Lecture 2-2: Fano resonance in metallic grating with small holes: embedded eigenvalues and coupled resonators

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Joint work with Junshan Lin (Auburn University)
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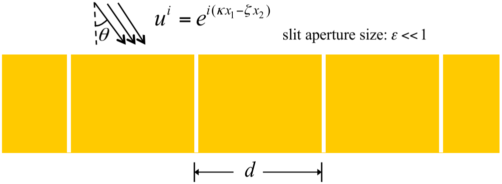
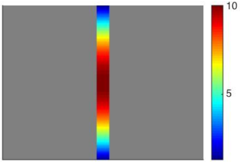
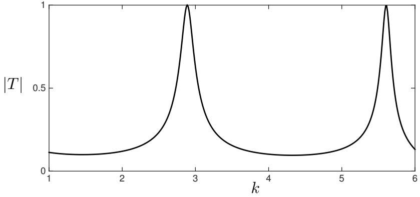
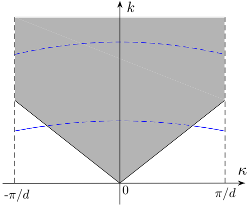
Scattering by a periodic array of narrow slits





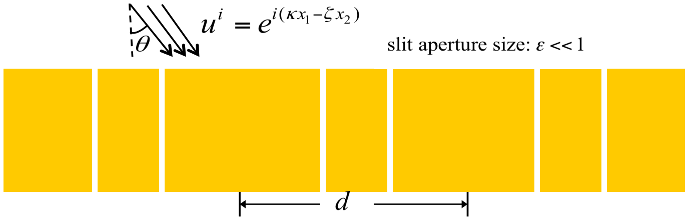


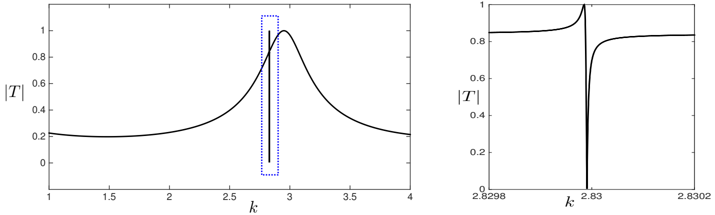


- Asymptotic expansion of real eigenvalues and resonances:

$$k_m(\kappa) = m\pi + 2m\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \gamma(\kappa)\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon), \quad m = 1, 2, 3, \cdots$$

Fano resonance for periodic slit holes





- Fano resonance: asymmetric spectral line shape (sharp transition from peak to dip).
- Applications: efficient optical switching devices, bio-sensing, and photonic devices with high quality factors, etc.

Fano resonance in a nutshell:

- Discovered first by Ettore Majorana in the experiment of scattering of electrons from helium, first theoretical explanation was given by Ugo Fano (1961).

  Basic principle: Interference between a wide-band background and a narrow-band (resonant) scattering process.

- Fano resonance in photonics: extensively explored since the last decade.
  M. Limonov, *et al.*, Nature Photonics (2017)

- Mathematically, Fano resonance can be attributed to two main mechanisms:

  (1) Embedded eigenvalues of the differential operator (bound states in the continuum).
  Existence of embedded eigenvalues: Bonnet-Bendia, Shipman, Volkov, etc.
  Perturbation theory: Shipman, Venakides, Lu, etc.

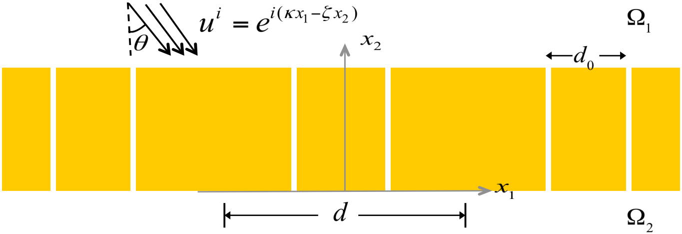
  (2) Coupled resonators with different resonance strength.

