Lecture 2-1: Anomalous Scattering by Periodic Subwavelength Slit Structures in the Homogenization Regiem

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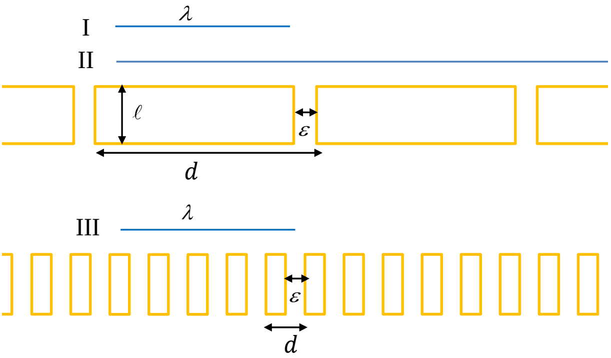
Three Configurations of Periodic Slits



- Normalization: $\ell = 1$.
- Three configurations of periodic slits:

(I) $\varepsilon \ll d \sim \lambda \sim O(1)$: diffraction regime.

(II) $\varepsilon \ll d \ll \lambda$: homogenization regime I

(III) $\varepsilon \sim d \ll \lambda \sim O(1)$: homogenization regime II

Summary of Results in Homogenization Regimes

- In the homogenization regime I, there exists no complex resonance or real eigenvalue. We show that although no enhancement is gained for the magnetic field, strong electric field is induced in the slits and on the slit apertures. We show that as the period d decreases and the coupling between the slits is stronger, the field enhancement becomes weaker.
- In the homogenization regime II, we demonstrate the existence of surface plasmon waves mimicking that of plasmonic metals. We show that total transmission through the structure can be achieved either at certain frequencies for all incident angles or for all frequencies at a specific incident angle.

Lemma

Assume that $\kappa \in (-\pi/d, \pi/d]$, and $\kappa \sim O(1)$ satisfying $|\kappa/\sqrt{k^2-\kappa^2}| \leq C$. Then the kernel $G_\varepsilon^e(X,Y)$ attains the following asymptotic expansion:

$$G_\varepsilon^e(X,Y) = \beta^e(k,\kappa,d,\varepsilon) + \rho^e(X,Y;k,\kappa) + r_\varepsilon^e(X,Y;k,\kappa), \tag{1}$$

where $r_\varepsilon^e(X,Y)$ is a bounded function with $r_\varepsilon^e \sim O(r(\varepsilon))$, and $r(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(1) In the homogenization regime I,

$$\beta^e(k,\kappa,d,\varepsilon) = \frac{1}{\pi}\left(\ln\varepsilon + \ln 2 + \ln\frac{\pi}{d}\right) + \left(\frac{1}{2\pi}\sum_{n\neq 0}\frac{1}{|n|} - \frac{i}{d}\sum_{n=-\infty}^{\infty}\frac{1}{\zeta_n(k)}\right), \tag{2}$$

and

$$\rho^e(X,Y;k,\kappa) = \frac{1}{\pi}\ln(|X-Y|). \tag{3}$$

In addition, $r(\varepsilon) = \varepsilon$ if $\kappa \neq 0$ and $r(\varepsilon) = \varepsilon^2\ln\varepsilon$ if $\kappa = 0$.

(2) In the homogenization regime II,

$$\beta^e(k,\kappa,d,\varepsilon) = \frac{1}{\pi}\ln 2 - \frac{i\eta}{\sqrt{k^2-\kappa^2}\,\varepsilon}, \tag{4}$$

$$\rho^e(X,Y;k,\kappa) = \frac{1}{\pi}\ln|\sin(\pi\eta(X-Y))| + \frac{\kappa\eta}{\sqrt{k^2-\kappa^2}}(X-Y), \tag{5}$$

where $\eta = \varepsilon/d$. In addition, $r(\varepsilon) = \varepsilon$ if $\kappa \neq 0$ and $r(\varepsilon) = \varepsilon^2$ if $\kappa = 0$.

Let us define

$$\gamma(k,\kappa,d) = \frac{1}{\pi}\left(3\ln 2 + \ln\frac{\pi}{d}\right) + \underline{\left(\frac{1}{2\pi}\sum_{n\neq 0}\frac{1}{|n|} - \frac{i}{d}\sum_{n=-\infty}^{\infty}\frac{1}{\zeta_n(k)}\right)}, \quad (6)$$

and

$$p(k;\kappa,d,\varepsilon) = \underline{\varepsilon + \left[\frac{\cot k}{k} + \frac{1}{k\sin k} + \varepsilon\gamma(k,\kappa,d)\right.} + \left.\frac{1}{\pi}\varepsilon\ln\varepsilon\right](\alpha + s(\varepsilon)), \quad (7)$$

$$q(k;\kappa,d,\varepsilon) = \varepsilon + \left[\frac{\cot k}{k} - \frac{1}{k\sin k} + \varepsilon\gamma(k,\kappa,d) + \frac{1}{\pi}\varepsilon\ln\varepsilon\right](\alpha + t(\varepsilon)), \quad (8)$$

where $s(\varepsilon), t(\varepsilon) \sim O(r(\varepsilon))$.

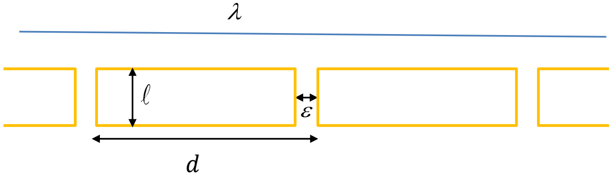
Lemma

In the slit region $S_\varepsilon^{(0),int}$, we have $u_\varepsilon(x_1,x_2) = u_0(x_2) + u_\infty(x_1,x_2)$, where

$$u_0(x_2) = \left[\alpha + O(r(\varepsilon))\right]\left[\frac{\cos(kx_2)}{k\sin k}\left(\frac{1}{\underline{p}} + \frac{1}{q}\right) + \frac{\cos(k(1-x_2))}{k\sin k}\left(\frac{1}{p} - \frac{1}{q}\right)\right], \quad (9)$$

and $u_\infty \sim O\left(e^{-1/\varepsilon}\right)$.

Homogenization Regime I. $\varepsilon \ll d \ll \lambda$



- No scattering resonance or eigenvalue exists if $k \ll 1$ (or $\lambda \gg 1$).

- If $\varepsilon \ll 1$ and $k = \varepsilon^{\sigma}$, in the reference slit,

$$u_\varepsilon(x) = \begin{cases} 2x_2 + O(\varepsilon^{2\sigma}) + O(\varepsilon^{1-\sigma}) & \text{if } 0 < \sigma < 1, \\ 1 + id \cdot \cos\theta \left(2x_2 - 1\right)\varepsilon^{\sigma-1} + O(\varepsilon^{\sigma+1}) + O(\varepsilon^{2(\sigma-1)}) & \text{if } \sigma > 1, \end{cases}$$

- No magnetic enhancement is gained.

- Electric field is enhanced.

$$E_{\varepsilon,1} = \begin{cases} \dfrac{2i}{\sqrt{\tau_0/\mu_0}} \cdot \dfrac{1}{\varepsilon^{\sigma}} + H.O.T \text{ if } 0 < \sigma < 1, \\ \dfrac{d\cos\theta}{\sqrt{\tau_0/\mu_0}} \cdot \dfrac{1}{\varepsilon} + H.O.T \quad \text{if } \sigma > 1, \end{cases} \quad \text{and} \quad E_{\varepsilon,2} \sim O(e^{-1/\varepsilon}).$$

Field enhancement in the near field for varying period

Lemma

If $\varepsilon \ll 1$, $k = \varepsilon^\sigma$ with $\sigma > 1$, and $d = O(\varepsilon^{1-\sigma-\delta})$ with $0 < \delta < 1$, then

$$u = 2x_2 + O(\varepsilon^\delta) \quad \text{and} \quad E_{\varepsilon,1} = \frac{2i}{\varepsilon^\sigma \sqrt{\tau_0/\mu_0}} + O(\varepsilon^{\delta-\sigma})$$

in the slits.

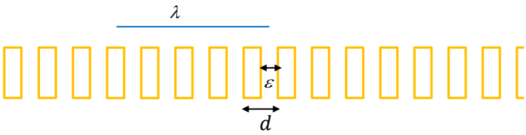
Lemma

If $\varepsilon \ll 1$, $k = \varepsilon^\sigma$, $\sigma > 0$, and $\underline{d = \varepsilon/\eta,}$ $0 < \eta < 1$, then

$$u = 1 + O(\varepsilon^\sigma) \quad \text{and} \quad E_{\varepsilon,1} = O(1)$$

in the slits. Therefore no enhancement is gained for such configuration.

Homogenization Regime II: $\varepsilon \sim d \ll \lambda$

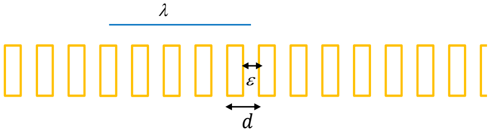


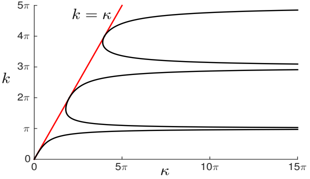
- $\eta := \varepsilon / d$, where $0 < \eta < 1$.

- The dispersion relation can be obtained by solving

$$
\begin{aligned}
p(k;\kappa,d,\varepsilon) &= \varepsilon + \left[\frac{\cot k}{k} + \frac{1}{k\sin k} + \varepsilon\gamma(k,\kappa,d) + \frac{1}{\pi}\varepsilon\ln\varepsilon\right](\alpha + s(\varepsilon)) = 0,\\
q(k;\kappa,d,\varepsilon) &= \varepsilon + \left[\frac{\cot k}{k} - \frac{1}{k\sin k} + \varepsilon\gamma(k,\kappa,d) + \frac{1}{\pi}\varepsilon\ln\varepsilon\right](\alpha + t(\varepsilon)) = 0.
\end{aligned}
$$

Homogenization Regime II: “Surface Spoof Plasmon"

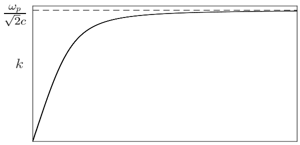




Theorem

There exist two groups of dispersion relations satisfying $|\kappa| > k$, and their leading orders are: $\kappa = k\sqrt{1+\eta^2\left(\dfrac{\sin k}{\cos k \pm 1}\right)^2}, \quad \eta = \varepsilon/d.$

- The associated eigenmodes u_ε^s are surface bound states.
- The dispersion relations and surface bound states resemble the ones for surface plasmon polaritons in the dielectric-metal configuration.



Surface plasmon polariton

We seek surface waves that may propagate along the surface $z=0$ with the field decaying away from $z=0$. W.L.O.G., we choose x-axis as the propagation direction. Write

$$E^{\pm} = (E_x^{\pm}, 0, E_z^{\pm}) e^{-\kappa^{\pm}|z|} e^{i(q\cdot x - \omega t)},$$
$$H^{\pm} = (0, E_y^{\pm}, 0) e^{-\kappa^{\pm}|z|} e^{i(q\cdot x - \omega t)},$$

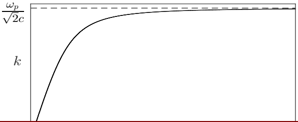
where $|q| > \frac{\omega}{c}$ and $q \in \mathbf{R}$. Plugging the ansatz into Maxwell equations, we obtain the surface plasmon condition

$$q(\omega) = \frac{\omega}{c}\sqrt{\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}}.$$

In the case of a Drude semi-infinite metal in vacuum, we take $\varepsilon_2 = 1$ and

$$\varepsilon_1 = 1 - \frac{\omega_p^2}{\omega(\omega + i0)}$$

Then the dispersion relation $q(\omega)$ is given as below



Lemma

For a given κ, the eigenmode outside the slit adopts the following expansion:

$$u_\varepsilon^s(x) = \frac{\eta}{\sqrt{\kappa^2 - (k_m^+)^2}} e^{i\kappa x_1 - \sqrt{\kappa^2 - (k_m^+)^2}\,|x_2 - 1|} + O(\varepsilon) \quad \text{for } x \in \Omega_1^+ \cap \Omega^{(0)}.$$

Similarly,

$$u_\varepsilon^s(x) = \frac{\eta}{\sqrt{\kappa^2 - (k_m^+)^2}} e^{i\kappa x_1 - \sqrt{\kappa^2 - (k_m^+)^2}\,|x_2|} + O(\varepsilon) \quad \text{for } x \in \Omega_1^- \cap \Omega^{(0)}.$$

Note that the eigenmode is a surface bound-state mode that decays exponentially above and below the slab. The same holds for eigenmode corresponding to $k = k_m^-$.

Lemma

For a given κ, the eigenmode in the slit region $S_\varepsilon^{(0),int} := \{x \in S_\varepsilon^{(0)} \mid x_2 \gg \varepsilon, 1 - x_2 \gg \varepsilon\}$ adopts the following asymptotical expansion:

$$u_\varepsilon(x) = a_0^\pm e^{ikx_2} + b_0^\pm e^{ik(1-x_2)} + O\left(e^{-1/\varepsilon}\right).$$

for the eigenvalue $k = k_m^\pm$.

As $\varepsilon \to 0$, one expects that the scattering by the slab with an array of slits is equivalent to the scattering by a homogenous effective slab. To this end, let us consider an incident wave $u^i = e^{i(\kappa x_1 - \zeta(x_2-1))}$, where $\kappa = k\sin\theta$ and $\zeta = k\cos\theta$, that impinges on the slab. We calculate the total field u_ε in the far-field zone.

Lemma

The total field above the slab has the following expansion

$$u_\varepsilon(x) = u^i(x) + R \cdot e^{i(\kappa x_1 + \zeta(x_2-1))} \quad \text{for } x \in \Omega_1^+ \cap \Omega^{(0)},$$

where the reflection coefficient

$$R = \frac{i\cdot(-\zeta^2 + \eta^2 k^2)\tan k}{-i\cdot(\zeta^2+\eta^2k^2)\tan k + 2\zeta\eta k} \cdot (1 + O(\varepsilon)).$$

The transmitted field below the slab has the expansion

$$u_\varepsilon(x) = T \cdot e^{i(\kappa x_1 - \zeta x_2)} \quad \text{for } x \in \Omega_1^- \cap \Omega^{(0)},$$

where the transmission coefficient

$$T = \frac{2\zeta\eta k}{-i\cdot(\zeta^2+\eta^2k^2)\sin k + 2\zeta\eta k\cos k} \cdot (1 + O(\varepsilon)).$$

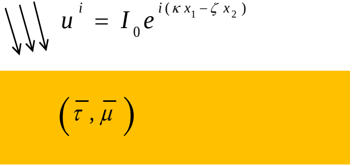
Homogenization and effective medium theory: II

We derive the effective slab as $\varepsilon \to 0$. Denote the relative permittivity and the permeability of the effective medium in the slab by $\bar{\tau}$ and $\bar{\mu}$ respectively, and consider the layered medium as below. The corresponding scattering problem is formulated as

$$\nabla \cdot \left(\frac{1}{\tau} \nabla u \right) + k^2 \mu u = 0, \tag{10}$$

where

$$\tau(x_1, x_2) = \begin{cases} 1, & x_2 > 1 \text{ or } x_2 < 0, \\ \bar{\tau}, & 0 < x_2 < 1. \end{cases} \quad \text{and} \quad \mu(x_1, x_2) = \begin{cases} 1, & x_2 > 1 \text{ or } x_2 < 0, \\ \bar{\mu}, & 0 < x_2 < 1. \end{cases}$$



We look for $\bar{\tau}$ and $\bar{\mu}$ such that the associated far-field u recovers the leading-order term of the far-field u_ε given by the slit structure.

Theorem

Let

$$\bar{\tau} = \begin{bmatrix} \infty & 0 \\ 0 & 1/\eta \end{bmatrix} \quad \text{and} \quad \bar{\mu} = \eta,$$

and let $u^i = e^{i(\kappa x_1 - \zeta(x_2-1))}$, where $\kappa = k\sin\theta$ and $\zeta = k\cos\theta$, be the incident wave. Then the total field has the following form

$$u(x_1,x_2) = \begin{cases} e^{i(\kappa x_1 - \zeta(x_2-1))} + \underline{R} e^{i(\kappa x_1 + \zeta(x_2-1))}, & x_2 > 1, \\ \underline{T} e^{i(\kappa x_1 - \zeta x_2)}, & x_2 < 0. \end{cases}$$

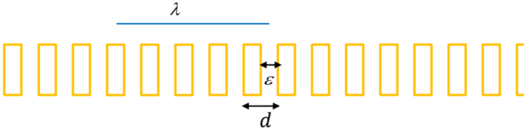
Theorem

If

$$\bar{\tau} = \begin{bmatrix} \infty & 0 \\ 0 & 1/\eta \end{bmatrix} \quad \text{and} \quad \bar{\mu} = \eta,$$

then the dispersion relation for the layered medium have two branches given by

$$\kappa = k\sqrt{1+\eta^2\tan^2(k/2)} \quad \text{and} \quad \kappa = k\sqrt{1+\eta^2\cot^2(k/2)}. \tag{11}$$

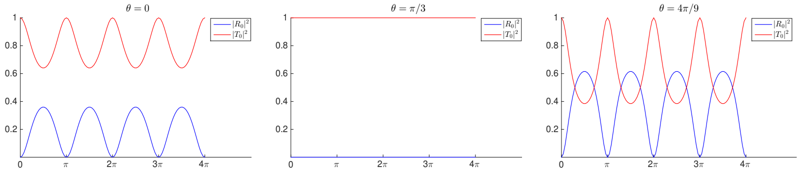


- Scattering by an incident plane wave $u^i = e^{i(\kappa x_1 - \zeta(x_2-1))}$, where $\kappa = k\sin\theta$, $\zeta = k\cos\theta$, and $|\kappa| < k$.

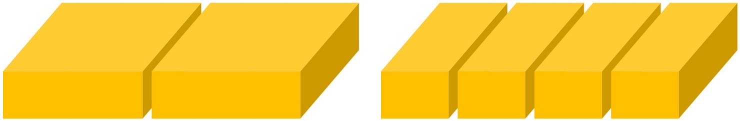
- The leading orders of the reflection and transmission coefficients are

$$R_0 = \frac{i\tan k \cdot (\eta^2 - \cos^2\theta)}{-i\tan k \cdot (\eta^2 + \cos^2\theta)) + 2\eta\cos\theta}, \quad T_0 = \frac{2\cos\theta \cdot \eta}{-i\sin k \cdot (\eta^2 + \cos^2\theta) + 2\cos\theta \cdot \eta\cos k}.$$

- Total transmission is achieved when $k = m\pi$ (Fabry-Perot resonance), and all frequencies for a special incident angle θ such that $\cos\theta = \eta$ (Brewster angle).



A Quick Summary



Field enhancement for PEC metals:

- An array of slits: resonant and non-resonant enhancement effects, surface bound states, “surface spoof plasmon", and total transmission.

Reference

Scattering by a periodic array of subwavelength slits II: surface bound state, total transmission and field enhancement in homogenization regimes, Junshan Lin and Hai Zhang, SIAM Journal on Multiscale Modeling and Simulation, 16(2), 954-990, 2018.