A short introduction to the asymptotic analysis of small obstacle problems

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This note aims at giving a modest glimps at the asymptotic treatment of elliptic boundary value problems perturbed by the presence of a small inclusion. We focus on the model case of a small inclusion in a Laplace problem with homogeneous Dirichlet boundary condition in a three dimensional bounded domain.

The analysis is based on the compound expansion method for which we derive the first two terms of the asymptotics. A more in depth course would have introduced theoretical tools for the analysis of corner singularities in boundary value problems. Also known as Kondratiev's calculus [5, 6], these are Fourier-like techniques that we shall avoid, keeping to a minimim the use of weighted Sobolev spaces.

General references on the asymptotic approach presented here for singular perturbation include [7]. The method of matched asympotics is detailed in [10, 4]. A very didactic introduction to such techniques is provided in [3] for a rounded corner problem which is close, although slightly different, to the setting of the present study. Other general references on small inclusion asymptotics include [1]

1 Dirichlet Laplace inclusion problem

Let $\Omega \subset \mathbb{R}^3$ refer to an open Lipschitz set that contains the origin $0 \in \Omega$, and denote $\delta > 0$ a small parameter that should be thought as $\delta \to 0$. Let $\omega \subset \mathbb{R}^3$ refer to another bounded Lipschitz domain and denote

$$\begin{aligned} &\omega_\delta := \delta\,\omega = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x}/\delta \in \omega\} \\ &\Omega_\delta := \Omega \setminus \overline{\omega}_\delta. \end{aligned}$$

The small obstacle ω_δ need not be smooth, star-shaped or connected. We shall systematically assume that δ is small enough that $\omega_\delta \subset \Omega$. Moreover, taking δ smaller if necessary, we may assume that

$$\begin{aligned} &\omega \subset \mathcal{B}_{1/2} \quad \text{where} \quad \mathcal{B}_\rho := \rho\,\mathcal{B}, \\ &\mathcal{B} := \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < 1\}. \end{aligned} \tag{1}$$

We have in particular $\omega_\delta \subset \mathcal{B}_{\delta/2} = (\delta/2)\mathcal{B}$. We shall also need to refer to the spherical coronna $\mathcal{C} := \mathcal{B} \setminus \overline{\mathcal{B}}_{1/2}$ and $\mathcal{C}_\rho := \rho\mathcal{C}$. As $\delta \to 0$, this small obstacle shrinks and Ω_δ "converges" to a punctured domain $\Omega_\star := \Omega \setminus \{0\}$.

As is customary, for any Lipschitz open set $\mathcal{O} \subset \mathbb{R}^3$, the space $\mathrm{L}^2(\mathcal{O})$ refers to square integrable measurable functions equipped with $\|v\|^2_{\mathrm{L}^2(\mathcal{O})} := \int_{\mathcal{O}} |v|^2 d\boldsymbol{x}$, and the Sobolev space is defined by $\mathrm{H}^1(\mathcal{O}) := \{v \in \mathrm{L}^2(\mathcal{O}) : \nabla v \in \mathrm{L}^2(\mathcal{O})\}$. For the sake of conciseness, the space $\mathrm{H}^1_0(\mathcal{O}) := \{v \in \mathrm{H}^1(\mathcal{O}) : v|_{\partial\mathcal{O}} = 0\}$ will be equipped with the norm

$$\|v\|_{\mathrm{H}^1_0(\mathcal{O})} := \|\nabla v\|_{\mathrm{L}^2(\mathcal{O})} \tag{2}$$

which is indeed a norm for $\mathrm{H}^1_0(\mathcal{O})$ (but not for $\mathrm{H}^1(\mathcal{O})$) according to Poincaré inequality. In this document, we wish to thoroughly describe the asymptotic behaviour as $\delta \to 0$ of the solution to the following boundary value problem:

$$\begin{aligned} & u_\delta \in \mathrm{H}^1_0(\Omega_\delta), \\ & \Delta u_\delta = -f \quad \text{in } \Omega_\delta. \end{aligned} \tag{3}$$

where $f \in \mathrm{L}^2(\Omega) \subset \mathrm{H}^{-1}(\Omega_\delta)$ is a given source term. To make things simple, we shall assume that f vanishes in a neighbourhood of the origin 0.

2 Basic asymptotic approximation

Because the small inclusion geometrically disappears in the limit $\delta \to 0$, it is reasonnable to expect a convergence toward the unique solution u_0 to the so-called limit problem i.e. the problem with no obstacle

$$\begin{aligned} & u_0 \in \mathrm{H}^1_0(\Omega) \\ & -\Delta u_0 = f \quad \text{in } \Omega. \end{aligned} \tag{4}$$

Extending by zero if necessary, we shall consider that $\mathrm{H}^1_0(\Omega_\delta) \subset \mathrm{H}^1_0(\Omega)$ and occasionally consider u_δ as an element of $\mathrm{H}^1_0(\Omega)$. Convergence of u_δ toward u_0 can be proved straighforwardly without resorting on any sophisticated asymptotic technique.

Lemma 2.1.
$\|u_\delta - u_0\|_{\mathrm{H}^1_0(\Omega)} = O(\delta^{1/2})$.

