Lecture 2: Dynamics and Diophantine approximation.

Given $x \in \mathbb{R}^d$, approximate x by rationals $\frac{p}{q} \in \mathbb{Q}^d$.

$\psi : [1,\infty) \longrightarrow (0,1)$ – nonincreasing function.

Def. $x \in \mathbb{R}^d$ is <u>ψ-approximable</u> if

$$\left\| x - \frac{p}{q} \right\| \leq \frac{\psi(q)}{q}$$

has infinitely many solutions $p \in \mathbb{Z}^d$, $q \in \mathbb{N}$.

$W_d(\psi) = \{ \psi\text{-approximable vectors in } \mathbb{R}^d \}$.

Clearly, $W_d(\psi) = \overline{\lim}\ B_{\frac{p}{q}}\left(\frac{\psi(q)}{q}\right)$.

Borel–Cantelli Lemma:

(X,μ) – prob. space, $B_n \subset X$, $n \geq 1$.

1) $\sum_{n\geq 1} \mu(B_n) < \infty \Rightarrow \overline{\lim}\, B_n$ has measure 0.] easy

2) $\sum_{n\geq 1} \mu(B_n) = \infty$,

$(*)$ $\sum_{m,n=1}^{N} \mu(B_m \cap B_n) \leq \left(\sum_{m=1}^{N} \mu(B_m)\right)^2 + C \cdot \left(\sum_{m=1}^{N} \mu(B_m)\right)$ $\Bigg| \Rightarrow$ $\overline{\lim}\, B_n$ has full measure.

Khinchin Thm

1) $\sum_{q\geq 1} \psi(q)^d < \infty \Rightarrow W_d(\psi)$ has measure 0.] easy

2) $\sum_{q\geq 1} \psi(q)^d = \infty \Rightarrow W_d(\psi)$ has full measure.

We prove Part (2) using dynamics on the space of lattices.

$\mathcal{L}_{d+1} = \{\text{unimodular lattices in } \mathbb{R}^{d+1}\} \simeq SL_{d+1}(\mathbb{Z})\backslash SL_{d+1}(\mathbb{R})$.

For $x \in \mathbb{R}^d$, set $L_x = \mathbb{Z}^{d+1}\left(\begin{array}{c|c} I & 0 \\ \hline -x & 1\end{array}\right) = \{(p-qx, q) : (p,q) \in \mathbb{Z}^{d+1}\}$,

$$a_t = \left(\begin{array}{c|c} e^t I & 0 \\ \hline 0 & e^{-dt}\end{array}\right).$$

Observation (Dani):

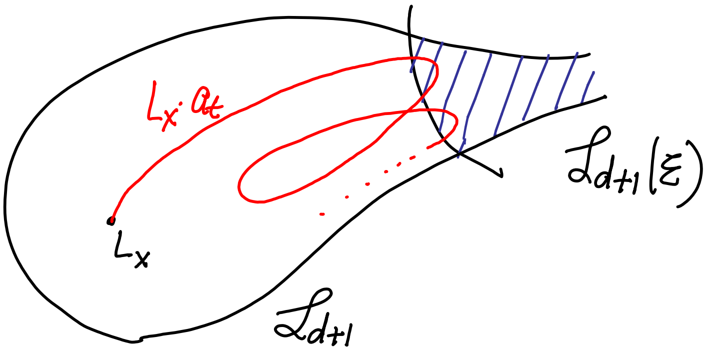
The lattice $L_x \cdot a_t$ has small vectors $\iff$ $x \in \mathbb{R}^d$ has good $\mathbb{Q}$-approximation.

$v = (e^t(p-qx), e^{-nt}q) \in L_x \cdot a_t$.

Suppose that $0 < \|v\| < \varepsilon$.

