Lecture 3: Counting $\mathbb{Z}/\mathbb{Q}$-points on varieties.

$$X = \{x \in \mathbb{C}^d : f_1(x) = \dots = f_e(x) = 0\} \text{ – rational surface}$$

$$f_i \in \mathbb{Q}[x_1, \dots x_d]$$

$\left[\ \underline{\text{Question:}}\ \text{Compute } \#\{x \in X(\mathbb{Z}) : \|x\| < T\}.\ \right]$

<u>Thm</u> (Borel–Harish-Chandra)

1) $G < GL_d(\mathbb{C})$ – reductive algebraic group, defined over $\mathbb{Q}$, G acts transitively on X.

Then $X(\mathbb{Z})$ = union of finitely many $G(\mathbb{Z})$-orbits.

2) If G has no characters defined over $\mathbb{Q}$, then $\mathrm{vol}\left(G(\mathbb{R})/G(\mathbb{Z})\right) < \infty$.

<u>ex.</u> $X = \left\{ \sum_{i,j=1}^{d} a_{ij} x_i x_j = b \right\}$ – nondegenerate quad. surface ($a_{ij} \in \mathbb{Q}$, $b \in \mathbb{Q}$, $b \neq 0$)

SO_Q = the orthogonal group.

Then $X(\mathbb{Z}) = \bigsqcup_{i=1}^{N} SO_Q(\mathbb{Z}) x_i$.

We consider $SO_Q(\mathbb{Z}) \cdot x_i$ as a subset of

$$SO_Q(\mathbb{R}) x_i \simeq SO_Q(\mathbb{R}) / \mathrm{Stab}_{SO_Q(\mathbb{R})}(x).$$

Counting $\longleftrightarrow$ Distribution of periods.

G – locally compact group

$\Gamma < G$ – discrete subgroup with $\mathrm{vol}(G/\Gamma) < \infty$.

$H < G$ – closed subgroup with $\mathrm{vol}(H/(H\cap\Gamma)) < \infty$

$B_T \subset G/H$ – family of compact increasing domains in G/H.

[**Question:** Compute asymptotics of $\#(\Gamma \cdot H \cap B_T)$ as $T \to \infty$.]

Let $F_T(g) = \sum_{\gamma \in \Gamma/\Gamma\cap H} \chi_{B_T}(g\gamma H)$.

Then F_T is a function on G/Γ, and $F_T(e) = \#(\Gamma \cdot H \cap B_T)$.

Step 1: Compute $\langle F_T, \varphi \rangle \underset{T\to\infty}{\sim}$? for $\varphi \in C_c(G/\Gamma)^+$.

Step 2: Deduce pointwise convergence.

$$\int_{G/\Gamma} F_T \cdot \varphi \, dm_{G/\Gamma} = \int_{G/\Gamma} \Big(\sum_{\gamma \in \Gamma/\Gamma\cap H} \chi_{B_T}(g\gamma H) \Big) \varphi(g\Gamma) \, dm_{G/\Gamma}(g)$$

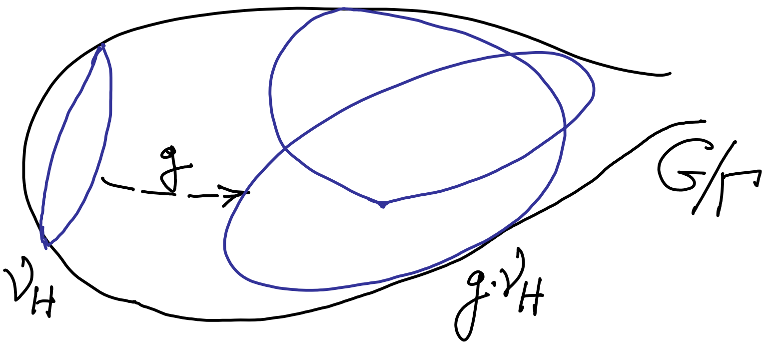
$$= \int_{G/\Gamma} \sum_{\gamma \in \Gamma/\Gamma\cap H} \chi_{B_T}(g\gamma H) \varphi(g\gamma\Gamma) \, dm_{G/\Gamma}(g)$$

$$= \int_{G/\Gamma\cap H} \chi_{B_T}(gH) \varphi(g\Gamma) \, dm_{G/\Gamma\cap H}(g)$$

$$= \int_{G/H} \left(\int_{H/\Gamma\cap H} \chi_{B_T}(ghH)\,\varphi(gh\Gamma)\,dm_{H/\Gamma\cap H}(h) \right) dm_{G/H}(g)$$

$$= \int_{B_T} \left(\int_{H/\Gamma\cap H} \varphi(gh\Gamma)\,dm_{H/\Gamma\cap H}(h) \right) dm_{G/H}(g)$$

$$= \int_{B_T} \underbrace{(g\cdot\nu_H)(\varphi)}_{\text{period integral}}\, dm_{G/H}(g)$$



ν_H = measure on $H/\Gamma\cap H \hookrightarrow G/\Gamma$.

We normalise: $\mathrm{vol}(G/\Gamma) = \mathrm{vol}(H/\Gamma\cap H) = 1$.

This also defines measure on G/H.

Equidistribution of $g\cdot\nu_H$.

Notation: G = simple noncompact Lie group $\subset GL_d(\mathbb{R})$, $v_0 \in \mathbb{R}^d$,
$H = \mathrm{Stab}_G(v_0)$, $B_T = \{gH \in G/H : \|gv_0\| < T\}$.

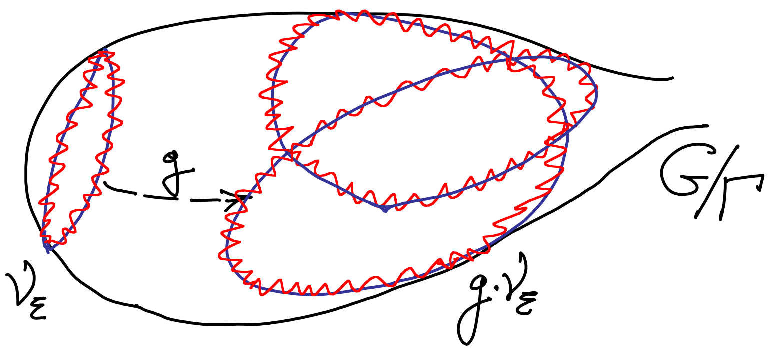
1) Harmonic analysis approach (Delsarte, ... Duke-Rudnick-Sarnak, Kroetz-Sayag-Schlichtkrull)

Show that $(g\cdot\nu_H)(\varphi_n) \xrightarrow[gH\to\infty]{} 0$ for a convinient "basis" of $L^2_0(G/\Gamma)$.

Thm (Duke-Rudnick-Sarnak)
Assume that H is symmetric (i.e, $\mathrm{Lie}(G)$ = the set of fixed points of an involution).

Then for $\delta>0$,
$$\#(\Gamma v_0 \cap B_T) = \mathrm{vol}(B_T) + O(\mathrm{vol}(B_T)^{1-\delta}).$$

2) **Mixing** (Margulis, Bartels, Eskin-McMullen, Maucourant, Benoist-Oh)



Let $\nu_\varepsilon = \psi_\varepsilon \, dm_{G/\Gamma}$ be an absolutely continuous prob. measure supported on ε-nbhd of $H\Gamma \subset G/\Gamma$ such that
$$\nu_H \approx \nu_\varepsilon.$$

By mixing,

$$\int_{G/\Gamma} \varphi(y)\, d(g\cdot\nu_\varepsilon)(y) = \int_{G/\Gamma} \varphi(gy)\,\psi_\varepsilon(y)\, dm_{G/\Gamma}(y)$$
$$\xrightarrow[g\to\infty]{} \left(\int_{G/\Gamma}\varphi\right)\cdot\left(\int_{G/\Gamma}\psi_\varepsilon\right) = \int_{G/\Gamma}\varphi.$$

We need to know that

$$g\cdot\nu_H \approx g\cdot\nu_\varepsilon \quad (\text{uniformly on } g),$$

which follows from the wavefront property:

<u>Wavefront property:</u> $\mathcal{O}_\varepsilon$ = nbhd of e in G.

$\forall \varepsilon\in(0,1)$ $\forall g\in G$: $\boxed{g\mathcal{O}_\varepsilon H \subset \mathcal{O}_{c\varepsilon}\, gH.}$

H-symmetric $\Rightarrow$ wavefront property.

<u>Thm.</u> Assume that H is symmetric.
Then for some $\delta>0$,

$$\#(\Gamma v_0 \cap B_T) = \mathrm{vol}(B_T) + O\left(\mathrm{vol}(B_T)^{1-\delta}\right).$$

3) <u>Unipotent flows</u> (Eskin–Margulis–Mozes)

Let ν_∞ be a weak* limit of $g\cdot\nu_H$ as $\|g v_0\|\to\infty$.

1) ν_∞ is invariant under 1-par. unipotent subgroup.
2) By Ratner's Thm, ν_∞ is "built" from homogeneous measures.
3) Using Dani–Margulis linearization techique, analyse accumulation of the sequence $g\cdot\nu_H$ on $\{\infty\}$ and on homogeneous submanifolds

Thm (special case) Assume that H is a maximal algebraic subgroup. Then

$$\#(\Gamma \cdot v_0 \cap B_T) \underset{T\to\infty}{\sim} \mathrm{vol}(B_T).$$

Ergodic Thm approach. (G.–Nevo)

Assume that H is symmetric and $\mathrm{vol}(H/H\cap\Gamma)<\infty$.

We construct $\widetilde{B}_T \subset G$ such that:

1) $\Gamma \cap \widetilde{B}_T \longrightarrow \Gamma\cdot v_0 \cap B_T$ is a bijection.

2) $\mathcal{O}_\varepsilon = \varepsilon$-nbhd of e in G. $\forall \varepsilon\in(0,1)$, $T\ge T_0$:

$$\mathrm{vol}(\mathcal{O}_\varepsilon \widetilde{B}_T \mathcal{O}_\varepsilon) \le (1+c\,\varepsilon^{1/3})\,\mathrm{vol}(\widetilde{B}_T),$$

$$\mathrm{vol}\Big(\bigcap_{u_1,u_2\in\mathcal{O}_\varepsilon} u_1 \widetilde{B}_T u_2\Big) \ge (1-c\,\varepsilon^{1/3})\,\mathrm{vol}(\widetilde{B}_T).$$

Idea: $(\Gamma\cdot v_0\cap B_T) \approx \int_{\widetilde{B}_T} f(b^{-1}y)\,db$, $\quad \mathrm{supp}(f)\subset \mathcal{O}_\varepsilon\Gamma$, $\; y\in\mathcal{O}_\varepsilon\Gamma$.

Mean Ergodic Thm: $\forall f\in L^2(G/\Gamma)$:

$$\left\| \frac{1}{\mathrm{vol}(\widetilde{B}_T)} \int_{\widetilde{B}_T} f(b^{-1}y)\,db - \int_{G/\Gamma} f \right\|_2 \ll_\varepsilon \mathrm{vol}(B_T)^{-\frac{1}{2n_e(p)}+\varepsilon}\, \|f\|_2, \quad \varepsilon>0.$$

where $p=$ the integrability exponent of $L^2_0(G/\Gamma)$,

$$n_e(p) = \begin{cases} \text{least even integer} \ge \frac{p}{2}, & \text{for } p>2,\\ 1, & p=2. \end{cases}$$

Thm. $$\#(\Gamma\cdot v_0 \cap B_T) = \mathrm{vol}(B_T) + O_\varepsilon\left(\mathrm{vol}(B_T)^{1-\frac{3}{n_e(p)(3+\dim(G))}+\varepsilon}\right),$$
$\varepsilon > 0$.

Counting rational points.

$X = \{x \in \mathbb{C}^d : f_1(x) = \ldots = f_\ell(x) = 0\}$ – rational surface
$f_i \in \mathbb{Q}[x_1, \ldots x_d]$

Height: $x = \left(\frac{p_1}{q}, \ldots \frac{p_d}{q}\right) \in X(\mathbb{Q})$, $\gcd(p_1, \ldots p_d, q) = 1$
Define $H(x) = \sqrt{p_1^2 + \ldots + p_d^2 + q^2}$.

[Question (Batyrev – Manin)
Compute asymptotics of $\#\{x \in X(\mathbb{Q}) : H(x) < T\}$.]

$X(\mathbb{Q}) \subset$ "classifying space" $= X(\mathbb{A})$
↑adeles

$\mathbb{A} = \{(x_p)_{p \leq \infty} : x_p \in \mathbb{Q}_p,\ x_p \in \mathbb{Z}_p \text{ for almost all } p\}$.

Then: 1) $X(\mathbb{Q}) \xhookrightarrow[\text{diagonally}]{} X(\mathbb{A})$ is discrete.

2) (Borel – Harish-Chandra)
If G is an algebraic group $/\mathbb{Q}$ without $\mathbb{Q}$-characters, then $\mathrm{vol}(G(\mathbb{A})/G(\mathbb{Q})) < \infty$.

Aproaches.

1) Harmonic analysis (Shalika-Takloo-Bighash-Tschinkel)
2) Mixing (G.-Maucourant-Oh)
3) Unipotent flows (Borovoi-G.-Oh)
4) Ergodic thms (G.-Nevo)