Proof:

Consider a cut-off function $\psi \in \mathscr{C}^\infty(\mathbb{R}^3)$ with $\psi(\boldsymbol{x}) = 1$ for $|\boldsymbol{x}| \leq 1/2$ and $\psi(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| \geq 1$, and denote $\psi_\delta(\boldsymbol{x}) := \psi(\boldsymbol{x}/\delta)$. By construction we have $\mathrm{supp}(\psi_\delta) \subset \mathcal{B}_\delta$. Observe in addition that $(1-\psi_\delta)v \in \mathrm{H}^1_0(\Omega_\delta)$ for any $v \in \mathrm{H}^1_0(\Omega)$. Taking account of the fact that u_δ (resp. u_0) is solution to (3) (resp. (4)), we have $\int_\Omega \nabla(u_\delta - u_0)\cdot\nabla u_\delta d\boldsymbol{x} = 0$ and $\int_\Omega \nabla(u_\delta - u_0)\cdot\nabla((1-\psi_\delta)u_0)d\boldsymbol{x} = 0$, which yields

$$\begin{aligned} & \|u_0 - u_\delta\|^2_{\mathrm{H}^1_0(\Omega)} = \int_\Omega \nabla(u_0 - u_\delta)\cdot\nabla(\psi_\delta \overline{u}_0)d\boldsymbol{x} \leq \|u_0 - u_\delta\|_{\mathrm{H}^1_0(\Omega)}\, \|\nabla(\psi_\delta u_0)\|_{\mathrm{L}^2(\Omega)} \\ & \|u_0 - u_\delta\|_{\mathrm{H}^1_0(\Omega)} \leq \|\psi_\delta \nabla u_0\|_{\mathrm{L}^2(\Omega)} + \|u_0 \nabla\psi_\delta\|_{\mathrm{L}^2(\Omega)} \end{aligned}$$

Note that $\Delta u_0 = f = 0$ in a neighbourhood of $\boldsymbol{x} = 0$ where, according to standard elliptic regularity, see e.g. [8, Chap.4], the function u_0 is of class $\mathscr{C}^\infty$: there exists $\mathcal{U} \subset \Omega$ a bounded open neighbourhood of 0 such that $u_0 \in \mathscr{C}^\infty(\overline{\mathcal{U}})$ and in particular $u_0 \in \mathrm{L}^\infty(\mathcal{U}), \nabla u_0 \in \mathrm{L}^\infty(\mathcal{U})$. We can use this in our previous estimate to obtain

$$\begin{aligned}\|u_0 - u_\delta\|_{\mathrm{H}^1_0(\Omega)} &\leq \|\nabla u_0\|_{\mathrm{L}^\infty(\mathcal{U})}\|\psi_\delta\|_{\mathrm{L}^2(\mathcal{B}_\delta)} + \|u_0\|_{\mathrm{L}^\infty(\mathcal{U})}\|\nabla\psi_\delta\|_{\mathrm{L}^2(\mathcal{B}_\delta)}\\ &\leq \|\nabla u_0\|_{\mathrm{L}^\infty(\mathcal{U})}|\mathcal{B}_\delta|^{1/2} + \|u_0\|_{\mathrm{L}^\infty(\mathcal{U})}\|\nabla\psi\|_{\mathrm{L}^\infty(\mathbb{R}^3)}|\mathcal{B}_\delta|^{1/2}/\delta\end{aligned}$$

We used the fact that $\mathrm{supp}(\nabla\psi_\delta) \subset \mathcal{B}_\delta$ and $\|\nabla\psi_\delta\|_{\mathrm{L}^\infty(\mathbb{R}^3)} = \delta^{-1}\|\nabla\psi\|_{\mathrm{L}^\infty(\mathbb{R}^3)}$. In the final inequality above, the predominant term is attached to $|\mathcal{B}_\delta|^{1/2}/\delta = O(\sqrt{\delta})$. □

Remark 2.2.

i) In the previous proof, through the terms $\|u_0\|_{\mathrm{L}^\infty(\mathcal{U})}, \|\nabla u_0\|_{\mathrm{L}^\infty(\mathcal{U})}$, we relied on elliptic regularity and the hypothesis that f vanishes in a neighbourhood of 0.

ii) The previous estimation strategy does not work as well in 2D as, in this case, the factor $|\mathcal{B}_\delta|/\delta = \pi$ does not converge to 0.

Exercise 2.3.
In this exercise, we consider the same geometrical setting as described in Section 1, but a different boundary value problem. Set $\mu_\delta(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \Omega_\delta := \Omega \setminus \overline{\omega}_\delta$, and $\mu_\delta(\boldsymbol{x}) = \mu_* > 0$ for $\boldsymbol{x} \in \omega_\delta$. Consider the problem: find $u_\delta \in \mathrm{H}^1_0(\Omega)$ such that $-\mathrm{div}(\mu_\delta\nabla u_\delta) = f$ in Ω where $f \in \mathrm{L}^2(\Omega)$ is defined as in Section 1. Prove that $\|u_\delta - u_0\|_{\mathrm{H}^1(\Omega)} = O(\sqrt{\delta})$.

3 Normalized inclusion problem

The previous approximation based on the limit field u_0 alone is not sharp. To obtain a more accurate asymptotic approximation, we need to study in more detail what occurs in the neighbourhood of the inclusion. Let us consider a rescaled geometry where the size of the inclusion is normalized. As $\delta \to 0$ the domain $\delta^{-1}\Omega_\delta = (\delta^{-1}\Omega) \setminus \overline{\omega}$ converges toward the unbounded domain

$$\Xi := \mathbb{R}^3 \setminus \overline{\omega}.$$

We need to properly understand the solution to Laplace problems in this unbounded geometry. To start with, the standard Sobolev space $\mathrm{H}^1(\Xi)$ is not appropriate a variational framework. The natural functional setting is dictated by Hardy's inequality. In the sequel $\mathscr{C}^\infty_{\mathrm{K}}(\mathbb{R}^3)$ refers to those $\varphi \in \mathscr{C}^\infty(\mathbb{R}^3)$ with bounded support.