Our goal:

- Investigate mathematically the two mechanisms for Fano resonance in the context of subwavelength holes.

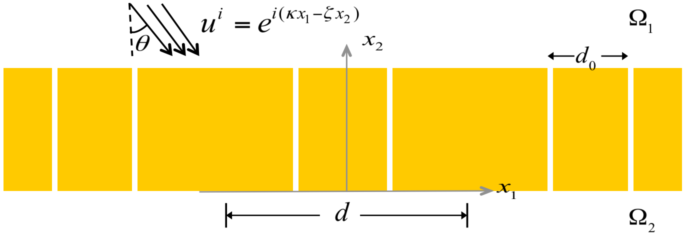
- Show new field amplification behavior at Fano resonance for subwavelength structures.

Setup I: weakly coupled slit holes



- Periodic array of slits, where each period consists of two identical silts $S_\varepsilon^- \cup S_\varepsilon^+$. Each has width ε and length 1.
- The exterior domain: $\Omega_\varepsilon = \Omega_1 \cup \Omega_2 \cup S_\varepsilon$.
- Transverse magnetic polarization: the incident magnetic field $H^i = (0, 0, u^i)$.
- The scattering problem: $\Delta u_\varepsilon + k^2 u_\varepsilon = 0$ in Ω_ε and $\partial_\nu u_\varepsilon = 0$ on $\partial\Omega_\varepsilon$.

Setup I: weakly coupled slit holes



- Look for quasi-periodic solutions such that $u_\varepsilon(x_1 + d, x_2) = e^{i\kappa d} u_\varepsilon(x_1, x_2)$.

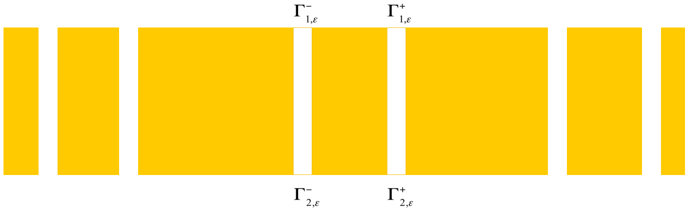
- Outgoing radiation condition: the scattered field

$$u_\varepsilon^s(x_1, x_2) = \sum_{n=-\infty}^{\infty} u_{n,j}^s \cdot e^{i\kappa_n x_1 \pm i\zeta_n x_2} \quad \text{in } \Omega_j \ (j = 1, 2),$$

where

$$\kappa_n = \kappa + \frac{2\pi n}{d} \quad \text{and} \quad \zeta_n(k) = \begin{cases} \sqrt{k^2 - \kappa_n^2}, & |\kappa_n| \le k, \\ i\sqrt{\kappa_n^2 - k^2}, & |\kappa_n| > k. \end{cases}$$

Boundary integral equations



Integral equation formulation over the slit apertures $\Gamma_1^\pm$ and $\Gamma_2^\pm$:

$$
\left\{
\begin{aligned}
&\int_{\Gamma^+_{1,\varepsilon}\cup\Gamma^-_{1,\varepsilon}} g^e(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y + \int_{\Gamma^-_{1,\varepsilon}} g^{i,-}_\varepsilon(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y - \int_{\Gamma^-_{2,\varepsilon}} g^{i,-}_\varepsilon(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y + u^i + u^r = 0, \quad \text{on } \Gamma^-_{1,\varepsilon},\\
&\int_{\Gamma^+_{1,\varepsilon}\cup\Gamma^-_{1,\varepsilon}} g^e(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y + \int_{\Gamma^+_{1,\varepsilon}} g^{i,+}_\varepsilon(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y - \int_{\Gamma^+_{2,\varepsilon}} g^{i,+}_\varepsilon(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y + u^i + u^r = 0, \quad \text{on } \Gamma^+_{1,\varepsilon},\\
&\int_{\Gamma^+_{2,\varepsilon}\cup\Gamma^-_{2,\varepsilon}} g^e(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y - \int_{\Gamma^-_{1,\varepsilon}} g^{i,-}_\varepsilon(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y + \int_{\Gamma^-_{2,\varepsilon}} g^{i,-}_\varepsilon(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y = 0, \quad \text{on } \Gamma^-_{2,\varepsilon},\\
&\int_{\Gamma^+_{2,\varepsilon}\cup\Gamma^-_{2,\varepsilon}} g^e(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y - \int_{\Gamma^+_{1,\varepsilon}} g^{i,+}_\varepsilon(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y + \int_{\Gamma^+_{2,\varepsilon}} g^{i,+}_\varepsilon(x,y)\frac{\partial u_\varepsilon(y)}{\partial y_2}ds_y = 0, \quad \text{on } \Gamma^+_{2,\varepsilon}.
\end{aligned}
\right.
$$

