_\rho u + \sigma^2(\rho,\theta)\partial_\theta^2 u + \rho\kappa'(\theta)\sigma^3(\rho,\theta)\partial_\theta u$
- in stretched coordinates $\widehat{\rho} := \rho/\varepsilon$, we have

$$\mathcal{L}_\varepsilon = -\partial_{\widehat{\rho}}^2 + \mathrm{Id} + \varepsilon\kappa(\theta)\sigma(\varepsilon\widehat{\rho},\theta)\partial_{\widehat{\rho}} - \varepsilon^2\sigma^2(\varepsilon\widehat{\rho},\theta)\partial_\theta^2 - \varepsilon\widehat{\rho}\kappa'(\theta)\sigma^3(\varepsilon\widehat{\rho},\theta)\partial_\theta$$

- expanding in power series of ε, we write

$$\mathcal{L}_\varepsilon = \sum_i \varepsilon^i L_i$$

$$L_0 = -\partial_{\widehat{\rho}}^2 + \mathrm{Id}, \qquad L_i = -\widehat{\rho}^{\,i-1} a_1^{i-1}\partial_{\widehat{\rho}} - \widehat{\rho}^{\,i-2} a_2^{i-2}\partial_\theta^2 - \widehat{\rho}^{\,i-2} a_3^{i-3}\partial_\theta, \qquad i \geq 1,$$

$$a_1^i = -\kappa^{i+1}, \qquad a_2^i = (i+1)\kappa^i, \qquad a_3^i = \frac{(i+1)(i+2)}{2}\kappa^i\kappa'$$

$$a_1^i = a_2^i = a_3^i = 0 \quad i < 0$$

- Ansatz for u^{inner}: $u^{inner} \sim \sum_i \varepsilon^i \widehat{U}_i(\widehat{\rho}, \theta)$.
- inserting condition $\mathcal{L}_\varepsilon u^{inner} \stackrel{!}{=} 0$ yields

$$\sum_i \varepsilon^i \sum_{j=0}^{i} L_j \widehat{U}_{i-j}(\widehat{\rho}, \theta) = 0$$

- $\rightsquigarrow$ recurrence relation for the $\widehat{U}_i$:

$$-\partial^2_{\widehat{\rho}} \widehat{U}_i + \widehat{U}_i = \widehat{F}_1 + \widehat{F}_2 + \widehat{F}_3,$$

$$\widehat{F}_1 = \sum_{j=0}^{i-1} \widehat{\rho}^j a_1^j \partial_{\widehat{\rho}} \widehat{U}_{i-1-j}, \quad \widehat{F}_2 = \sum_{j=0}^{i-2} \widehat{\rho}^j a_2^j \partial^2_\theta \widehat{U}_{i-2-j}, \quad \widehat{F}_3 = \sum_{j=0}^{i-3} \widehat{\rho}^{j+1} a_3^j \partial_\theta \widehat{U}_{i-3-j},$$

- boundary conditions:

$$\widehat{U}_i(\widehat{\rho}, \theta) \to 0 \quad \text{for } \widehat{\rho} \to \infty$$

$$\widehat{U}_i(0, \theta) = G_i := \begin{cases} g - f|_{\partial\Omega} & \text{for } i = 0, \\ \Delta^{(i/2)} f|_{\partial\Omega} & \text{if } i \text{ is even}, \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

asymptotic expansions for smooth $\partial\Omega$

- induction $\Longrightarrow \widehat{U}_i(\widehat{\rho},\theta) = \left(\sum_{j=0}^{i} \Theta_{ij}(\theta)\widehat{\rho}^j\right)e^{-\widehat{\rho}}$ with smooth fcts Θ_{ij}
- truncated outer expansion: $u_M^{outer} := \sum_{i=0}^{M} \varepsilon^{2i}\Delta^i f$
- truncated inner expansion: $u_M^{inner} := \left(\sum_{i=0}^{2M+1} \varepsilon^i \widehat{U}_i(\rho/\varepsilon,\theta)\right) \circ \psi^{-1}$
- By construction: $u_M^{outer} + u_M^{inner} = g$ on $\partial\Omega$
- By construction: $\mathcal{L}_\varepsilon u_M^{inner} = O(\varepsilon^{2M+2})$ in a neighborhood of $\partial\Omega$
- $\chi =$ be a cut-off function supported by a tubular neighborhood of $\partial\Omega$; $\chi \equiv 1$ near $\partial\Omega$
- u_M^{inner} decays exponentially away from $\partial\Omega \Longrightarrow$ the function χu_M^{inner} is defined on Ω and satisfies
$$\|\mathcal{L}_\varepsilon(\chi u_M^{inner})\|_{L^\infty(\Omega)} = O(\varepsilon^{2M+2})$$
- $\Longrightarrow$ representation $u_\varepsilon = u_M^{outer} + \chi u_M^{inner} + r_M$

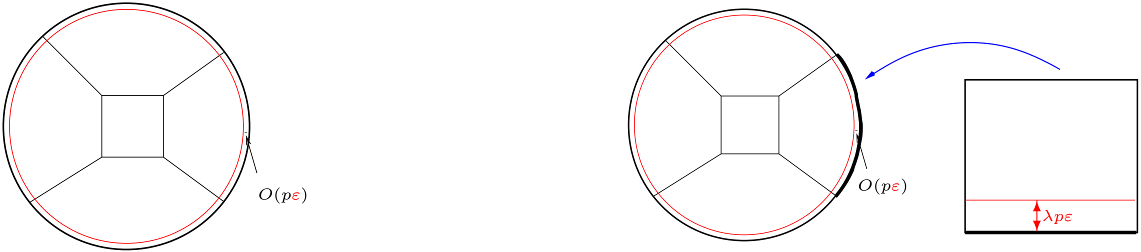
regularity of the asymptotic expansion

Theorem (Melenk & Schwab '99, Melenk '02)

Let f, g be analytic and let $\partial\Omega$ be analytic. Then for the choice $M \sim 1/\varepsilon$

$$
\begin{aligned}
\|\nabla^n u_M^{outer}\|_{L^\infty(\Omega)} &\leq C\gamma^n n! \qquad \forall n \in \mathbb{N}_0, \\
\|\partial_\theta^m \partial_\rho^n u_M^{inner}\|_{L^\infty(\mathcal{U})} &\leq C\gamma^{n+m} m! \max\{n, \varepsilon^{-1}\}^n \qquad \forall (n,m) \in \mathbb{N}_0^2, \\
\|r_M\|_{H^1(\Omega)} &\leq C e^{-b/\varepsilon}.
\end{aligned}
$$

convergence on spectral boundary layer mesh $\mathcal{T}(\varepsilon, p)$

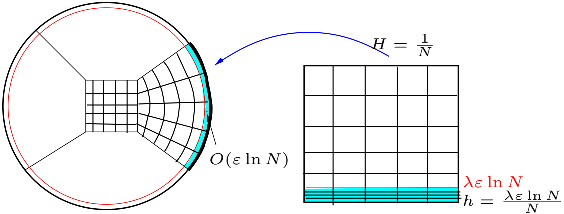


Theorem (Melenk & Schwab '98)

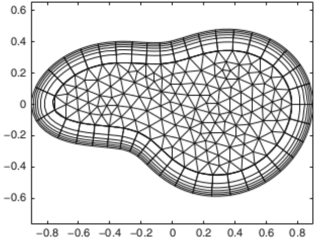
Let f, g, $\partial\Omega$ be analytic. Let $\mathcal{T}^0$ be a fixed mesh consisting of quadrilaterals, and let the spectral boundary layer mesh $\mathcal{T}(\varepsilon, p)$ be constructed by refining the elements at the boundary. Then there is $\lambda_0 > 0$ such that for $0 < \lambda \leq \lambda_0$ there are C, $b > 0$ such that

$$\inf_{v \in S_0^{p,1}(\mathcal{T}(\varepsilon,p))} \|u_\varepsilon - v\|_\varepsilon \leq C \frac{1}{\sqrt{\lambda}} e^{-\lambda b p}.$$

convergence on Shishkin meshes



Shishkin mesh near $\partial\Omega$



Bakhvalov mesh near $\partial\Omega$

Theorem

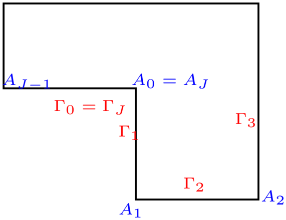
Let f, g, $\partial\Omega$ be analytic. Let $\mathcal{T}^0$ be a fixed mesh consisting of quadrilaterals, and let the Shishkin mesh $\mathcal{T}^{Shishkin}$ be constructed by refining the elements at the boundary. Then for $\lambda > 0$ sufficiently large there is $C > 0$ such that

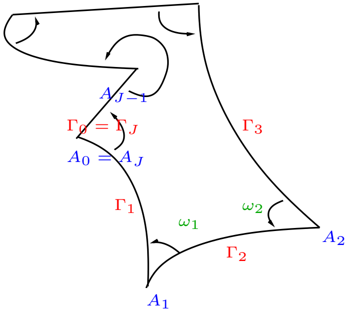
$$\inf_{v \in S_0^{1,1}(\mathcal{T}^{Shishkin})} \|u_\varepsilon - v\|_\varepsilon \leq C N^{-1} \ln^{3/2} N$$

corner domains

curvilinear polygon

- J vertices A_i, $i = 0, \ldots, J-1$
- J edges; Γ_i and Γ_{i+1} meet at A_{i+1}
- interior angles ω_i
- arcs Γ_i analytic (straight arcs = special case)



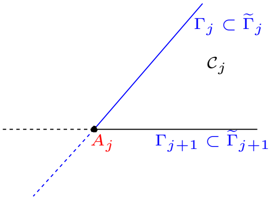


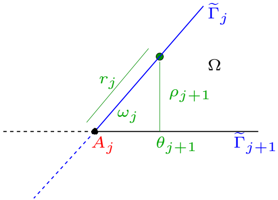
asymptotic expansions for corner domains

$$-\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon = f \quad \text{in } \Omega, \qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega$$

- elliptic equations in corner domains have corner singularities $\rightarrow$ need terms ("corner layers") that represent these
- definition of asymptotic require smoothness of the data! $\rightarrow$ boundary layer expansions can only be defined in a piecewise manner. $\rightarrow$ connecting these pieces requires "corner layers"

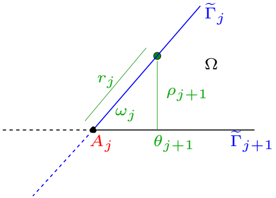
convex corners





- $\widetilde{\Gamma}_j :=$ half-line emating from A_j in direction of Γ_j
- $\widetilde{\Gamma}_{j+1} :=$ half-line emating from A_j in direction of Γ_{j+1}
- $\mathcal{C}_j :=$ cone with apex A_j and edges $\widetilde{\Gamma}_j$, $\widetilde{\Gamma}_{j+1}$
- $(\rho_j, \theta_j) =$ local coordinates associated with Γ_j
- $(\rho_{j+1}, \theta_{j+1}) =$ local coord. associated with Γ_{j+1}
- $r_j(\cdot) := \mathrm{dist}(\cdot, A_j)$
- parametrize $\widetilde{\Gamma}_j$ and $\widetilde{\Gamma}_{j+1}$ by r_j
- on the half-line $\widetilde{\Gamma}_j$: $r_j = \frac{\rho_{j+1}}{\sin \omega_j}$, on the half-line $\widetilde{\Gamma}_{j+1}$: $r_j = \frac{\rho_j}{\sin \omega_j}$

convex corners



- on $\widetilde{\Gamma}_j$: $r_j = \frac{\rho_{j+1}}{\sin\omega_j}$
- on $\widetilde{\Gamma}_{j+1}$: $r_j = \frac{\rho_j}{\sin\omega_j}$

- outer expansion: $u_M^{outer} := \sum_{i=0}^{M} \varepsilon^{2i}\Delta^i f$
- $\rightsquigarrow$ boundary correction needed

- for Γ_j, define the inner expansion $u_{M,j}^{inner} = \sum_{i=0}^{2M+1} \varepsilon^i \widehat{U}_{i,j}(\rho_j/\varepsilon, \theta_j)$ as for the smooth case above in the local coordinates (ρ_j, θ_j)
- for Γ_{j+1}, define the inner expansion $u_{M,j+1}^{inner} = \sum_{i=0}^{2M+1} \varepsilon^i \widehat{U}_{i,j+1}(\rho_{j+1}/\varepsilon, \theta_{j+1})$ as for the smooth case above in the local coordinates $(\rho_{j+1}, \theta_{j+1})$
- The terms $\widehat{U}_{i,j+1}(\cdot, \theta_{j+1})$ and $\widehat{U}_{i,j}(\cdot, \theta_j)$ decay exponentially
- mismatch on $\widetilde{\Gamma}_{j+1}$: $\widehat{\widehat{U}}_{i,j}(r_j/\varepsilon) := \widehat{U}_{i,j}(\rho_j/\varepsilon, \theta_j) = \widehat{U}_{i,j}(r_j/(\varepsilon\sin\omega_j), \theta_j)$
- mismatch on $\widetilde{\Gamma}_j$: $\widehat{\widehat{U}}_{i,j+1}(r_j/\varepsilon) := \widehat{U}_{i,j+1}(\rho_{j+1}/\varepsilon, \theta_j) = \widehat{U}_{i,j+1}(r_j/(\varepsilon\sin\omega_j), \theta_j)$
- functions $\widehat{\widehat{U}}_{i,j}$, $\widehat{\widehat{U}}_{i,j+1}$ decay exponentially (if extended suitably)

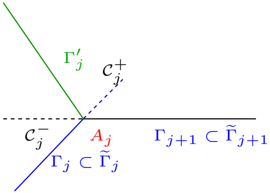
convex corners

- $\rightsquigarrow$ define the corner layer functions $\widehat{U}_{i,j}^{CL}$ as the solutions of

$$
\begin{aligned}
-\Delta \widehat{U}_{i,j}^{CL} + \widehat{U}_{i,j}^{CL} &= 0 \quad \text{in cone } \mathcal{C}_i \\
\widehat{U}_{i,j}^{CL} &= -\widehat{\widehat{U}}_{i,j} \quad \text{on } \widetilde{\Gamma}_{j+1} \\
\widehat{U}_{i,j}^{CL} &= -\widehat{\widehat{U}}_{i,j+1} \quad \text{on } \widetilde{\Gamma}_{j} \\
\widehat{U}_{i,j}^{CL} &\to 0 \quad \text{as } |x| \to \infty \text{ in } \mathcal{C}_i
\end{aligned}
$$

- upshot: $u_M^{outer} + u_{M,j}^{inner} + u_{M,j+1}^{inner} + \sum_{i=0}^{2M+1} \widehat{U}_{i,j}^{CL}((x - A_j)/\varepsilon)$ is a good approximation to u_ε in a neighborhood of A_j
- $\rightsquigarrow$ localize with a cut-off function χ_j^{CL}

non-convex corners



- Γ_j' = bisector
- $\mathcal{C}_j$ is split into two sectors $\mathcal{C}_j^+$ and $\mathcal{C}_j^-$

- for Γ_j, define the inner expansion $u_{M,j}^{inner} = \sum_{i=0}^{2M+1} \varepsilon^i \widehat{U}_{i,j}(\rho_j/\varepsilon, \theta_j)$ as for the smooth case above in the local coordinates (ρ_j, θ_j)
- for Γ_{j+1}, define the inner expansion $u_{M,j+1}^{inner} = \sum_{i=0}^{2M+1} \varepsilon^i \widehat{U}_{i,j+1}(\rho_{j+1}/\varepsilon, \theta_{j+1})$ as for the smooth case above in the local coordinates $(\rho_{j+1}, \theta_{j+1})$
- define u_M^{inner} near A_j by

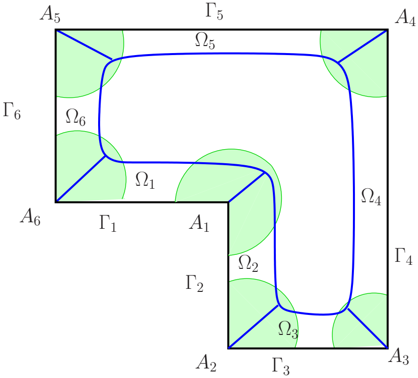
$$u_M^{inner} = \begin{cases} u_{M,j+1}^{inner} & \text{on } \mathcal{C}_j^+ \\ u_{M,j}^{inner} & \text{on } \mathcal{C}_j^- \end{cases}$$

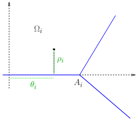
- correct the jump and the jump of the normal derivative of u_M^{inner} across Γ_j' with a corner layer u_j^{CL}, i.e., the solution of a suitable transmission problem
- Remark: construction also works for convex corners

Decomposition of u_ε via asymptotic expansions

For each ε, the exact solution u_ε can be decomposed as

$$u_\varepsilon = w_\varepsilon + \chi\, u_\varepsilon^{BL} + \widehat{\chi}\, u_\varepsilon^{CL} + r_\varepsilon \tag{12}$$





- χ = cut-off function supported by blue region near $\partial\Omega$
- $\widehat{\chi}$ = cut-off function supported by green region near vertices

Regularity of the decomposition

Theorem (Melenk '02)

There are C, γ, $\alpha > 0$, $\beta \in [0,1)$ *s.t. for each* $\varepsilon \in (0,1]$ *there is a decomposition of the form* $u_\varepsilon = w_\varepsilon + \chi\, u_\varepsilon^{BL} + \widehat{\chi}\, u_\varepsilon^{CL} + r_\varepsilon$ *with*

1. w_ε *is analytic on* $\overline{\Omega}$ *and* $\|\nabla^p w_\varepsilon\|_{L^\infty(\Omega)} \le C\gamma^p p!, \qquad \forall p \in \mathbb{N}_0$.
2. *remainder* $r_\varepsilon \in H_0^1(\Omega)$ *and* $\|r_\varepsilon\|_\varepsilon \le C \exp(-\alpha/\varepsilon)$.
3. *On* Ω_j *the boundary layer function* u_ε^{BL} *satisfies*

$$|\partial_{\rho_j}^r \partial_{\theta_j}^s u^{BL}(\rho_j,\theta_j)| \le C \max\{r!, \varepsilon^{-r}\}\gamma^{r+s} s!\, e^{-\alpha\rho_j/\varepsilon}, \quad \rho_j \ge 0.$$

4. *The corner layer* u_ε^{CL} *satisfies on* $(\Omega_j \cap B_j) \cup (\Omega_{j+1} \cap B_j)$

$$|\nabla^p u^{CL}(x)| \le C\varepsilon^{\beta-1}\, r_j^{1-p-\beta} \exp(-\alpha r_j/\varepsilon) \qquad p \in \mathbb{N}_0$$

$(B_j =$ *neighborhood of* A_j, $r_j = \mathrm{dist}(x, A_j))$

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1. w_ε *is analytic on* $\overline{\Omega}$ *and* $\|\nabla^p w_\varepsilon\|_{L^\infty(\Omega)} \le C\gamma^p p!, \qquad \forall p \in \mathbb{N}_0$.
2. *remainder* $r_\varepsilon \in H^1_0(\Omega)$ *and* $\|r_\varepsilon\|_\varepsilon \le C\exp(-\alpha/\varepsilon)$.
3. *On* Ω_j *the boundary layer function* u_ε^{BL} *satisfies*
$$|\partial^r_{\rho_j}\partial^s_{\theta_j} u^{BL}(\rho_j,\theta_j)| \le C\max\{r!, \varepsilon^{-r}\}\gamma^{r+s} s!\, e^{-\alpha\rho_j/\varepsilon}, \quad \rho_j \ge 0.$$
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$$|\nabla^p u^{CL}(x)| \leq C\varepsilon^{\beta-1}\, r_j^{1-p-\beta} \exp(-\alpha r_j/\varepsilon) \qquad p \in \mathbb{N}_0$$

(B_j *= neighborhood of* A_j*,* $r_j = \operatorname{dist}(x, A_j)$*)*

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3. *On* Ω_j *the boundary layer function* u_ε^{BL} *satisfies*

$$|\partial_{\rho_j}^r \partial_{\theta_j}^s u^{BL}(\rho_j, \theta_j)| \le C \max\{r!, \varepsilon^{-r}\}\gamma^{r+s} s!\, e^{-\alpha\rho_j/\varepsilon}, \quad \rho_j \ge 0.$$

4. *The corner layer* u_ε^{CL} *satisfies on* $(\Omega_j \cap B_j) \cup (\Omega_{j+1} \cap B_j)$

$$|\nabla^p u^{CL}(x)| \le C\varepsilon^{\beta-1}\, r_j^{1-p-\beta} \exp(-\alpha r_j/\varepsilon) \qquad p \in \mathbb{N}_0$$

$(B_j =$ *neighborhood of* A_j, $r_j = \operatorname{dist}(x, A_j))$

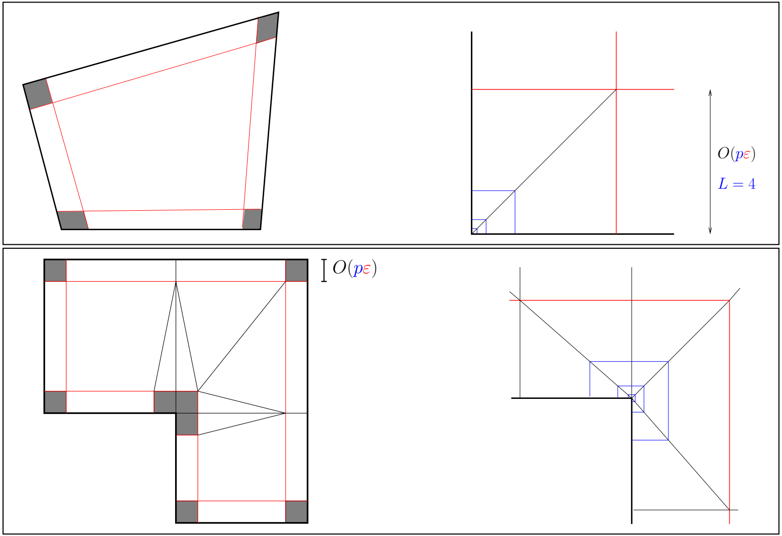
Regularity of the decomposition

Theorem (Melenk '02)

There are C, γ, $\alpha > 0$, $\beta \in [0,1)$ *s.t. for each* $\varepsilon \in (0,1]$ *there is a decomposition of the form* $u_\varepsilon = w_\varepsilon + \chi\, u_\varepsilon^{BL} + \widehat{\chi}\, u_\varepsilon^{CL} + r_\varepsilon$ *with*

1. w_ε *is analytic on* $\overline{\Omega}$ *and* $\|\nabla^p w_\varepsilon\|_{L^\infty(\Omega)} \le C\gamma^p p!, \qquad \forall p \in \mathbb{N}_0.$
2. *remainder* $r_\varepsilon \in H_0^1(\Omega)$ *and* $\|r_\varepsilon\|_\varepsilon \le C \exp(-\alpha/\varepsilon)$.
3. *On* Ω_j *the boundary layer function* u_ε^{BL} *satisfies*
$$|\partial_{\rho_j}^r \partial_{\theta_j}^s u^{BL}(\rho_j,\theta_j)| \le C \max\{r!, \varepsilon^{-r}\}\gamma^{r+s} s!\, e^{-\alpha\rho_j/\varepsilon}, \quad \rho_j \ge 0.$$
4. *The corner layer* u_ε^{CL} *satisfies on* $(\Omega_j \cap B_j) \cup (\Omega_{j+1} \cap B_j)$
$$|\nabla^p u^{CL}(x)| \le C\varepsilon^{\beta-1}\, r_j^{1-p-\beta} \exp(-\alpha r_j/\varepsilon) \qquad p \in \mathbb{N}_0$$
(B_j = *neighborhood of* A_j*,* $r_j = \operatorname{dist}(x, A_j)$*)*

Geometric meshes in $O(p\varepsilon)$ neighborhoods of vertices



hp-FEM approximation result

Theorem (Melenk '02)

Let $\mathcal{T}(\varepsilon, p)$ be a mesh with

- *needle elements of width $\lambda p \varepsilon$ at the boundary*
- *a geometric mesh with L layers in the $O(\lambda p \varepsilon)$ neighborhood of the vertices.*

Then there exist C, $b > 0$ independent of ε, p, L such that

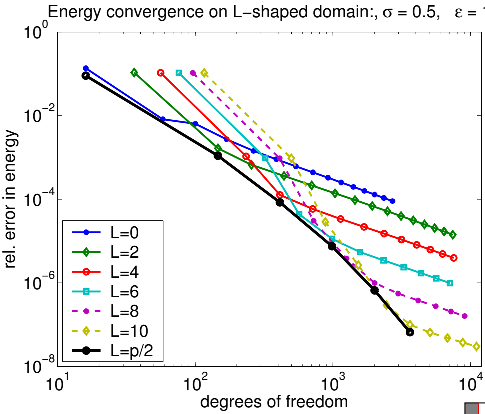
$$\boxed{\|u_\varepsilon - u_N\|_\varepsilon \leq C\left[e^{-bp} + \varepsilon(\lambda p)^2 e^{-bL}\right]}$$

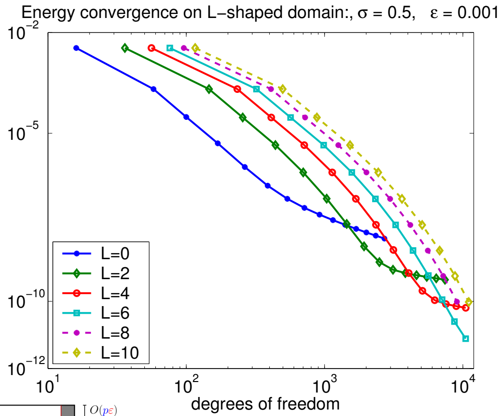
provided that λ is sufficiently small.
For $L \sim p$, we get $\dim S_0^p(\mathcal{T}(\varepsilon, p)) =: N \sim p^3$ and therefore

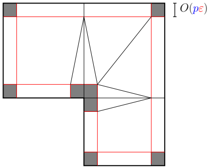
$$\boxed{\|u_\varepsilon - u_N\|_\varepsilon \leq C e^{-b' N^{1/3}}.}$$

Remark: energy norm of corner layer is $O(\varepsilon) \implies$ very few layers suffice

$p = 1, \ldots, 15$, $\varepsilon = 1.0$ and $\varepsilon = 10^{-3}$







geometric meshes instead of the 2-element mesh

- issue: seek robustness (ε may not be known precisely, parameter λ needs to be chosen ...)
- observe: (anisotropic) geometric refinement towards the boundary leads to meshes that resolve boundary layer
- observe: geometric meshes are also effective in resolving (algebraic) singularities

meshes for multi-scale singular perturbation problems

$-\varepsilon^2 \Delta u + u = f,$ ε is in a range: $\varepsilon_{min} \leq \varepsilon \leq 1$

Theorem (Banjai, Melenk, Schwab '20+)

f analytic on $\overline{\Omega}$ and $L \sim p \Longrightarrow$
hp-FEM converges exponentially (in p) in the energy norm uniformly in $\varepsilon_{min} \leq \varepsilon \leq 1$.

meshes for multi-scale singular perturbation problems

$$-\varepsilon^2 \Delta u + u = f, \qquad \varepsilon \text{ is in a range: } \qquad \varepsilon_{min} \leq \varepsilon \leq 1$$

1D mesh $\mathcal{T}_{geo}$ (L layers of refinement)

$\mathcal{T}_{geo}$

scale resolution requirement

L such that $\sigma^L \approx \varepsilon_{min} \rightarrow$ all scales are resolved

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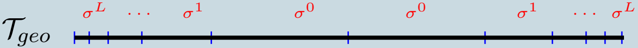
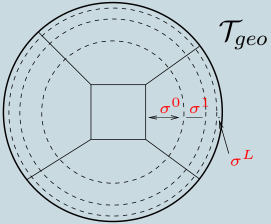


scale resolution requirement

L such that $\sigma^L \approx \varepsilon_{min}$ $\rightarrow$ all scales are resolved

2D analytic $\partial\Omega$



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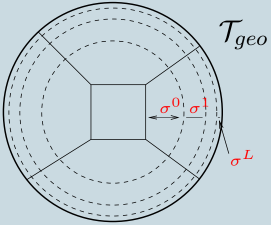

scale resolution requirement

L such that $\sigma^L \approx \varepsilon_{min}$ $\rightarrow$ all scales are resolved

- polygons: geometric refinement towards edges and corners (next slide)

Theorem (Banjai, Melenk, Schwab '20+)

f analytic on $\overline{\Omega}$ and $L \sim p \Longrightarrow$
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meshes for multi-scale singular perturbation problems

$$-\varepsilon^2 \Delta u + u = f, \qquad \varepsilon \text{ is in a range:} \qquad \varepsilon_{min} \leq \varepsilon \leq 1$$

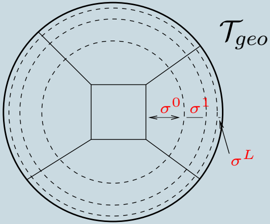
1D mesh $\mathcal{T}_{geo}$ (L layers of refinement)

$\mathcal{T}_{geo}$

scale resolution requirement

L such that $\sigma^L \approx \varepsilon_{min} \to$ all scales are resolved

2D analytic $\partial\Omega$



- polygons: geometric refinement towards edges and corners (next slide)

Theorem (Banjai, Melenk, Schwab '20+)

f analytic on $\overline{\Omega}$ and $L \sim p \Longrightarrow$
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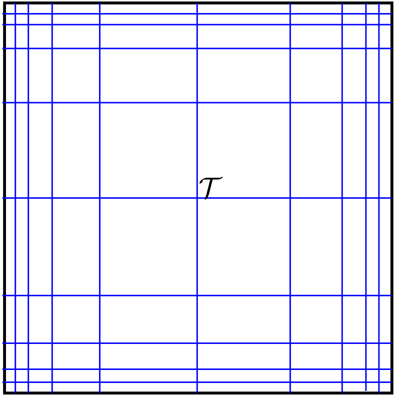
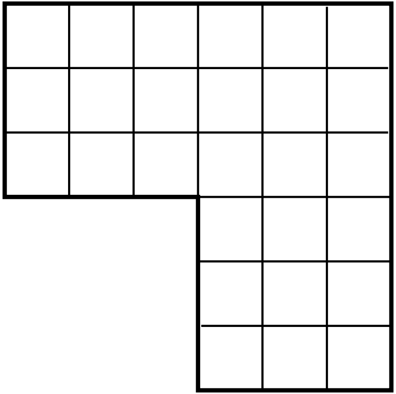
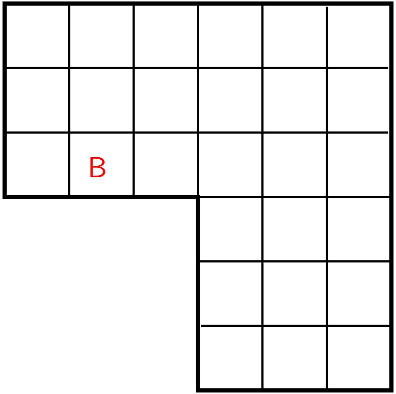
mesh construction via mesh patches

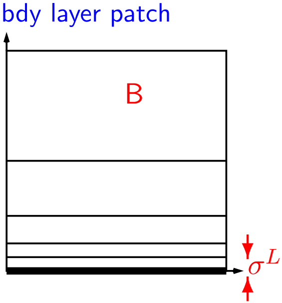


mesh construction via mesh patches



mesh construction via mesh patches





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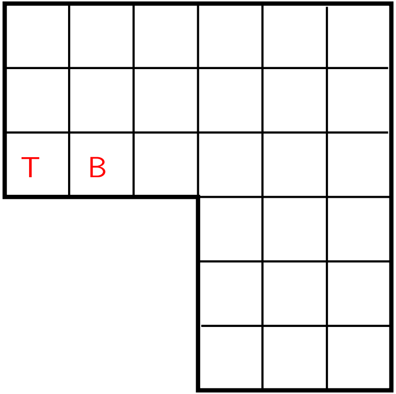
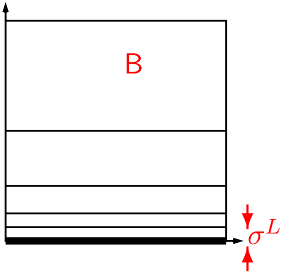
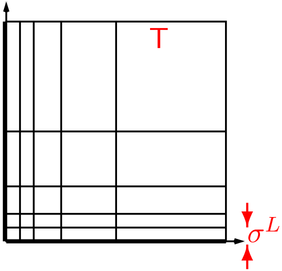


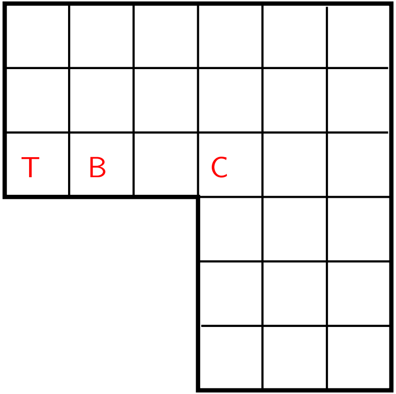
bdy layer patch



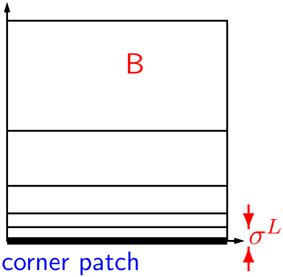
tensor patch



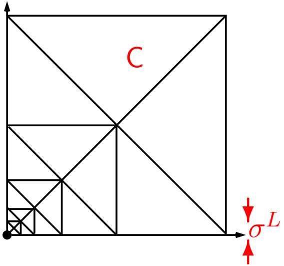
mesh construction via mesh patches



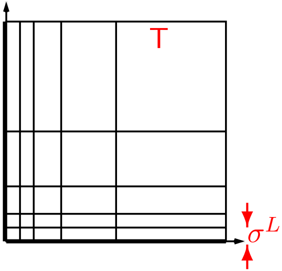
bdy layer patch



corner patch



tensor patch



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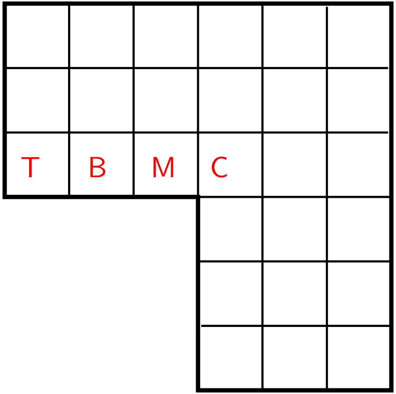
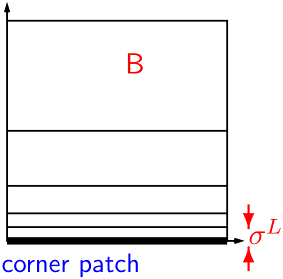
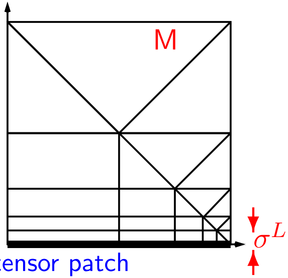


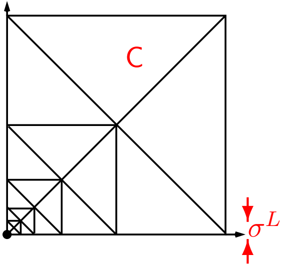
bdy layer patch



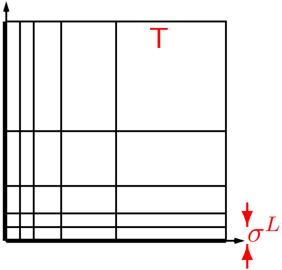
mixed patch



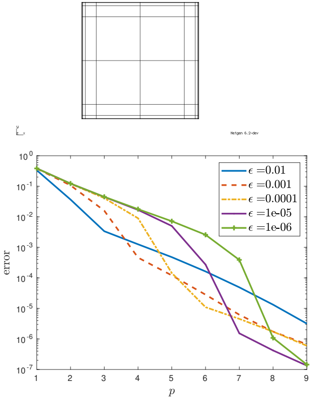
corner patch

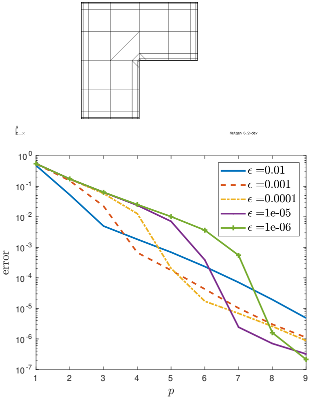


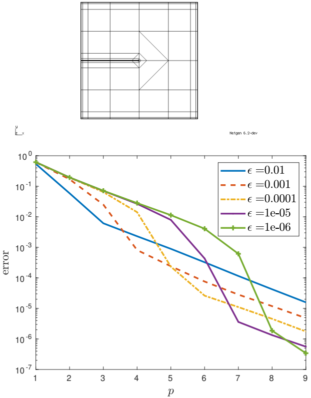
tensor patch



exponential convergence on Netgen-generated geometric meshes

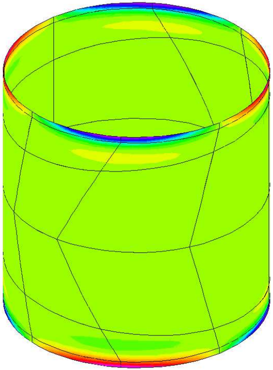


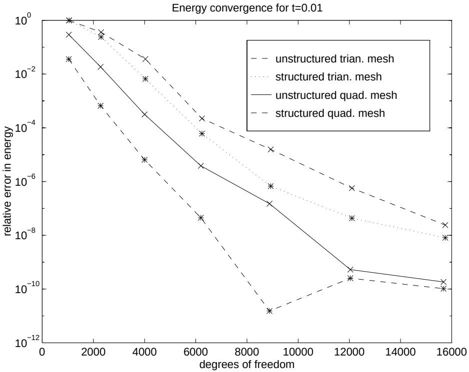




www.ngsolve.org (Schöberl et al.)

Example: Naghdi shell



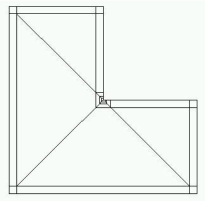


from: Gerdes, Matache, Schwab, ZAMM

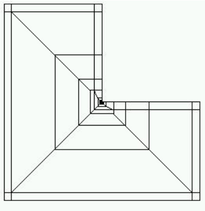
boundary layers on scale $O(t)$ and $O(\sqrt{t})$

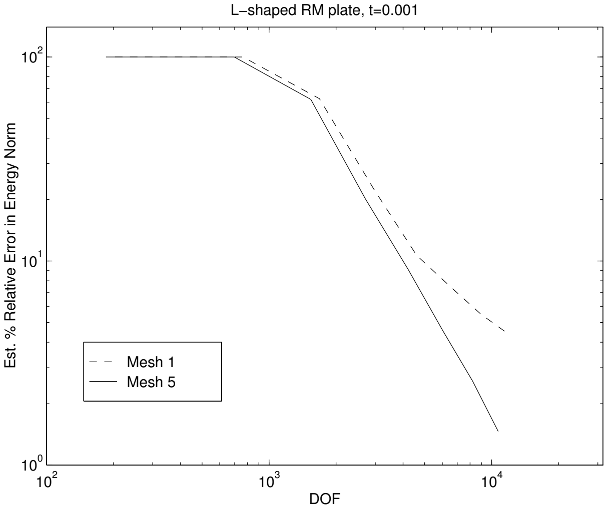
simply supported Reissner-Mindlin plate

mesh 1



mesh 5





taken from: C. Xenophontos, CNME, '98

boundary layer on $O(t)$; non-smooth limit equation $\rightarrow$ geometric mesh

the convection-diffusion problem

convection-diffusion equation

$$L_\varepsilon u := -\varepsilon\Delta u + \mathbf{b}(x)\cdot\nabla u + c(x)u = f \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0 \tag{13}$$

- associated bilinear form: $B_\varepsilon(u,v) := \int_\Omega \varepsilon\nabla u\cdot\nabla v + \mathbf{b}\cdot\nabla u v + cuv, \quad \ell(v) = \int_\Omega fv$

Lemma

Assume that $c - \frac{1}{2}\operatorname{div}\mathbf{b} \geq c_0 > 0$. *Then*

$$\|u\|^2_{\sqrt{\varepsilon}} = \varepsilon|u|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\Omega)} \lesssim B_\varepsilon(u,u) \qquad \forall u \in H^1_0(\Omega)$$

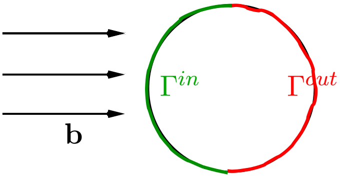
Proof: key observation is that by integration by parts $\int_\Omega \mathbf{b}\cdot\nabla u u = -\frac{1}{2}\int_\Omega \operatorname{div}\mathbf{b}\, u^2$

- $\rightarrow$ Galerkin method possible
- issue: bilinear form B_ε is not continuous in $\|\cdot\|_{\sqrt{\varepsilon}}$ uniformly in ε! ($\rightarrow$ no quasi-optimality!)
- technique: modify (in a consistent way) the bilinear form for stability in a stronger norm

limit problem and layers

limit problem

$$\mathbf{b} \cdot \nabla u + cu = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \Gamma^{in} := \{x \in \partial\Omega \,|\, \mathbf{b} \cdot \mathbf{n} < 0\}$$



- $\rightsquigarrow$ expect layers at outflow boundary
- (also internal layers if $\partial\Omega$ is non-smooth)
- Galerkin method performs quite well if meshes are used that appropriately resolve the layers (careful analysis!)
- problem: Galerkin method fails if layers are not resolved (oscillations everywhere)
- stabilized methods such as SDFEM or GLSFEM have much better stability properties: even if layers are not resolved, the approximation is good away from the (unresolved) layers