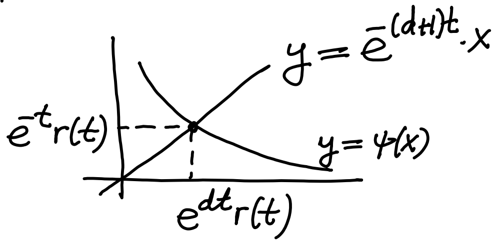
$$\|v\| < \varepsilon \Rightarrow \begin{cases} e^t\|p-qx\| < \varepsilon \\ e^{-nt}|q| < \varepsilon \end{cases} \Rightarrow |q| \cdot \|p-qx\|^n < \varepsilon^{n+1} \Rightarrow \left\|x - \frac{p}{q}\right\| < \frac{\varepsilon^{1+\frac{1}{n}}}{|q|^{1+\frac{1}{n}}}.$$

Let $\mathcal{L}_{d+1}(\varepsilon) = \{ L \in \mathcal{L}_{d+1} : L \cap \{\|v\| < \varepsilon\} \neq \{0\} \}$
(nbhds of ∞ in $\mathcal{L}_{d+1}$).



<u>Dani Correspondence</u> (Kleinbock – Margulis)

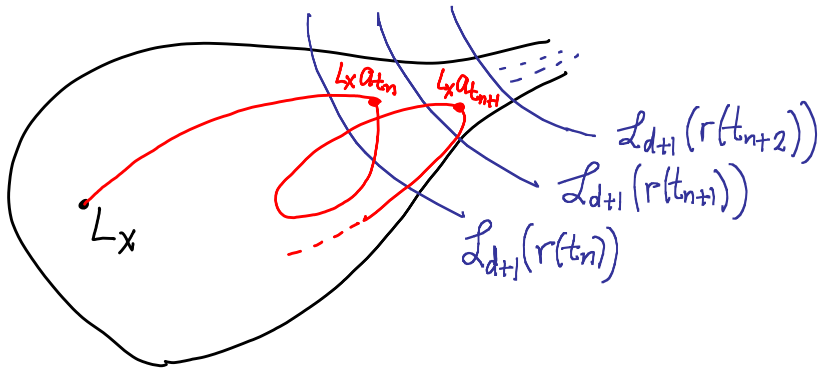
Define the function $r(t)$ by:



Then

$$x \in W_d(\psi) \iff \left[L_x a_{t_n} \in \mathcal{L}_{d+1}(r(t_n)) \text{ for a sequence } t_n \to \infty. \right]$$

Shrinking Target Property



"Dynamical" Khinchin Thm $(\Longrightarrow$ Khinchin Thm$)$
(Kleinbock – Margulis)

Let $\varepsilon_n \to 0$ be a decreasing sequence. Then

$$\sum_{n\geq 1} \mathrm{vol}(\mathcal{L}_{d+1}(\varepsilon_n)) = \infty \Longrightarrow \{L \in \mathcal{L}_{d+1} : L \cdot a_n \in \mathcal{L}_{\varepsilon_n} \text{ infinitely often}\}$$

has full measure in $\mathcal{L}_{d+1}$.

Sketch of the proof:

Observe that the set is $\overline{\lim}\,(\mathcal{L}_{\varepsilon_n} a_n^{-1})$.

By Borel-Cantelli Lemma (2), we just need to show:

$$\sum_{m,n=1}^{N} \mathrm{vol}(\mathcal{L}_{\varepsilon_n} a_n^{-1} \cap \mathcal{L}_{\varepsilon_m} a_m^{-1}) \leq \left(\sum_{m=1}^{N} \mathrm{vol}(\mathcal{L}_m)\right)^2 + C \cdot \left(\sum_{m=1}^{N} \mathrm{vol}(\mathcal{L}_m)\right).$$

$$\| $$

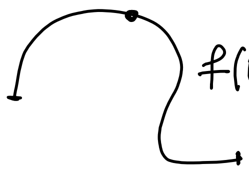
$$\langle \chi_{\mathcal{L}_{\varepsilon_n}} \cdot a_n , \chi_{\mathcal{L}_{\varepsilon_n}} \cdot a_n \rangle$$

This estimate is deduced from exponential mixing. $\square$

Diophantine approximation with dependent quantities.