Lemma 3.1 (Hardy inequality).
For any $v \in \mathrm{H}^1_{loc}(\mathbb{R}^3)$, the function defined by $\boldsymbol{x} \mapsto v(\boldsymbol{x})/|\boldsymbol{x}|$ is locally square integrable. Moreover we have the bound

$$\sup_{\varphi\in\mathscr{C}^\infty_{\mathrm{K}}(\mathbb{R}^3)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^3} |\varphi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}/|\boldsymbol{\xi}|^2}{\displaystyle\int_{\mathbb{R}^3} |\nabla\varphi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}} \leq 4. \tag{5}$$

Proof:

The present proof is taken from [9, §16]. We only need to establish the bound (5) and the rest of the lemma follows by density. Pick an arbitrary $\varphi \in \mathscr{C}^\infty_{\mathrm{K}}(\mathbb{R}^3)$. Applying Green's formula, we obtain

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^3} \Big|\nabla\varphi + \frac{\boldsymbol{\xi}\varphi(\boldsymbol{\xi})}{2|\boldsymbol{\xi}|^2}\Big|^2 d\boldsymbol{\xi} &= \int_{\mathbb{R}^3} |\nabla\varphi|^2 d\boldsymbol{\xi} + \frac{1}{4}\int_{\mathbb{R}^3} \frac{|\varphi(\boldsymbol{\xi})|^2}{|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} + \int_{\mathbb{R}^3} \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \Re e\{\overline{\varphi}\nabla\varphi\} d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^3} |\nabla\varphi|^2 d\boldsymbol{\xi} + \frac{1}{4}\int_{\mathbb{R}^3} \frac{|\varphi(\boldsymbol{\xi})|^2}{|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} + \frac{1}{2}\int_{\mathbb{R}^3} \frac{\boldsymbol{\xi}\cdot\nabla|\varphi|^2}{|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^3} |\nabla\varphi|^2 d\boldsymbol{\xi} - \frac{1}{4}\int_{\mathbb{R}^3} \frac{|\varphi(\boldsymbol{\xi})|^2}{|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} \end{aligned}$$

□

From the point of view of variational treatment of Laplace equation in Ξ, the inequality above should be regarded as a weighted version of Poincaré inequality. For any (potentially unbounded) Lispschitz open set $\mathcal{O}$, let us define $\mathscr{V}(\mathcal{O})$ as the closure of $\mathscr{C}^\infty_{\mathrm{K}}(\overline{\mathcal{O}}) = \{\varphi|_{\mathcal{O}} : \varphi \in \mathscr{C}^\infty_{\mathrm{K}}(\mathbb{R}^3)\}$ for the following weighted norm

$$\|v\|^2_{\mathscr{V}(\mathcal{O})} := \int_{\mathcal{O}} |\nabla v(\boldsymbol{\xi})|^2 + \frac{|v(\boldsymbol{\xi})|^2}{|\boldsymbol{\xi}|^2}\, d\boldsymbol{\xi}. \tag{6}$$

We have $\mathrm{H}^1(\mathcal{O}) \subset \mathscr{V}(\mathcal{O}) \subset \mathrm{H}^1_{\mathrm{loc}}(\overline{\mathcal{O}})$. For $\mathcal{O}$ unbounded, we have $\mathrm{H}^1(\mathcal{O}) \neq \mathscr{V}(\mathcal{O})$ (strict embedding) which can be seen by considering $\boldsymbol{\xi} \mapsto (1+|\boldsymbol{\xi}|^2)^{-1/2}$, and the space $\mathscr{V}(\mathcal{O})$ only contains those functions that decay sufficiently fast at infinity: in particular, it does not contain the functions that predominantly behave as a constant at $|\boldsymbol{\xi}| \to \infty$. Since we are dealing with Dirichlet boundary conditions, we shall consider the subspaces

$$\mathscr{V}_0(\mathcal{O}) := \{v \in \mathscr{V}(\mathcal{O}) : v|_{\partial\mathcal{O}} = 0\}.$$

Extending by zero wherever necessary, we can consider that $\mathscr{V}_0(\mathcal{O}) \subset \mathscr{V}(\mathbb{R}^3)$. Hence, as a direct consequence of Lemma 3.1, we obtain a coercivity inequality for the variational form of the Laplace equation in $\Xi = \mathbb{R}^3 \setminus \overline{\omega}$ i.e. $(1/5)\|v\|^2_{\mathscr{V}(\Xi)} \leq \|\nabla v\|^2_{\mathrm{L}^2(\Xi)}$ for all $v \in \mathscr{V}_0(\Xi)$, which yields the following result by applying Lax-Milgram's lemma.

Corollary 3.2.
For any boundedly supported $g \in \mathrm{L}^2(\Xi)$, there exists a unique $v \in \mathscr{V}(\Xi)$ such that $-\Delta v = g$ in Ξ and $v = 0$ on $\partial\Xi$. It solves the variational problem

$$v \in \mathscr{V}_0(\Xi) \quad \textit{and} \quad \int_{\Xi} \nabla v \cdot \nabla w \, d\boldsymbol{\xi} = \int_{\Xi} g w \, d\boldsymbol{\xi} \quad \forall w \in \mathscr{V}_0(\Xi).$$

We shall also be interested in solutions to the homogeneous Dirichlet-Laplace problem that admit a specified extra-variational behaviour at infinity. Such solution are commonly called "profile functions".

Lemma 3.3.
There exists $\mathrm{S} \in \mathrm{H}^1_{loc}(\Xi)$ unique such that $\mathrm{S} - 1 \in \mathscr{V}(\Xi)$, $\Delta\mathrm{S} = 0$ in Ξ and $\mathrm{S} = 0$ on $\partial\Xi$.

Proof:

Clearly there is uniqueness since if S' is another solution to the boundary value problem above, then $\mathrm{S}-\mathrm{S}' \in \mathscr{V}_0(\Xi)$ and $\Delta(\mathrm{S}-\mathrm{S}') = 0$ so that $\mathrm{S}-\mathrm{S}' = 0$ according to Corollary 3.2. To prove existence, take $\chi \in \mathscr{C}^\infty(\mathbb{R}^3)$ satisfying $\chi(\boldsymbol{\xi}) = 0$ for $|\boldsymbol{\xi}| \leq 1/2$ and $\chi(\boldsymbol{\xi}) = 1$ for $|\boldsymbol{\xi}| \geq 1$. In particular $\Delta\chi$ has compact support so we can apply Corollary 3.2 to obtain existence of $\widetilde{\mathrm{S}} \in \mathscr{V}_0(\Xi)$ such that $\Delta(\widetilde{\mathrm{S}}+\chi) = 0$ in Ξ. There only remains to set $\mathrm{S} = \widetilde{\mathrm{S}}+\chi$. Since $\chi = 0$ on $\partial\Xi$ thanks to (1), we conclude that $\mathrm{S} = 0$ on $\partial\Xi$. □

