Lecture 1-2: Anomalous Scattering by Periodic Subwavelength Slit Structures in the Diffraction Regime

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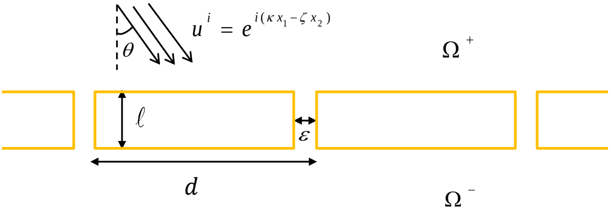


- A periodic array of slits: $S_\varepsilon = \bigcup_{n=-\infty}^{\infty} \left(S_\varepsilon^{(0)} + nd \right)$.
- The scattering problem: $\Delta u_\varepsilon + k^2 u_\varepsilon = 0$ in Ω_ε and $\partial_\nu u_\varepsilon = 0$ on $\partial\Omega_\varepsilon$.
- Look for quasi-periodic solutions such that $u_\varepsilon(x_1 + d, x_2) = e^{i\kappa d} u_\varepsilon(x_1, x_2)$.
- Outgoing radiation condition: the scattered field

$$u_\varepsilon^s(x_1, x_2) = \sum_{n=-\infty}^{\infty} u_n^{s,\pm} e^{i\kappa_n x_1 \pm i\zeta_n x_2} \quad \text{in } \Omega^\pm,$$

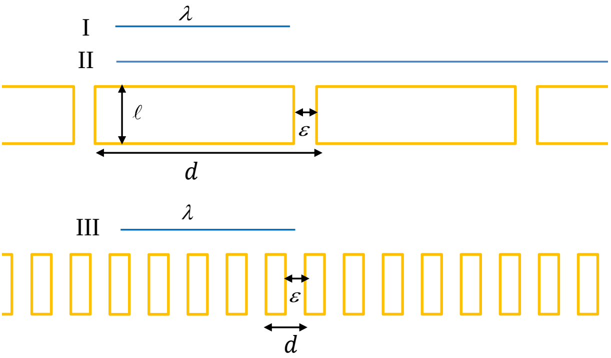
where

$$\kappa_n = \kappa + \frac{2\pi n}{d} \quad \text{and} \quad \zeta_n(k) = \begin{cases} \sqrt{k^2 - \kappa_n^2}, & |\kappa_n| \le k, \\ i\sqrt{\kappa_n^2 - k^2}, & |\kappa_n| > k. \end{cases}$$

New Features of Scattering by Periodic Structures

1. The solution to the scattering problem may not be unique. The homogeneous problem with $u^i = 0$ may attain non-trivial solutions (k, u_ε) for $k \in \mathbf{R}$. Such real-valued k is called a real eigenvalue or singular frequency and u_ε is the corresponding surface bound state that decays exponentially away from the periodic structure;
2. There may exist complex-valued k's such that the homogeneous problem attains nontrivial solutions. Such k's are called resonances of the scattering problem, and the associated nontrivial solutions are called leaky modes (or quasi-normal modes).
3. Near the cut-off frequency where $k = \pm(\kappa + 2\pi n/d)$ for some integer n, the propagating spatial harmonic mode becomes an evanescent mode or vice versa. As such the diffracted field will exhibit anomalous behaviors and this is the so-called Rayleigh anomaly.

Three Configurations of Periodic Slits



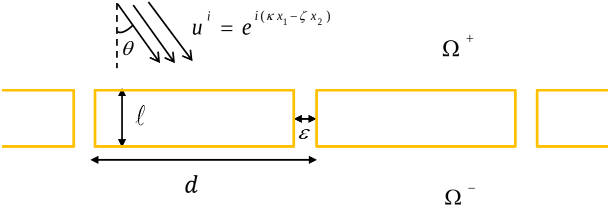
- Normalization: $\ell = 1$.
- Three configurations of periodic slits:

(I) $\varepsilon \ll d \sim \lambda \sim O(1)$: diffraction regime.