Homogeneous problem and the condition for singular frequencies

- Boundary integral equations in the scaled domain ($x_1 = \varepsilon X,\ y_1 = \varepsilon Y$):

$$\mathbb{T}(k;\kappa,\varepsilon)\,\varphi = \begin{bmatrix} T^e + T^i & T^{e,-} & \tilde{T}^i & 0 \\ T^{e,+} & T^e + T^i & 0 & \tilde{T}^i \\ \tilde{T}^i & 0 & T^e + T^i & T^{e,-} \\ 0 & \tilde{T}^i & T^{e,+} & T^e + T^i \end{bmatrix} \begin{bmatrix} \varphi_1^- \\ \varphi_1^+ \\ \varphi_2^- \\ \varphi_2^+ \end{bmatrix} = \begin{bmatrix} 2f^- \\ 2f^+ \\ 0 \\ 0 \end{bmatrix},$$

where T^e, $T^{e,\pm}$, T^i, and $\tilde{T}^i$ are integral operators with Green functions kernels, $\varphi_1^\pm$ and $\varphi_2^\pm$ are Neumann data.

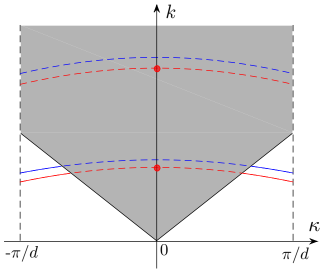
- The set of singular frequencies (eigenvalues and resonances) $\sigma(\mathbb{T})$: we solve for k such that $\mathbb{T}(k;\kappa,\varepsilon)\,\varphi = 0$ attains non-trivial solutions.

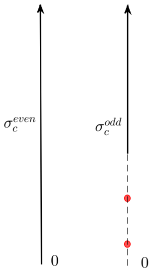
- Mathematical tools:
  - Gohberg-Sigal theory: reduce the solution of singular frequencies to the roots of nonlinear equations.
  - Asymptotic analysis and complex analysis.

Condition for singular frequencies

$\sigma(\mathbb{T})$ are the roots of $\lambda_{j,\pm}(k;\kappa,\varepsilon) = 0$ $(j = 1,2)$, where $\lambda_{j,\pm}$ are eigenvalues of certain 2×2 matrices $\mathbb{M}_\pm(k;\kappa,\varepsilon)$.

Asymptotic expansion of singular frequencies: $\kappa = 0$





Theorem ($\kappa = 0$)

If $\kappa = 0$, the singular frequencies of the scattering problem attain the expansions

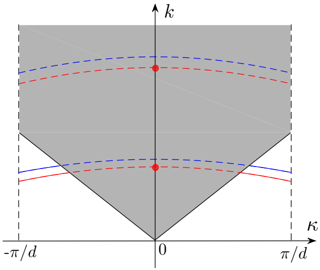
$$k_m^{(1)} = m\pi + 2m\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \gamma + \hat{\beta}\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon);$$

$$k_m^{(2)} = m\pi + 2m\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \gamma - \hat{\beta}\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon)$$

for $m < 2/d$. In the above, $\text{Im}\, k_m^{(1)} = O(\varepsilon)$ and $\text{Im}\, k_m^{(2)} = 0$.