In the problem caracterizing this profile function, the equation $\mathrm{S} - 1 \in \mathscr{V}(\Xi)$ should be understood as an inhomogeneous Dirichlet condition at infinity. Because $\Delta\mathrm{S}(\boldsymbol{\xi}) = 0$ for $|\boldsymbol{\xi}|$ sufficiently large, the expansion of S can be described by means of separation of variables, which yields the existence of a constant $\mathrm{cap}(\omega) \in \mathbb{R}$, such that

$$\mathrm{S}(\boldsymbol{\xi}) = 1 - \mathrm{cap}(\omega)/|\boldsymbol{\xi}| + \underset{|\boldsymbol{\xi}|\to\infty}{O}(|\boldsymbol{\xi}|^{-2}). \tag{7}$$

The constant $\mathrm{cap}(\omega)$ is commonly referred to as the capacity of the normalized inclusion ω and plays an important role in potential theory and asymptotic analysis, see e.g. [2, Eq.(2.48)], [8, Chap.8], [7, §8.1.4].

Exercise 3.4.
Compute $\mathrm{cap}(\omega)$ *for the case where* ω *is a ball.*

Exercise 3.5.

1) *Defining* $\Xi^\sharp := \{\boldsymbol{x}/|\boldsymbol{x}|^2 : \boldsymbol{x} \in \Xi \setminus \{0\}\}$, *show that the function* $\mathrm{S}^\sharp(\boldsymbol{x}) := \mathrm{S}(\boldsymbol{x}/|\boldsymbol{x}|^2)/|\boldsymbol{x}|$ *satisfies* $\Delta\mathrm{S}^\sharp = 0$ *in* $\Xi^\sharp$ *and that* $\boldsymbol{x} \mapsto \mathrm{S}^\sharp(\boldsymbol{x}) - 1/|\boldsymbol{x}|$ *belongs to* $\mathrm{H}^1_{\mathrm{loc}}(\Xi^\sharp)$.

2) *Verify with direct calculus that* $\boldsymbol{x} \mapsto 1/|\boldsymbol{x}|$ *is harmonic in* $\mathbb{R}^3$. *Then show that* $\boldsymbol{x} \mapsto \mathrm{S}^\sharp(\boldsymbol{x}) - 1/|\boldsymbol{x}|$ *is of class* $\mathscr{C}^\infty$ *in a neighbourhood of* 0.

3) *Using a Taylor expansion of* $\mathrm{S}^\sharp(\boldsymbol{x}) - 1/|\boldsymbol{x}|$ *in a neighbourhood of* 0, *prove the expansion formula* (7).

4 Refined low order asymptotics

We come back to the asymptotic analysis of Problem (3). The estimate of Lemma 2.1 shows that u_0 can legitimately be taken as a lowest order approximation of u_δ which we write formally as

$$u_\delta(\boldsymbol{x}) \underset{\delta\to 0}{\sim} u_0(\boldsymbol{x}) + \dots \tag{8}$$

This approximation is not satisfactory because u_0 does not carry any information related to the small inclusion ω_δ, which motivates the derivation of a sharper approximation. The limit solution u_0 does not fit the perturbed geometry Ω_δ because it does not vanish at $\partial\omega_\delta$. Since ω_δ shrinks to $\boldsymbol{x} = 0$, we examine the behaviour of this function at 0. Because $\Delta u_0 = -f$ vanishes in a neighbourhood of 0, we know that u_0 is $\mathscr{C}^\infty$ in a neighbourhood of 0 and thus admit a Taylor expansion of the form

$$\begin{aligned} & u_0(\boldsymbol{x}) \underset{|\boldsymbol{x}|\to 0}{\sim} u_0(0) + \boldsymbol{x}\cdot\nabla u_0(0) + \dots \quad \text{hence} \\ & \limsup_{\delta\to 0} \big(\, \delta^{-3/2}\|u_0 - u_0(0)\|_{\mathscr{V}(\mathcal{C}_\delta)} \,\big) < +\infty. \end{aligned} \tag{9}$$

where we recall that $\mathcal{C}_\delta := \mathcal{B}_\delta \setminus \overline{\mathcal{B}}_{\delta/2}$. Let us emphasize that we made explicit use of the assumption that f vanishes in a neighbourhood of 0. To refine Approximation (8), a natural idea consists in adjusting u_0 so that it vanishes at $\partial\omega_\delta$, which we shall do by introducing a cut-off function. Consider $\chi \in \mathscr{C}^\infty(\mathbb{R}^3)$ satisfying

$$\begin{aligned} &\chi(\boldsymbol{x}) = 0 \quad \text{for} \quad |\boldsymbol{x}| \leq 1/2 \\ &\chi(\boldsymbol{x}) = 1 \quad \text{for} \quad |\boldsymbol{x}| \geq 1 \end{aligned} \tag{10}$$

For any $\delta > 0$ we set $\chi_\delta(\boldsymbol{x}) := \chi(\boldsymbol{x}/\delta)$. The adjustment in our asymptotic approximation may simply consist in considering $\chi_\delta u_0 \in \mathrm{H}^1_0(\Omega_\delta)$ instead of u_0 to approximate u_δ. How well would that fit Problem (3)? If we feed the Laplace operator with $\chi_\delta u_0$ we obtain $\Delta(\chi_\delta u_0) = \chi_\delta \Delta u_0 + 2\nabla\chi_\delta \cdot \nabla u_0 + u_0\Delta\chi_\delta$ and, since $\chi_\delta\Delta u_0 = -\chi_\delta f = -f$ because f vanishes in a neighbourhood of 0, we obtain

$$\Delta(\chi_\delta u_0) = -f + 2\nabla\chi_\delta \cdot \nabla u_0 + u_0\Delta\chi_\delta \tag{11}$$