Let $f: \mathcal{U} (\underset{\text{open}}{\subset} \mathbb{R}^k) \to \mathbb{R}^d$ be C^ℓ-map.

What are the Diophantine properties of vectors x in the surface $f(\mathcal{U}) \subset \mathbb{R}^d$?



Conjecture (Mahler, Sprindzuk)

If $f(\mathcal{U})$ is "curved", then $f(y) \notin W_d(x^{\frac{1}{d}+\varepsilon})$, $\varepsilon > 0$, for almost every $y \in \mathcal{U}$.

Thm (Kleinbock–Margulis)

Assume that $\left\langle \frac{\partial^\alpha f}{\partial x^\alpha}(x) : |\alpha| \le \ell \right\rangle = \mathbb{R}^d$ for all $x \in \mathcal{U}$.

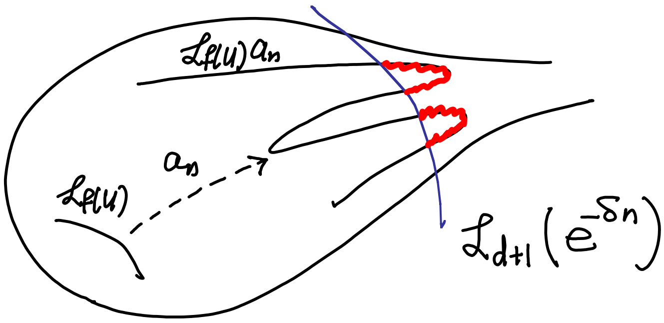
Then $\{ y \in \mathcal{U} : f(y) \in W_d(x^{\frac{1}{d}+\varepsilon}) \}$ has measure 0.

Sketch of the proof:

By the Dani correspondence, Thm would follow if we show that for $\delta = \delta(\varepsilon) > 0$, the set

$$\left\{ y \in \mathcal{U} : L_{f(y)} \cdot a_n \in \mathcal{L}_{d+1}(e^{-\delta n}) \text{ for infinitely many } n \right\}$$

has measure 0.



This is deduced from quantitative nondivergence of unipotent flows (combined with Borel-Cantelli Lemma (1)):

Thm (Kleinbock-Margulis) $\exists\, C, \delta > 0$ and open cover $\mathcal{U} = \bigcup_i V_i$ such that $\forall \varepsilon > 0$: $\forall t \geq 0$

$$\mathrm{Vol}(\{y \in V : L_{f(y)} a_t \in \mathcal{L}_{d+1}(\varepsilon)\}) \leq C \cdot \varepsilon^{\delta} \cdot \mathrm{vol}(V).$$

Diophantine approximation on varieties

$$X = \{x \in \mathbb{R}^d : f_1(x) = \ldots = f_\ell(x) = 0\}, \quad f_i \in \mathbb{Q}[x_1, \ldots x_d].$$

Def. $x \in X(\mathbb{R})$ is $(X(\mathbb{Z}[\tfrac{1}{p}]), \psi)$-approximable if

$$\|x - r\| \leq \psi(\mathrm{den}(r))$$

has infinitely many solutions $r \in X(\mathbb{Z}[\tfrac{1}{p}])$

$$W(X(\mathbb{Z}[\tfrac{1}{p}]), \psi) = \{ (X(\mathbb{Z}[\tfrac{1}{p}]), \psi)\text{-approximable points} \}$$

Note that $W(X(\mathbb{Z}[\frac{1}{p}]), \psi) = \overline{\lim}_{r \in X(\mathbb{Z}[\frac{1}{p}])} B^{X}_{\psi(den(r))}(r)$,

where $B^X_\varepsilon(x)$ denotes ε-ball in X.

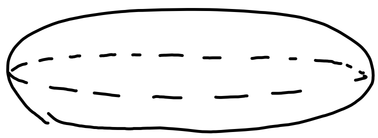
Hence, by Borel-Cantelli Lemma (1),

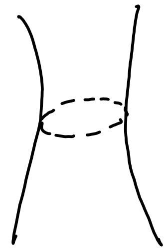
$$\sum_{\substack{r \in X(\mathbb{Z}[\frac{1}{p}]) \\ \|r\| \le c}} \mathrm{vol}\left(B^X_{\psi(den(r))}(r)\right) < \infty \Rightarrow W(X(\mathbb{Z}[\tfrac{1}{p}]), \psi) \cap \{\|x\| \le c\}$$

has measure 0.

Example.

$X = \{x : \sum_{i,j=1}^{3} a_{ij} x_i x_j = b\}$ – nondegenerate rational quad. surface in $\mathbb{R}^3$.





Thm (Ghosh – G. – Nevo) Assume that:

1) $X(\mathbb{Z}[\frac{1}{p}]) \subset X(\mathbb{R})$ is not discrete,
2) $X(\mathbb{R})$ is compact or Ramanujan-Petterson conjecture holds.

Then:

$$\sum_{\substack{r \in X(\mathbb{Z}[\frac{1}{p}]) \\ \|r\| \le c}} \mathrm{vol}\left(B^X_{\psi(den(r))}(r)\right) < \infty \Rightarrow W(X(\mathbb{Z}[\tfrac{1}{p}]), \psi) \cap \{\|x\| \le c\}$$

has full measure in $X(\mathbb{R})$.

Dynamical correspondence.

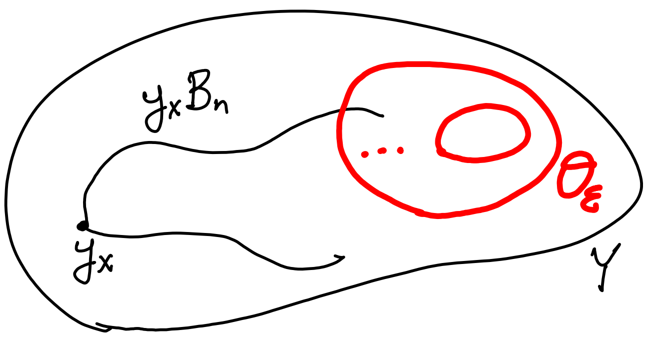
$G = SO_Q$ – the orthogonal group

$Y = G(\mathbb{Z}[\frac{1}{p}]) \backslash (G(\mathbb{R}) \times G(\mathbb{Q}_p))$

Dynamical system: $Y \circlearrowleft G(\mathbb{Q}_p)$

$B_n = \{ g \in G(\mathbb{Q}_p) : \|g\|_p \le p^n \}$.

Dioph. approximation	Dynamics
$x \in X(\mathbb{R})$	$y_x = G(\mathbb{Z}[\frac{1}{p}])(x, e) \in Y$
$\|x - r\| < \varepsilon$	$\mathcal{O}_\varepsilon \subset Y$ shrinking open sets
$\begin{cases} \|x-r\| < \varepsilon \\ \mathrm{den}(r) \le p^n \end{cases}$ has solution	$y_x \cdot B_n \cap \mathcal{O}_\varepsilon \neq \emptyset$



Quantitative Ergodic Thm

$\mathcal{H} = \{ f \in L^2(Y) : f \perp \text{characters of } G(\mathbb{Q}_p) \}$

$\pi : G(\mathbb{Q}_p) \curvearrowright \mathcal{H}$ – unitary representation

$q = q(\pi) =$ integrability exponent (as in Lecture 1)

Then $\forall f \in \mathcal{H}$:

$$\boxed{\left\| \frac{1}{\mathrm{vol}(B_n)} \int_{B_n} f(y.b)\, db \right\|_2 \ll_\varepsilon \mathrm{vol}(B_n)^{-\frac{1}{q}+\varepsilon} \|f\|_2, \quad \varepsilon > 0.}$$