Remark: $k_m^{(2)}$ is an embedded eigenvalue, and the eigenmode is odd w.r.t. x_1.

Asymptotic expansion of singular frequencies: $\kappa \neq 0$



Theorem ($\kappa \neq 0$)

If $|\kappa| \ll 1$, the singular frequencies of the scattering problem attain the expansions

$$k_m^{(1)} = m\pi + 2m\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \gamma + \frac{1}{2}(\beta^- + \beta^+)\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon);$$

$$k_m^{(2)} = m\pi + 2m\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \gamma - \frac{1}{2}(\beta^- + \beta^+)\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon)$$

for $m < 2/d$. In the above, Im $k_m^{(1)} = O(\varepsilon)$ and Im $k_m^{(2)} = O(\kappa^2\varepsilon)$.

Fano resonance I

Lemma (Reflected and transmitted wave)

If $0 < k < 2\pi/d$, the reflected and transmitted wave adopts the expansion

$$u_\varepsilon^r(x) = R(k,\kappa,\varepsilon) \cdot e^{i\kappa x_1 + i\zeta_0(x_2-1)} \quad \text{and} \quad u_\varepsilon^t(x) = T(k,\kappa,\varepsilon) \cdot e^{i\kappa x_1 - i\zeta_0 x_2},$$

where

$$R(k,\kappa,\varepsilon) = 1 - \frac{i\varepsilon\big(\alpha + O(\varepsilon+\kappa^2)\big)}{2d\,\zeta_0\,(1+\eta)} \cdot \left[-\mu_+^2\left(\frac{1}{\lambda_{1,+}} + \frac{1}{\lambda_{1,-}}\right) + \mu_-^2\left(\frac{1}{\lambda_{2,+}} + \frac{1}{\lambda_{2,-}}\right)\right],$$

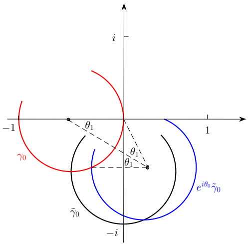
$$T(k,\kappa,\varepsilon) = -\frac{i\varepsilon\big(\alpha + O(\varepsilon+\kappa^2)\big)}{2d\,\zeta_0\,(1+\eta)} \cdot \left[-\mu_+^2\left(\frac{1}{\lambda_{1,+}} - \frac{1}{\lambda_{1,-}}\right) + \mu_-^2\left(\frac{1}{\lambda_{2,+}} - \frac{1}{\lambda_{2,-}}\right)\right],$$

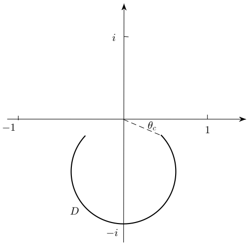
$$\eta = O(\kappa), \quad \mu^+ = 1 + O(\kappa), \quad \mu^- = O(\kappa^2).$$

Fano resonance II

Theorem (Fano resonance)

Set $k_* = \text{Re } k_m^{(2)}$. There exists $k_1, k_2 \in [k_* - c\kappa^2\varepsilon, k_* + c\kappa^2\varepsilon]$ such that $|T(k_1)| \lesssim \varepsilon$ and $|T(k_2)| \gtrsim 1 - \varepsilon$.