(II) $\varepsilon \ll d \ll \lambda$: homogenization regime I

(III) $\varepsilon \sim d \ll \lambda \sim O(1)$: homogenization regime II

Diffraction Regime: $\varepsilon \ll d \sim \lambda$



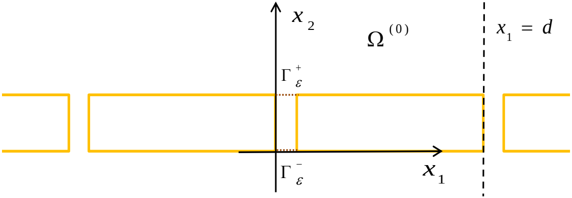
- $\kappa^2 + \eta^2 = k^2$, $\kappa \in (-\pi/d, \pi/d]$.
- Exterior Green's function in $\Omega^{\pm}$: $g_\sharp^e(x,y) = g_\sharp^d(x,y) + g_\sharp^d(x',y)$, where

$$g_\sharp^d(x,y) = -\frac{i}{2d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} e^{i\kappa_n(x_1-y_1)+i\zeta_n(k)|x_2-y_2|},$$

and

$$\kappa_n = \kappa + \frac{2\pi n}{d} \quad \text{and} \quad \zeta_n(k) = \begin{cases} \sqrt{k^2-\kappa_n^2}, & |\kappa_n| \le k, \\ i\sqrt{\kappa_n^2 - k^2}, & |\kappa_n| > k. \end{cases}$$

Diffraction Regime: Integral Equation and Asymptotic Expansion



- Integral equation formulation over one reference period:

$$\begin{cases} \displaystyle\int_{\Gamma_\varepsilon^+} g_\sharp^e(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y + \int_{\Gamma_\varepsilon^+\cup\Gamma_\varepsilon^-} g_\varepsilon^i(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y = -(u^i+u^r), \quad \text{on } \Gamma_\varepsilon^+, \\ \displaystyle\int_{\Gamma_\varepsilon^-} g_\sharp^e(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y + \int_{\Gamma_\varepsilon^+\cup\Gamma_\varepsilon^-} g_\varepsilon^i(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y = 0, \quad \text{on } \Gamma_\varepsilon^-. \end{cases}$$

- Boundary integral equation after scaling:

$$\begin{bmatrix} T_\sharp^e + T^i & \tilde{T}^i \\ \tilde{T}^i & T_\sharp^e + T^i \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} f/\varepsilon \\ 0 \end{bmatrix}.$$

$T_\sharp^e$ is the integral operator with kernel $G_{\sharp,\varepsilon}^e$:

$$G_{\sharp,\varepsilon}^e(X,Y) = \frac{1}{\pi}\left(\ln\varepsilon + \ln 2 + \ln\frac{\pi}{d}\right) + \frac{1}{2\pi}\sum_{n\neq 0}\frac{1}{|n|} - \frac{i}{d}\sum_{n=-\infty}^{\infty}\frac{1}{\zeta_n(k)} + \frac{1}{\pi}\ln|X-Y| + O(\varepsilon|X-Y|).$$

- Asymptotics of integral operators and the resonance condition can be obtained!

- Rayleigh anomaly frequencies: $k = \kappa_n = \kappa + 2\pi n/d$ or $\zeta_n = 0$ for some n.

Note that the scattered field

$$u^s_\varepsilon(x_1,x_2) = \sum_{n=-\infty}^{\infty} u_n^{s,\pm} e^{i\kappa_n x_1 \pm i\zeta_n x_2}, \quad \zeta_n(k) = \begin{cases} \sqrt{k^2-\kappa_n^2}, & |\kappa_n| \le k, \\ i\sqrt{\kappa_n^2-k^2}, & |\kappa_n| > k. \end{cases}$$

- Resonances away from the Rayleigh anomaly frequencies: Let $\delta = O(\varepsilon^{2\tau})$. Consider the domain

$$D_{\kappa,\delta,M} := (\mathbf{C}\backslash B_{\kappa,\delta}) \cap \{z \mid |z| \le M\}, \quad \text{where } B_{\kappa,\delta} := \bigcup_{n=-\infty}^{\infty} B_\delta(\kappa + 2\pi n/d).$$