which means that the Laplace equation is satisfied up to a residual. However the residual is not small because of the terms $\nabla\chi_\delta$ and $\Delta\chi_\delta$ that are of order $O(\delta^{-1})$ and $O(\delta^{-2})$. Let us take a closer look at the higher order term. Re-arranging it by means of the so-called "rapid" change of variables $\boldsymbol{\xi} := \boldsymbol{x}/\delta$, we obtain

$$\begin{aligned} u_0(\boldsymbol{x})(\Delta\chi_\delta)(\boldsymbol{x}) &= \delta^{-2}u_0(\boldsymbol{x})(\Delta\chi)(\boldsymbol{x}/\delta) \\ &= \delta^{-2}u_0(\delta\boldsymbol{\xi})(\Delta\chi)(\boldsymbol{\xi}) \\ &\simeq \delta^{-2}u_0(0)(\Delta\chi)(\boldsymbol{\xi}) + O(\delta^{-1}) \end{aligned}$$

To compensate this residual term, a natural idea consists in adding another solution to the Dirichlet Laplace equation, that depends on the variable $\boldsymbol{\xi} = \boldsymbol{x}/\delta$. We localize this so-called "near field correction" by means of another cut-off function

$$\psi(\boldsymbol{x}) := 1 - \chi(\boldsymbol{x}/r) \tag{12}$$

where $r > 0$ is a strictly positive parameter that is fixed once and for all, chosen as small as desired, so as to garantee in particular that $\mathrm{supp}(\psi) \subset \Omega$. With the properties of χ given by (10), we have $\psi(\boldsymbol{x}) = 1$ for $|\boldsymbol{x}| < r/2$ and $\psi(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| > r$ hence $\mathrm{supp}(\psi) \subset \mathcal{B}_r$ and $\mathrm{supp}(\nabla\psi) \subset \mathcal{C}_r$. We look for an expansion of the form

$$u_\delta(\boldsymbol{x}) \sim \chi(\frac{\boldsymbol{x}}{\delta})u_0(\boldsymbol{x}) + \psi(\boldsymbol{x})U_0(\frac{\boldsymbol{x}}{\delta}) + \dots \tag{13}$$

where the "near field" term $U_0(\boldsymbol{\xi})$ satisfies $\Delta_{\boldsymbol{\xi}}(U_0 + u_0(0)\chi) = 0$ as well as the Dirichlet condition $U_0 = 0$ at $\partial\Xi$. Of course the presence of the cut-off function ψ comes into play at some point, but we shall discuss this later. To summarize, observing that $\Delta\chi$ has compact support in $\Xi := \mathbb{R}^3 \setminus \overline{\omega}$, the near field would be defined in the normalized geometry $\boldsymbol{\xi} \in \Xi$ and would solve the boundary value problem

$$\begin{aligned} &U_0 \in \mathscr{V}(\Xi) \quad \text{such that} \\ &\Delta U_0 = -u_0(0)\Delta\chi \qquad \text{in } \Xi = \mathbb{R}^3 \setminus \overline{\omega}, \\ &U_0 = 0 \qquad\qquad\quad\;\; \text{on } \partial\Xi = \partial\omega. \end{aligned} \tag{14}$$

For this boundary value problem, we selected $\mathscr{V}(\Xi)$ as functional setting because it offers an existence/uniqueness result see Corollary 3.2. We need to keep in mind that this choice has an impact on the behaviour of $U_0(\boldsymbol{\xi})$ at $|\boldsymbol{\xi}|\to\infty$. Problem (14) can be reformulated variationally as

$$\begin{aligned}
&U_0\in\mathscr{V}_0(\Xi)\quad\text{and}\\
&\int_{\Xi}\nabla(U_0+u_0(0)\chi)\cdot\nabla V d\boldsymbol{\xi}=0\quad\forall V\in\mathscr{V}_0(\Xi).
\end{aligned}\tag{15}$$

Comparing this variational formulation with Lemma 3.3 and its proof, we realize that $U_0(\boldsymbol{\xi})+u_0(0)\chi(\boldsymbol{\xi})=u_0(0)\mathrm{S}(\boldsymbol{\xi})$. Consequently, unlike what it seems, the function $U_0+u_0(0)\chi$ does *not* depend on the choice of cut-off function χ. Besides, (7) leads to the following asymptotic behaviour at infinity

$$\begin{aligned}
&U_0(\boldsymbol{\xi})\underset{|\boldsymbol{\xi}|\to\infty}{\sim}-u_0(0)\mathrm{cap}(\omega)/|\boldsymbol{\xi}|+\dots\quad\text{hence}\\
&\limsup_{\delta\to 0}\big(\delta^{-1/2}\|U_0\|_{\mathscr{V}(\mathcal{C}_{r/\delta})}\big)<+\infty.
\end{aligned}\tag{16}$$

Let us come back to the expansion (13). For the sake of conciseness, we shall denote $U_{0,\delta}(\boldsymbol{x}):=U_0(\boldsymbol{x}/\delta)$. We have the following error estimate.

Lemma 4.1.
Denote $\tilde{u}_\delta:=\chi_\delta u_0+\psi U_{0,\delta}$ where u_0 is the unique solution to (4) *and U_0 is the unique solution to* (15)*. Then $\|u_\delta-\tilde{u}_\delta\|_{\mathrm{H}^1_0(\Omega)}=O(\delta)$.*