Sketch of proof. It can be calculated that $R = 1 + T + O(\varepsilon)$, then from the conservation of energy $|R|^2 + |T|^2 = 1$, we deduce

$$|T(k) + 1|^2 + |T(k)|^2 = 1 + O(\varepsilon). \tag{1}$$

On the other hand, it can be shown that

$$T(k) = t_1(k_*) + \frac{e^{i\theta_0}}{c_1 s + i c_2} + O(\varepsilon), \quad \text{where } k - k_* = s \cdot \kappa^2 \varepsilon. \tag{2}$$

$$\theta_1 = \arctan \frac{|\Im t_1(k_*)|}{|\Re t_1(k_*)|}$$

Field amplification at Fano resonant frequency

Field enhancement

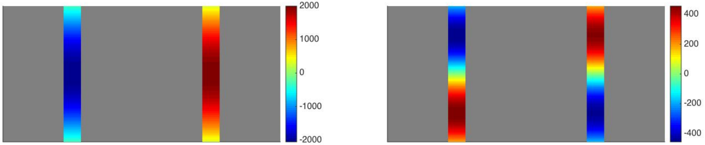
The wave field attains the following asymptotic expansion

$$u_\varepsilon(x) = \left[\pm\frac{c_{odd}}{\kappa\varepsilon} + O\left(\frac{1}{\varepsilon}\right)\right]\cdot\cos(k(x_2 - 1/2)) + O(1)$$

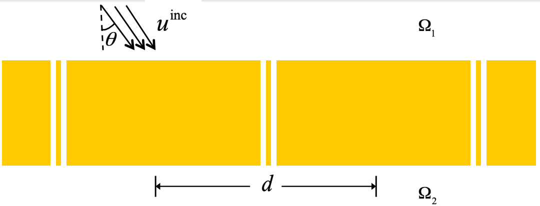
$$u_\varepsilon(x) = \left[\pm\frac{c_{even}}{\kappa\varepsilon} + O\left(\frac{1}{\varepsilon}\right)\right]\cdot\sin(k(x_2 - 1/2)) + O(1)$$

at the Fano resonant frequency $k = \Re k_m^{(2)}$ where m is odd and even respectively.

The wave field at the first two Fano resonance frequencies: $d = 1$, $\varepsilon = 0.05$, $\kappa = 0.1$.



Setup II: strongly coupled slit holes



- The distance between two slit holes in each period is $O(\varepsilon)$.
- Boundary integral equation formulation:

$$\begin{bmatrix} T^e+T^i & T^{e,-} & \tilde{T}^i & 0 \\ T^{e,+} & T^e+T^i & 0 & \tilde{T}^i \\ \tilde{T}^i & 0 & T^e+T^i & T^{e,-} \\ 0 & \tilde{T}^i & T^{e,+} & T^e+T^i \end{bmatrix} \begin{bmatrix} \varphi_1^- \\ \varphi_1^+ \\ \varphi_2^- \\ \varphi_2^+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

- Decomposition of the integral operators:

$$T^{\mathrm{e}}+T^{\mathrm{i}} = (\beta_{\mathrm{e}}+\beta_{\mathrm{i}})P + S + S^{\infty}, \quad T^{\mathrm{e},\pm} = \beta_{\mathrm{e}}P + S^{\pm} + S^{\infty,\pm}, \quad \tilde{T}^{\mathrm{i}} = \tilde{\beta}P + \tilde{S}^{\infty},$$

where $P\varphi = \langle \varphi, 1\rangle 1$, S and $S^{\pm}$ are integral operators with logarithm kernels. β_e may be complex-valued, but β_{i} and $\tilde{\beta}$ are real.

- The leading terms in the resonant conditions $\lambda_{1,\pm}(k)=0$ and $\lambda_{2,\pm}(k)=0$ are

$$\lambda_{1,\pm}(k) \approx (\beta_{\mathrm{e}}+\beta_{\mathrm{i}} \pm \tilde{\beta} + \beta_{\mathrm{e}})(\alpha+\tilde{\alpha}), \quad \lambda_{2,\pm}(k) \approx (\beta_{\mathrm{e}}+\beta_{\mathrm{i}} \pm \tilde{\beta} - \beta_{\mathrm{e}})(\alpha-\tilde{\alpha}).$$

Asymptotic expansion of singular frequencies

Theorem

The singular frequencies of the scattering problem attain the expansions

$$k_m^{(1)} = m\pi + 2m\pi\left[\frac{2}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha+\tilde{\alpha}} + \gamma(\kappa, m\pi)\right)\varepsilon\right] + k_{m,h}^{(1)},$$

$$k_m^{(2)} = m\pi + 2m\pi\left(\frac{1}{\alpha-\tilde{\alpha}} + \frac{2\ln 2}{\pi}\right)\varepsilon + k_{m,h}^{(2)}$$

for $m\varepsilon \ll 1$.