Remark 3.2 *δ denotes the distance from the Rayleigh cut-off frequencies. When $\tau = 0$ and $\delta = O(1)$, the frequency in $\mathbf{R}^+\backslash B_{\kappa,\delta}$ is then away from the Rayleigh cut-off frequencies. On the other hand, if $0 < \tau < 1$, a frequency in $\mathbf{R}^+\backslash B_{\kappa,\delta}$ can be close to the Rayleigh cut-off frequencies. The assumption $0 \le \tau < 1$ is essential for a uniform asymptotic expansion of the Green function. The treatment for $\tau > 1$ is more complicated.*

Lemma

Let $\kappa = k \sin\theta$, $\delta = O(\varepsilon^{2\tau})$ where $0 \le \tau < 1$. Then

$$\varphi \quad = \quad K^{-1}1 \cdot \left[\kappa \cdot O(1) \cdot \mathbf{e}_1 + \frac{\alpha}{p}(\mathbf{e}_1 + \mathbf{e}_2) + \frac{\alpha}{q}(\mathbf{e}_1 - \mathbf{e}_2)\right] + \left(\frac{\alpha}{p} + \frac{\alpha}{q}\right) \cdot O(k\varepsilon^{1-\tau}) + O(k\varepsilon^{1-\tau}),$$

where

$$p(k;\kappa,d,\varepsilon) = \varepsilon + \left[\frac{\cot k}{k} + \frac{1}{k \sin k} + \varepsilon\gamma(k,\kappa,d) + \frac{1}{\pi}\varepsilon \ln \varepsilon\right] \left(\alpha + O(k\varepsilon^{1-\tau})\right),$$

$$q(k;\kappa,d,\varepsilon) = \varepsilon + \left[\frac{\cot k}{k} - \frac{1}{k \sin k} + \varepsilon\gamma(k,\kappa,d) + \frac{1}{\pi}\varepsilon \ln \varepsilon\right] \left(\alpha + O(k\varepsilon^{1-\tau})\right),$$

$$\gamma(k,\kappa,d) = \frac{1}{\pi}\left(3\ln 2 + \ln\frac{\pi}{d}\right) + \left(\frac{1}{2\pi}\sum_{n\neq 0}\frac{1}{|n|} - \frac{i}{d}\sum_{n=-\infty}^{\infty}\frac{1}{\zeta_n(k)}\right).$$

(i) $p(k;\kappa,d,\varepsilon)=0$ or $q(k;\kappa,d,\varepsilon)=0$ attain complex roots k with negative imaginary part and real part Re $k>|\kappa|$. Such k are called resonances and the corresponding modes are called quasi-modes or leaky modes. If the incident frequency coincides with the resonant frequency, then field enhancement will occur.

(ii) $p(k;\kappa,d,\varepsilon)=0$ or $q(k;\kappa,d,\varepsilon)=0$ attain real roots k with $k<|\kappa|$. Such k are called real eigenvalues of the scattering operator, and the corresponding eigenmodes are called Rayleigh-Bloch surface bound states that are confined near the periodic structure. The surface bound-state modes can couple with nearby sources through near field interaction, but not with a plane incident wave that is considered here.

(iii) $p(k;\kappa,d,\varepsilon)=0$ or $q(k;\kappa,d,\varepsilon)=0$ attain real roots k with $k>|\kappa|$. In such scenario, the periodic slab structures possesses certain finite bound state embedded in the continuum states (or a point spectrum embedded in the continuous spectrum).

(iv) The function $\gamma=\gamma(k,\kappa,d)$ has a branch cut at the triplet (k,κ,d) such that $k=|\kappa+2\pi n/d|$ and $\zeta_n(k)=0$. This corresponds to the Rayleigh anomaly, where the propagating mode $e^{i\kappa_n x_1 \pm i\zeta_n x_2}$ becomes an evanescent mode or vice versa.

Diffraction Regime: Resonances and Eigenvalues

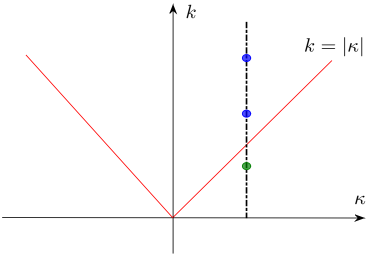
Theorem

For each $\kappa \in (-\pi/d, \pi/d]$, if $m\pi \in D_{\kappa,\delta,M}$, there exists a resonance or an eigenvalue k_m in the neighborhood of $m\pi$.

- If $m\pi > |\kappa|$, k_m is a resonance. Otherwise, k_m is an eigenvalue.
- The following asymptotic expansion holds for k_m if $m\varepsilon \ll 1$:

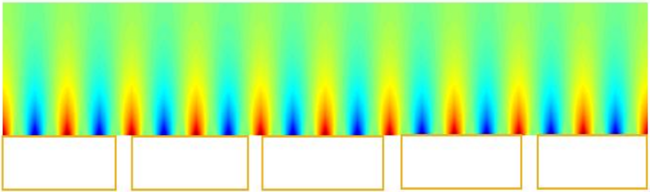
$$k_m = m\pi + 2m\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \gamma(m\pi,\kappa,d)\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon),$$

Here $\alpha = \langle K^{-1}1, 1\rangle$, $\gamma(k,\kappa,d) = \frac{1}{\pi}\left(3\ln 2 + \ln\frac{\pi}{d}\right) + \left(\frac{1}{2\pi}\sum_{n\neq 0}\frac{1}{|n|} - \frac{i}{d}\sum_{n=-\infty}^{\infty}\frac{1}{\zeta_n(k)}\right)$.



- Im $\gamma(m\pi,\kappa,d) = -\frac{1}{d}\sum_{|\kappa_n|<m\pi}\frac{1}{\zeta_n(m\pi)} < 0$ if $m\pi > |\kappa|$, and the resonance has an imaginary part of $O(\varepsilon)$.
- Im $\gamma(m\pi,\kappa,d) = 0$ if $m\pi < |\kappa|$.
- The eigenvalue occurs only if $d < 1$.
- The eigenmode u_ε^s is a surface bound state (decaying exponetial away from the grating surface).

Surface Bound State



Diffraction regime: Enhancement at Resonant Frequencies I

- Recall that

$$\varphi = K^{-1}1 \cdot \left[\kappa \cdot O(1) \cdot \mathbf{e}_1 + \frac{\alpha}{p}(\mathbf{e}_1 + \mathbf{e}_2) + \frac{\alpha}{q}(\mathbf{e}_1 - \mathbf{e}_2) \right] + \text{H.O.T},$$

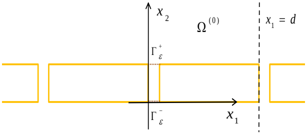
- Away from the resonant frequencies, $\varphi \sim O(1)$.

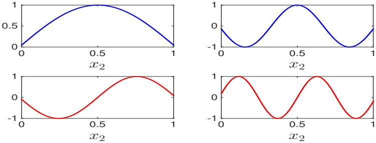
Scattering solution at resonant frequencies

At the odd and even resonant frequencies $k = \text{Re}\, k_{m,1}$ and $k = \text{Re}\, k_{m,2}$,

$$\varphi \sim O(1/\varepsilon).$$

Field Enhancement at Resonant Frequencies II





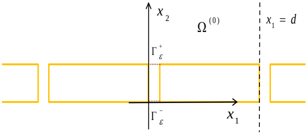
In the slit $S_\varepsilon^{(0)}$

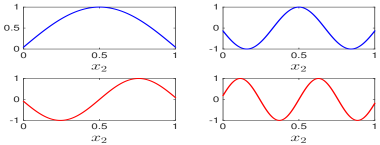
At the odd and even resonances respectively:

$$u_\varepsilon(x) = \frac{1}{\varepsilon} \cdot \frac{i}{\operatorname{Im} \gamma(m\pi, \kappa, d) \cdot k \sin(k/2)} \cdot \cos(k(x_2 - 1/2)) + O(\ln^2 \varepsilon)$$

$$u_\varepsilon(x) = -\frac{1}{\varepsilon} \cdot \frac{i}{\operatorname{Im} \gamma(m\pi, \kappa, d) \cdot k \cos(k/2)} \cdot \sin(k(x_2 - 1/2)) + O(\ln^2 \varepsilon).$$

Field Enhancement at Resonant Frequencies II





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$$u_\varepsilon(x) = -\frac{1}{\varepsilon} \cdot \frac{i}{\operatorname{Im}\gamma(m\pi,\kappa,d)\cdot k\cos(k/2)} \cdot \sin(k(x_2 - 1/2)) + O(\ln^2\varepsilon).$$

Transmitted field

At the odd and even resonances respectively:

$$u_\varepsilon(x) = \frac{i}{\operatorname{Im}\gamma(m\pi,\kappa,d)} \cdot g^e(x,(0,0)) + O(\varepsilon\ln^2\varepsilon)$$

$$u_\varepsilon(x) = -\frac{i}{\operatorname{Im}\gamma(m\pi,\kappa,d)} \cdot g^e(x,(0,0)) + O(\varepsilon\ln^2\varepsilon)$$

Field Enhancement at a Resonant Frequency that is near the Rayleigh cutoff frequency I

We consider a pair (k^0, κ^0) with $k^0 = \kappa^0 + 2\pi n_0/d$ for some integer n_0 (k^0 is the cut-off frequency). For clarity of presentation, let us assume that $\kappa^0 \neq 0$ and $\kappa^0 \neq \pi/d$ so that only one diffracted mode is converted into evanescent mode near (k^0, κ^0). Let $kd = k^0 d + \delta = \kappa^0 d + 2\pi n_0 + \delta$, where $\delta = O(\varepsilon^{2\tau})$ and $0 < \tau < 1$. It is clear that $e^{i\kappa_{n_0} x_1 \pm i\zeta_n x_2}$ is a propagating mode if $\delta > 0$ and an evanescent mode if $\delta < 0$. A direct expansion yields

$$\frac{1}{d \cdot \zeta_{n_0}(k)} = \frac{1}{\sqrt{|\delta|}} e^{-\frac{1}{2} i \arg \delta} \left(\frac{1}{\sqrt{2k_0 d}} + O(\delta) \right).$$

and that

$$\begin{aligned} \gamma(k, \kappa^0, d) &= \frac{1}{\pi}\left(3\ln 2 + \ln \frac{\pi}{d}\right) + \left(\frac{1}{2\pi} \sum_{n \neq 0} \frac{1}{|n|} - \frac{i}{d} \sum_{n \neq n_0} \frac{1}{\zeta_n(k)} \right) - \frac{i}{\sqrt{|\delta|}} e^{-\frac{1}{2} i \arg \delta} \left(\frac{1}{\sqrt{2k_0 d}} + O(\delta) \right) \\ &= \gamma_0 - \frac{i}{\sqrt{|\delta|}} e^{-\frac{1}{2} i \arg \delta} \left(\frac{1}{\sqrt{2k_0 d}} + O(\delta) \right). \end{aligned}$$

Field Enhancement at a Resonant Frequency that is near the Rayleigh cutoff frequency II

Proposition

If the resonant frequency $m\pi$ is $\varepsilon^{2\tau}$-close to a cut-off frequency k^0, then the amplitude of the near-field wave is

$$O(\varepsilon^{\tau-1}) \quad \text{and} \quad O(\varepsilon^{\tau}\ln\varepsilon)$$

inside the silts $S_\varepsilon^{(0),int}$ and on the slit apertures $\Gamma_\varepsilon^{\pm}$, respectively.

Remark: in contrast to the case when the resonant frequency is away from the Rayleigh cut-off frequencies, for which the field is enhanced by an order of $O(\varepsilon^{-1})$, the field enhancement becomes weaker if the resonant frequency is close to one of the Rayleigh cut-off frequencies. In addition, it is observed that the wave field at the resonance frequency has a phase difference of π for $\delta > 0$ and $\delta < 0$.

Reference

Scattering by a periodic array of subwavelength slits I: field enhancement in the diffraction regime,
Junshan Lin and Hai Zhang, SIAM Journal on Multiscale Modeling and Simulation, 16(2), 922-953, 2018.