Proof:
Setting $v=u_\delta-\tilde{u}_\delta\in\mathrm{H}^1_0(\Omega_\delta)$, we have

$$\begin{aligned}
\|u_\delta-\tilde{u}_\delta\|^2_{\mathrm{H}^1_0(\Omega)}&=\int_\Omega\nabla(u_\delta-\tilde{u}_\delta)\cdot\nabla\overline{v}d\boldsymbol{x}=\int_\Omega f\overline{v}-\nabla(\chi_\delta u_0+\psi U_{0,\delta})\cdot\nabla\overline{v}d\boldsymbol{x}\\
&=\int_\Omega f\overline{v}-\nabla u_0\nabla(\chi_\delta\overline{v})d\boldsymbol{x}+\int_\Omega\nabla\chi_\delta\cdot(\overline{v}\nabla u_0-u_0\nabla\overline{v})d\boldsymbol{x}\\
&\quad-\int_\Omega\nabla U_{0,\delta}\cdot\nabla(\psi\overline{v})d\boldsymbol{x}+\int_\Omega\nabla\psi\cdot(\overline{v}\nabla U_{0,\delta}-U_{0,\delta}\nabla\overline{v})d\boldsymbol{x}
\end{aligned}$$

First note that $\int_\Omega f\overline{v}-\nabla u_0\nabla(\chi_\delta\overline{v})d\boldsymbol{x}=\int_\Omega f\overline{v}(1-\chi_\delta)d\boldsymbol{x}$ since u_0 solves (4). Besides, since f vanishes close to 0, we deduce that $f(1-\chi_\delta)=0$ for δ sufficiently small. So the first integral in the right hand side above vanishes.

In addition, observe that $\mathrm{supp}(\nabla\chi_\delta)\subset\mathcal{B}_\delta$ and $\psi(\boldsymbol{x})=1$ for $\boldsymbol{x}\in\mathcal{B}_\delta$ hence $\overline{v}\nabla\chi_\delta=\psi\overline{v}\nabla\chi_\delta$ and $\nabla\chi_\delta\nabla\overline{v}=\nabla\chi_\delta\nabla(\psi\overline{v})$. Setting $\tilde{u}_0(\boldsymbol{x}):=u_0(\boldsymbol{x})-u_0(0)$, we can re-arrange the second term in the right hand side above, which yields

$$\begin{aligned}
\|u_\delta-\tilde{u}_\delta\|^2_{\mathrm{H}^1_0(\Omega)}=&-\int_\Omega\nabla(U_{0,\delta}+u_0(0)\chi_\delta)\cdot\nabla(\psi\overline{v})d\boldsymbol{x}\\
&+\int_\Omega\nabla\chi_\delta\cdot(\overline{v}\nabla\tilde{u}_0-\tilde{u}_0\nabla\overline{v})d\boldsymbol{x}+\int_\Omega\nabla\psi\cdot(\overline{v}\nabla U_{0,\delta}-U_{0,\delta}\nabla\overline{v})d\boldsymbol{x}
\end{aligned}\tag{17}$$

Now we take a closer look at the first term in the right hand side of (17). Consider $V(\boldsymbol{\xi}):=\psi(\delta\boldsymbol{\xi})\overline{v}(\delta\boldsymbol{\xi})$ which is exactly equivalent to $V(\boldsymbol{x}/\delta):=\psi(\boldsymbol{x})\overline{v}(\boldsymbol{x})$. Extending by zero wherever necessary, we have $V \in \mathscr{V}_0(\Xi)$. Hence using the change of variable $\boldsymbol{x} = \delta\boldsymbol{\xi}$ and applying (15), we obtain

$$\int_{\Omega} \nabla(U_{0,\delta} + u_0(0)\chi_\delta)\cdot\nabla(\psi\overline{v})d\boldsymbol{x} = \delta\int_{\Xi}\nabla(U_0 + u_0(0)\chi)\cdot\nabla\overline{V}d\boldsymbol{\xi} = 0 \tag{18}$$

Hence the first term in the right hand side of (17) vanishes, and it remains to estimate each of the other two terms. We have $(\nabla\chi_\delta)(\boldsymbol{x}) = \delta^{-1}(\nabla\chi)(\boldsymbol{x}/\delta)$ so that $\limsup_{\delta\to 0}\sup_{\mathbb{R}^3}|\delta\nabla\chi_\delta| < +\infty$. Consequently there exists constants $C_1, C_2 > 0$ independent of δ such that

$$\begin{aligned}\Big|\int_{\Omega}\nabla\chi_\delta\cdot(\overline{v}\nabla\tilde{u}_0 - \tilde{u}_0\nabla\overline{v})d\boldsymbol{x}\Big| &\leq C_1\int_{\mathcal{C}_\delta}|v\nabla\tilde{u}_0| + |\tilde{u}_0\nabla v|\,\frac{d\boldsymbol{x}}{|\boldsymbol{x}|}\\ &\leq C_1\|v\|_{\mathscr{V}(\Omega)}\|\tilde{u}_0\|_{\mathscr{V}(\mathcal{C}_\delta)} \leq C_2\delta^{3/2}\|v\|_{\mathrm{H}^1_0(\Omega)}.\end{aligned} \tag{19}$$

where we used (9). As regards the near field term in (17), recall that $\mathrm{supp}(\nabla\psi)\subset\mathcal{C}_r$. Hence applying the change of variable $\boldsymbol{x} = \delta\boldsymbol{\xi}$, and taking account of (16), there exist two constants $C_3, C_4 > 0$ independent of δ such that

$$\begin{aligned}\Big|\int_{\Omega}\nabla\psi\cdot(\overline{v}\nabla U_{0,\delta} - U_{0,\delta}\nabla\overline{v})d\boldsymbol{x}\Big| &\leq C_3\|v\|_{\mathrm{H}^1_0(\Omega)}\|U_{0,\delta}\|_{\mathscr{V}(\mathcal{C}_r)}\\ &\leq C_3\delta^{1/2}\,\|v\|_{\mathrm{H}^1_0(\Omega)}\|U_0\|_{\mathscr{V}(\mathcal{C}_{r/\delta})} \leq C_4\delta\|v\|_{\mathrm{H}^1_0(\Omega)}.\end{aligned} \tag{20}$$

where we have taken account of (16) in the last inequality above. Now since $\|v\|_{\mathrm{H}^1_0(\Omega)} = \|u_\delta - \tilde{u}_\delta\|_{\mathrm{H}^1_0(\Omega)}$, there only remains to plug (18)-(19)-(20) into (17) and we obtain the desired estimate. □

Exercise 4.2.
With the same geometry as described in Section 1, and given constants $\alpha_{\mathrm{I}}, \alpha_{\mathrm{E}} \in \mathbb{R}$, we consider a different problem compared to (3) consisting in finding $u_\delta \in \mathrm{H}^1(\Omega_\delta)$ such that

$$\begin{aligned}&\Delta u_\delta - iu_\delta = 0 \quad \text{in } \Omega_\delta,\\ &u_\delta = \alpha_{\mathrm{I}} \;\; \text{on } \partial\omega_\delta, \quad u_\delta = \alpha_{\mathrm{E}} \;\; \text{on } \partial\Omega.\end{aligned} \tag{21}$$

Derive a low order asypmptotic approximation of the solution u_δ to this alternative boundary value problem, following the same methodology as described in Section 4, and establish an error estimate.

5 Higher order asymptotics

We will push the construction that we presented in the previous section one order further so as to exhibit the influence of the small inclusion in the far field part of the asymptotic expansion. We look for an expansion of the form

$$\begin{aligned}u_\delta(\boldsymbol{x}) \sim\;\; &\chi(\frac{\boldsymbol{x}}{\delta})\Big(\,u_0(\boldsymbol{x}) + \delta u_1(\boldsymbol{x})\,\Big)\\ &+\psi(\boldsymbol{x})\Big(\,U_0(\frac{\boldsymbol{x}}{\delta}) + \delta U_1(\frac{\boldsymbol{x}}{\delta})\,\Big) + \cdots\end{aligned} \tag{22}$$

We will define u_1 so as to compensate the residuals of the Dirichlet-Laplace problem induced by U_0, and U_1 will be defined so as to compensate one additional term in (11). We start by an elementary result for the construction of a singular solution to far field Dirichlet-Laplace problem.

Lemma 5.1.
There exists a unique vector field $\sigma \in \mathrm{H}^1_{loc}(\Omega \setminus \{0\})$ *such that* $\sigma(\boldsymbol{x}) - 1/|\boldsymbol{x}| \in \mathrm{H}^1(\Omega)$, $\Delta\sigma = 0$ *in* $\Omega \setminus \{0\}$ *and* $\sigma = 0$ *on* $\partial\Omega$.

This result is an easy consequence of the fact that $\Delta(1/|\boldsymbol{x}|) = 0$ in $\Omega \setminus \{0\}$. Although the prescribed asymptotic behaviour is different, it should be understood as a far field counterpart of Lemma 3.3. By means of separation variables, we find that there is $\beta(\Omega) \in \mathbb{R}^3$ such that

$$\sigma(\boldsymbol{x}) = 1/|\boldsymbol{x}| + \beta(\Omega) + \underset{|\boldsymbol{x}|\to 0}{O}(|\boldsymbol{x}|) \tag{23}$$

Exercise 5.2.
Compute $\beta(\Omega)$ *for the case where* Ω *is a ball.*

Exercise 5.3.
Prove that there exists a unique (vector valued) $\boldsymbol{\sigma}' \in \mathrm{H}^1_{loc}(\Omega)^3$ *such that* $\boldsymbol{\sigma}'(\boldsymbol{x}) - \boldsymbol{x}/|\boldsymbol{x}|^3 \in \mathrm{H}^1(\Omega)^3$, $\Delta\boldsymbol{\sigma}' = 0$ *in* $\Omega \setminus \{0\}$ *and* $\boldsymbol{\sigma}' = 0$ *on* $\partial\Omega$. *Show that this function admits an expansion of the form* $\boldsymbol{\sigma}'(\boldsymbol{x}) = \boldsymbol{x}/|\boldsymbol{x}|^3 + \boldsymbol{\beta}' + O_{|\boldsymbol{x}|\to 0}(|\boldsymbol{x}|)$ *for some* $\boldsymbol{\beta}' \in \mathbb{R}^3$.

Now let us point that the near field term $\psi U_{0,\delta}$ is not exactly harmonic i.e. $\Delta(\psi U_{0,\delta}) = 2\nabla\psi \cdot \nabla U_{0,\delta} + U_{0,\delta}\Delta\psi \neq 0$. However $\Delta(\psi U_{0,\delta})$ is small because U_0 decays at infinity. More precisely, formally replacing $U_0(\boldsymbol{x}/\delta)$ by its expansion at infinity given by (16), we have

$$\Delta(\psi U_{0,\delta}) \underset{\delta\to 0}{\sim} -\delta\, u_0(0)\mathrm{cap}(\omega)\Delta(\psi(\boldsymbol{x})/|\boldsymbol{x}|)$$

In spite of $1/|\boldsymbol{x}|$ being singular at $\boldsymbol{x} \to 0$, $\Delta(\psi(\boldsymbol{x})/|\boldsymbol{x}|)$ is a $\mathscr{C}^\infty$ function supported away from 0 because $\Delta(1/|\boldsymbol{x}|) = 0$. The predominant behaviour above is what the new far field term aims at compensating. We thus define u_1 as the unique solution to the problem

$$\begin{aligned} & u_1 \in \mathrm{H}^1_0(\Omega) \\ & -\Delta u_1 = -u_0(0)\mathrm{cap}(\omega)\Delta(\psi(\boldsymbol{x})/|\boldsymbol{x}|) \quad \text{in } \Omega \setminus \{0\}. \end{aligned} \tag{24}$$

Comparing this problem with Lemma 5.1, we conclude that $u_1(\boldsymbol{x}) - u_0(0)\psi(\boldsymbol{x})\mathrm{cap}(\omega)/|\boldsymbol{x}| = -u_0(0)\mathrm{cap}(\omega)\sigma(\boldsymbol{x})$ which does not depend on the choice of the cut-off function ψ. Observe that, in contrast with the limit far field u_0, the presence of the small inclusion has an influence on the definition of u_1 through the constant $\mathrm{cap}(\omega)$.

Because the right hand side in (24) is $\mathscr{C}^\infty$, the function u_1 admits a Taylor expansion at 0 with $u_1(0) = -u_0(0)\mathrm{cap}(\omega)\beta(\Omega)$, and we have the bound

$$\limsup_{\delta\to 0} \left(\, \delta^{-3/2}\|u_1 - u_1(0)\|_{\mathscr{V}(\mathcal{C}_\delta)} \,\right) < +\infty. \tag{25}$$

To construct the second near field term U_1, we take a look at the predominant terms in the far field residual, plugging in the rapid change of variables $\boldsymbol{x} = \delta\boldsymbol{\xi}$, which yields

$$\begin{aligned} \Delta_{\boldsymbol{x}}\{\chi_\delta(\boldsymbol{x})(u_0(\boldsymbol{x}) + \delta u_1(\boldsymbol{x}))\} &= \delta^{-2}\Delta_{\boldsymbol{\xi}}\{\chi(\boldsymbol{\xi})(u_0(\delta\boldsymbol{\xi}) + \delta u_1(\delta\boldsymbol{\xi}))\} \\ &\simeq \delta^{-2}\Delta_{\boldsymbol{\xi}}\{\chi(\boldsymbol{\xi})(u_0(0) + \delta\boldsymbol{\xi}\cdot\nabla u_0(0) + \delta u_1(0))\} + \cdots \\ &\simeq \delta^{-2}\Delta_{\boldsymbol{\xi}}\{\chi(\boldsymbol{\xi})u_0(0)\} + \delta^{-1}\Delta_{\boldsymbol{\xi}}\{\chi(\boldsymbol{\xi})(u_1(0) + \boldsymbol{\xi}\cdot\nabla u_0(0))\} + \cdots \end{aligned} \tag{26}$$

We construct U_1 so as to compensate the term attached to δ^{-1} in the right hand side of (26). This leads to define it as the unique solution to the following Dirichlet-Laplace boundary value problem in normalized geometry

$$\begin{aligned} & U_1 \in \mathscr{V}_0(\Xi) \quad \text{and} \\ & -\Delta U_1 = \Delta_{\boldsymbol{\xi}}\big(\chi(\boldsymbol{\xi})(u_1(0) + \boldsymbol{\xi}\cdot\nabla u_0(0))\big) \quad \text{in } \Xi. \end{aligned} \tag{27}$$

The right hand side in (27) has bounded support, since the functions $\boldsymbol{\xi} \mapsto u_1(0) + \boldsymbol{\xi}\cdot\nabla u_0(0)$ is harmonic. As a consequence Problem (27) fits the setting of Corollary 3.2. Like for U_0, the function $U_1(\boldsymbol{\xi}) + \chi(\boldsymbol{\xi})(u_1(0) + \boldsymbol{\xi}\cdot\nabla u_0(0))$ can be proved independent of the actual choice of cut-off function χ. Because $(\Delta U_1)(\boldsymbol{\xi}) = 0$ for $|\boldsymbol{\xi}|$ sufficiently large, we can use separation of variables to show that $U_1(\boldsymbol{\xi}) = O_{|\boldsymbol{\xi}|\to\infty}(|\boldsymbol{\xi}|^{-1})$, and more precisely

$$\limsup_{\delta\to 0}\big(\delta^{-1/2}\|U_1\|_{\mathscr{V}(\mathcal{C}_{r/\delta})}\big) < +\infty. \tag{28}$$

Now that all terms of Expansion (22) have been properly defined, error estimates can be established. We omit the proof of the next result because it rests on tedious calculations that nevertheless follow the same lines of argument as for Lemma 4.1. We denote $U_{1,\delta}(\boldsymbol{x}) := U_1(\boldsymbol{x}/\delta)$.

Lemma 5.4.
Set $\tilde{u}_\delta := \chi_\delta(u_0 + \delta u_1) + \psi(U_{0,\delta} + \delta U_{1,\delta})$ where u_0, u_1 are the unique solutions to (4),(24), and U_0, U_1 are the unique solutions to (15), (27). Then $\|u_\delta - \tilde{u}_\delta\|_{\mathrm{H}^1_0(\Omega)} = O(\delta^2)$.

Exercise 5.5.
Prove the previous error estimate.

Ton conclude this study let us re-arrange the asymptotic approximation $\tilde{u}_\delta = \chi_\delta(u_0 + \delta u_1) + \psi(U_{0,\delta} + \delta U_{1,\delta})$ trying to move the cut-off function ψ away from the far field contributions. Taking account of the formula $u_1(\boldsymbol{x}) = u_0(0)\psi(\boldsymbol{x})\mathrm{cap}(\omega)/|\boldsymbol{x}| - u_0(0)\mathrm{cap}(\omega)\sigma(\boldsymbol{x})$, leads to

$$\begin{aligned} \tilde{u}_\delta(\boldsymbol{x}) = & \chi(\frac{\boldsymbol{x}}{\delta})\big(\, u_0(\boldsymbol{x}) - \delta u_0(0)\mathrm{cap}(\omega)\sigma(\boldsymbol{x})\,\big) \\ & + \psi(\boldsymbol{x})\big(\, U_0(\frac{\boldsymbol{x}}{\delta}) + \delta U_1(\frac{\boldsymbol{x}}{\delta}) + \delta\chi(\frac{\boldsymbol{x}}{\delta})u_0(0)\frac{\mathrm{cap}(\omega)}{|\boldsymbol{x}|}\,\big) \end{aligned}$$

This formula suggests that the asymptotic expansion of u_δ, away from the small inclusion, takes the form $u_0(\boldsymbol{x}) - \delta u_0(0)\mathrm{cap}(\omega)\sigma(\boldsymbol{x}) =$ "limit field"$+\delta\times$ "singular correction". In the vicinity of the small inclusion, the asymptotic expansion is more ellaborate, and there is a so-called *boundary layer* effect which includes additional near field corrections. Let us emphasize that the cut-off function ψ can be chosen as localized as desired, see (12), choosing the parameter $r > 0$ smaller if required.

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