Asymptotic expansion of singular frequencies

Theorem

The singular frequencies of the scattering problem attain the expansions

$$\begin{aligned} k_m^{(1)} &= m\pi + 2m\pi\left[\frac{2}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha+\tilde{\alpha}} + \gamma(\kappa, m\pi)\right)\varepsilon\right] + k_{m,h}^{(1)}, \\ k_m^{(2)} &= m\pi + 2m\pi\left(\frac{1}{\alpha-\tilde{\alpha}} + \frac{2\ln 2}{\pi}\right)\varepsilon + k_{m,h}^{(2)} \end{aligned}$$

for $m\varepsilon \ll 1$.

Theorem

The imaginary parts of $k_m^{(1)}$ and $k_m^{(2)}$ attain the following orders:

$$\Im\, k_m^{(1)} = O(\varepsilon), \quad \Im\, k_m^{(2)} = \begin{cases} O(\kappa\varepsilon^2) & \text{if } (\kappa, m\pi) \text{ in the first radiation continuum;} \\ O(\varepsilon^2) & \text{if } (\kappa, m\pi) \text{ above the first radiation continuum.} \end{cases}$$

Fano resonance

- The transmitted field

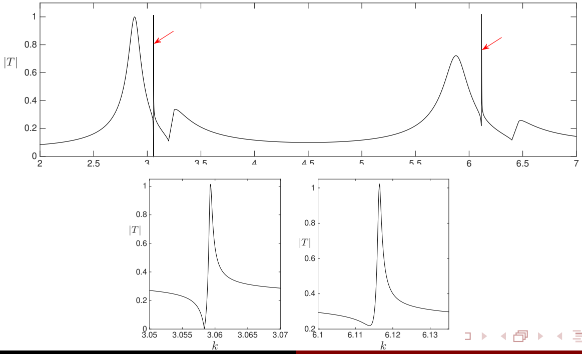
$$u_\varepsilon(x) = \varepsilon \hat{t}_- \cdot \Big(g_2\big(x,(-\varepsilon,0)\big) + O(\varepsilon) \Big) + \varepsilon \hat{t}_+ \cdot \Big(g_2\big(x,(\varepsilon,0)\big) + O(\varepsilon) \Big),$$

where the transmission coefficients

$$\hat{t}_-(\kappa,k,\varepsilon) \quad := \quad = -\frac{1}{\lambda_{1,+}}\Big(\eta_+ + O(\delta)\Big) + \frac{1}{\lambda_{1,-}}\Big(\eta_- + O(\delta)\Big) + \frac{w_{1,+}}{\lambda_{2,+}} - \frac{w_{1,-}}{\lambda_{2,-}},$$

$$\hat{t}_+(\kappa,k,\varepsilon) \quad := \quad -\frac{1}{\lambda_{1,+}}\Big(\eta_+ + O(\delta)\Big) + \frac{1}{\lambda_{1,-}}\Big(\eta_- + O(\delta)\Big) + \frac{w_{2,+}}{\lambda_{2,+}} - \frac{w_{2,-}}{\lambda_{2,-}}.$$

Top: Transmission $|T|$ for $k \in (2,7)$ when $d = 1.3$, $\varepsilon = 0.02$. The incident angle $\theta = \pi/6$.