Gerdes, Melenk, Schötzau, Schwab, '01

Johnson, Schatz, Wahlbin, '87

SDFEM

$$B_\varepsilon(u,v) = \int_\Omega \varepsilon \nabla u \cdot \nabla v + \mathbf{b} \cdot \nabla u v + cuv, \qquad \ell(v) = \int_\Omega f v$$

$$B_{SD}(u,v) = B_\varepsilon(u,v) + \sum_{K\in\mathcal{T}} \delta_K \int_K (-\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu)\mathbf{b}\cdot\nabla v,$$

$$\ell_{SD}(v) = \ell(v) + \sum_{K\in\mathcal{T}} \delta_K \int_K f\mathbf{b}\cdot\nabla v$$

$$\|u\|_{SD}^2 := \|u\|_{\sqrt{\varepsilon}}^2 + \sum_{K\in\mathcal{T}} \delta_K \|\mathbf{b}\cdot\nabla u\|_{L^2(K)}^2$$

Theorem

Let $c - \frac{1}{2}\operatorname{div}\mathbf{b} \geq c_0$. *Let* $\mathcal{T}$ *be a shape-regular mesh. Then* $\exists\ \delta_0' > 0$ *s.t. for* $\delta_K \leq \delta_0' \min\{1, h_K^2/\varepsilon\}$

$$\|u\|_{SD}^2 \lesssim B_{SD}(u,u) \qquad \forall S_0^{1,1}(\mathcal{T})$$

choice of δ_K:

$$\delta_K := \begin{cases} \delta_0 h_K & \text{if } Pe_K > 1 \text{ (convection dominated)} \\ \delta_1 \frac{h_K^2}{\varepsilon} & \text{if } Pe_K \leq 1 \text{ (diffusion dominated)} \end{cases} \qquad Pe_K := \frac{\|\mathbf{b}\|_{L^\infty(K)} h_K}{\varepsilon} \text{ (= local Péclet number)}$$

GLSFEM

$$B_\varepsilon(u,v) = \int_\Omega \varepsilon \nabla u \cdot \nabla v + \mathbf{b} \cdot \nabla u v + cuv, \qquad \ell(v) = \int_\Omega fv$$

$$B_{GLS}(u,v) = B_\varepsilon(u,v) + \sum_{K \in \mathcal{T}} \delta_K \int_K L_\varepsilon u \, L_\varepsilon v$$

$$\ell_{SD}(v) = \ell(v) + \sum_{K \in \mathcal{T}} \delta_K \int_K f \, L_\varepsilon v$$

$$\|u\|^2_{GLS} := \|u\|^2_{\sqrt{\varepsilon}} + \sum_{K \in \mathcal{T}} \delta_K \|L_\varepsilon u\|^2_{L^2(K)}$$

Theorem

Let $c - \frac{1}{2} \operatorname{div} \mathbf{b} \geq c_0$. *Then:*

$$\|u\|^2_{GLS} \lesssim B_{GLS}(u,u) \qquad \forall S_0^{1,1}(\mathcal{T})$$

DG methods: discretization of limiting transport equation

- instead of working with H^1-conforming elements work with discontinuous piecewise polynomials
- $S^{1,0}(\mathcal{T}) := \{u \in L^2(\Omega) \,|\, u|_K \in \mathcal{P}_1 \quad \forall K \in \mathcal{T}\}$

upwind DG discretization for $\mathrm{b} \cdot \nabla u + cu$

$$B_{DG}^{trans}(u,v) := \sum_{K \in \mathcal{T}} \int_K (\mathbf{b} \cdot \nabla u + cu)v + \int_{\partial K} \mathbf{b} \cdot \mathbf{n}_K (\widehat{u} - u)v$$

with the upwind flux $\mathbf{b} \cdot \mathbf{n}_K \widehat{u}$ given by

$$\widehat{u} := \begin{cases} u|_K & \text{if } \mathbf{b} \cdot \mathbf{n}_K > 0 \\ u|_{K'} & \text{for neighboring element } K' \text{ if } \mathbf{b} \cdot \mathbf{n}_K < 0 \end{cases}$$

Lemma

Let $c - \frac{1}{2} \operatorname{div} \mathbf{b} \geq c_0 > 0$. With the jump $[\![\cdot]\!]$ across an edge $e \in \mathcal{E}$

$$B_{DG}^{trans}(u,u) \gtrsim \|u\|_{L^2(\Omega)}^2 + \sum_{e \in \mathcal{E}} \| \, |\mathbf{b} \cdot \mathbf{n}_K| [\![u]\!] \|_{L^2(e)}^2$$

DG methods: discretization of elliptic part

DG discretization of $-\varepsilon\Delta u$ (SIP variant)

$$B^{ell}_{DG}(u,v) := \sum_{K\in\mathcal{T}} \int_K \varepsilon \nabla u \cdot \nabla v + \sum_{e\in\mathcal{E}} \varepsilon \int_e -[\![u]\!]\{\nabla v\} - \{\nabla u\}[\![v]\!] + \frac{\alpha}{h_e}[\![u]\!][\![v]\!]$$

with $[\![\cdot]\!]$ denoting the jump and $\{\cdot\}$ the average on the edge $e\in\mathcal{E}$

Lemma

For $\alpha > 0$ sufficiently large (independent of ε)

$$B^{ell}_{DG}(u,u) \gtrsim \varepsilon \sum_{K\in\mathcal{T}} \|\nabla u\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}} \varepsilon \frac{\alpha}{h_e} \|[\![u]\!]\|^2_{L^2(e)} \qquad \forall u \in S^{1,0}(\mathcal{T})$$

$$B_{DG}(u,v) := B_{DG}^{trans}(u,v) + B_{DG}^{ell}(u,v)$$

Lemma

Let $c - \frac{1}{2}\operatorname{div}\mathbf{b} \geq c_0 > 0$. *For* $\alpha > 0$ *sufficiently large there holds*

$$B_{DG}(u,v) \gtrsim \varepsilon \sum_{K\in\mathcal{T}} \|\nabla u\|^2_{L^2(K)} + \|u\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}} \varepsilon \frac{\alpha}{h_e} \|[\![u]\!]\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}} \|\,|\mathbf{b}\cdot\mathbf{n}_K|[\![u]\!]\|^2_{L^2(e)}$$

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on singular perturbations

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hp-FEM

- Schwab, p- and hp-FEM, Oxford University Press 1998

work referenced in the lectures:

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