Summary

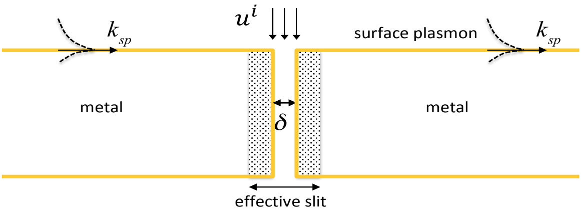
Investigate mathematically two mechanisms for generating the Fano resonance in a periodic array of small holes:

- Provide quantitative analysis of the embedded eigenvalues/resonances for the homogeneous scattering problem.
- Present a rigorous proof of Fano transmission line.
- Characterize the corresponding wave field enhancement at Fano resonance.

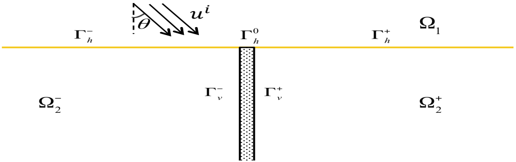
References

1. Fano resonance for a periodic array of perfectly conducting narrow slits, with Junshan Lin and Stephen P Shipman, SIAM Journal on Applied Mathematics, 2020.
2. Fano resonance in metallic grating via strongly coupled subwavelength resonators, Junshan Lin and Hai Zhang, European Journal of Applied Mathematics, 2021.

Outlook: Field enhancement for a single slit in a real metallic slab



1. Multiscale problem: size of slit aperture δ, skin depth of metal δ_m, thickness of slab d, and wavelength λ;
2. The skin depth effect weakens the Fabry-Perot resonance, and induces small shifts of the FP resonance;
3. The slit structure can excite plasmonic surface waves (plasmonic resonance) along the metal interface;
4. The plasmonic resonance can interact with the FP resonance, an vice visa.



$$\begin{cases} \nabla \cdot \left(\frac{1}{\varepsilon(x)} \nabla u \right) + k^2 u = 0 & \text{in } \Omega_1 \cup \Omega_2^+ \cup \Omega_2^-, \\ [u] = 0, \left[\frac{1}{\varepsilon} \frac{\partial u}{\partial \nu} \right] = 0 & \text{on } \Gamma_\delta^- \cup \Gamma_\delta^+, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\delta^0 \cup \Gamma_v^- \cup \Gamma_v^+, \end{cases}$$

where

$$\varepsilon(x) = \begin{cases} 1 & x \in \Omega_1, \\ \varepsilon_m & x \in \Omega_2, \end{cases} \qquad \varepsilon_m(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}$$

The even incident case

Let $\varepsilon_m = \varepsilon_m' + i\,\varepsilon_m''$ with the following scaling:

$$|\varepsilon_m'| = O(\delta^\alpha), \quad |\varepsilon_m''| = O(\delta^\beta).$$

Theorem

Consider the even case. Assume that $-4 \le \alpha < 0$ and $\beta < 2$. The solution admits the decomposition $\varphi = \varphi_0 + \varphi_1$, where

$$\varphi_0 = \varphi_{00} \cdot \chi_{(\delta,\infty)}$$

In addition,

$$\left\| \frac{\chi_\Delta(\xi)}{\rho_0(\xi)} \widehat{E\varphi_1}(\xi) \right\|_{L^1(\mathbb{R})} \lesssim \delta(1+\delta^{-\beta/2}), \quad \left\| \frac{1-\chi_\Delta(\xi)}{\sqrt{1+|\xi|}} \widehat{E\varphi_1}(\xi) \right\|_{L^2(\mathbb{R})} \lesssim \delta^{1-\alpha}\sqrt{\ln\frac{1}{\delta}}.$$

Remark: similar results for the odd case.

Reference: Mathematical analysis of surface plasmon resonance by a nano-gap in the plasmonic metal, Junshan Lin and Z, SIAM Math. Anal, 2019.

Outlook

1. Part II: The interaction between plasmonic resonance and FP resonance;
2. Part III: The mechanism of EOT and LFE;
3. Future work: 3D subwavelength structures: