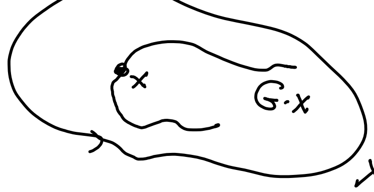
Lecture 1: Spectral gap on homogeneous spaces.

Spectral gap property →

- **Ergodic Theory** (quantitative ergodic thms, exponential mixing)
- **Number Theory** (Langland's programme, Diophantine approximation, counting $\mathbb{Z}/\mathbb{Q}$-points on varieties)

Problem of distribution of orbits:



G – locally compact group,

(X,μ) – prob. space

$G \curvearrowright (X,\mu)$ – measure-preserving action.

Consider averaging operators: for $f: X \to \mathbb{C}$ and $B \subset_{cpt} G$,

$$f \longmapsto \frac{1}{\mathrm{vol}(B)} \int_B f(b^{-1} \cdot x)\, db$$

What is the asymptotic behaviour as B exhasts G?

Note that we have unitary representation:

$$G \curvearrowright L^2_0(X) = \{ f \in L^2(X) : \int_X f = 0 \}$$

$$f \overset{g}{\longmapsto} f(g^{-1}x).$$

More generally:

β = absolutely continuous symmetric prob. measure on G with $\overline{\langle \mathrm{supp}(\mu) \rangle} = G$.

$\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$ – unitary representation

Averaging operator: $\pi(\beta) : \mathcal{H} \longrightarrow \mathcal{H}$

$$v \longmapsto \int_G \pi(g) v \, d\beta(g).$$

<u>Def.</u> π has <u>spectral gap</u> (SG) if $\boxed{\|\pi(\beta)\| < 1.}$

<u>Results.</u>

1) <u>Amenable groups</u> ($\mathbb{Z}^k$, $\mathbb{R}^k$, ...)
(del Junco, Rosenblatt)

G = countable amenable group

$G \curvearrowright (X, \mu)$ – nonatomic prob. space.

Then $G \curvearrowright L^2_0(X)$ has no (SG.).

<u>Rmk</u>: $\mathbb{R}^d \curvearrowright \mathbb{R}^d/\mathbb{Z}^d$ has (SG).

2) <u>Tori/nilmanifolds</u> (Bekka-Guivarc'h)

X = torus/nilmanifold,

G = countable subgroup of $\mathrm{Aff}(X)$.

Then $G \curvearrowright X$ has no (SG) $\Leftrightarrow \exists$ (nontrivial) factor

$$\begin{array}{ccc} G & \curvearrowright & X \\ \downarrow & & \downarrow \\ \overline{G} & \curvearrowright & \overline{X} \end{array}$$

$\overline{G}$ amenable

<u>Question</u>: Is this criterion true for general hom. spaces of Lie groups?

3) <u>Isometric actions</u> (Bourgain – Gamburd)

G = fin. generated dense subgroup of $SU(n)$

$G \subset M_n(\overline{\mathbb{Q}})$

Then $G \curvearrowright SU(n)$ has (SG).

4) <u>Transitive actions</u> (Bekka – Cornullier)

$X = G/\Lambda$ where G is a connected Lie group, Λ is a closed subgroup, $vol(G/\Lambda) < \infty$.

Then $G \curvearrowright G/\Lambda$ has (SG).

5) <u>Semisimple spaces</u> (Nevo, Shalom)

$X = G/\Lambda$ where G is a simple Lie group, Λ is a closed subgroup, $vol(G/\Lambda) < \infty$.

$H \subset G$ – closed nonamenable subgroup.

Then $H \curvearrowright X$ has (SG).

Refinements.

G = simple (noncompact) Lie group (e.g. $G = SL(n,\mathbb{R})$).

$\pi: G \to \mathcal{U}(\mathcal{H})$ – unitary representation.

1) Integrability exponents (Cowling, Borel-Wallach, Howe-Moore)

Define $q(\pi) = \inf\{ q > 0 : \langle \pi(g)v_1, v_2\rangle \in L^q(G)$
for $v_1, v_2 \in$ dense subspace of $\mathcal{H}\}$.

Thm. π has (SG) $\Leftrightarrow$ $q(\pi) < \infty$.

2) Exponential mixing (Cowling, Haagerup, Howe, Moore)

Thm. If π is a representation with (SG),
then $\exists \delta > 0$: $\forall v_1, v_2 \in \mathcal{H}$:

$$|\langle \pi(g)v_1, v_2\rangle| \le c \cdot e^{-\delta \cdot d(g,e)} \cdot S(v_1) S(v_2)$$

where $S(v_1)$, $S(v_2)$ are Sobolev norms.

3) Quantitative ergodic thm.

Let $B_t \underset{cpt}{\subset} G$ satisfy: $\exists c > 0$: $\forall \varepsilon \in (0,1)$: $\forall t \ge t_0$:

- $\mathcal{O}_\varepsilon B_t \mathcal{O}_\varepsilon \subset B_{t+c\varepsilon}$ ($\mathcal{O}_\varepsilon$ = nbhd of e in G),
- $\mathrm{vol}(B_{t+\varepsilon}) \le (1+c\varepsilon)\,\mathrm{vol}(B_t)$.

Thm (Margulis–Nevo–Stein)

If $G \curvearrowright (X,\mu)$ has (SG), then $\exists \delta>0$: $\forall f\in L^2(X)$, $\forall$a.e. $x\in X$:

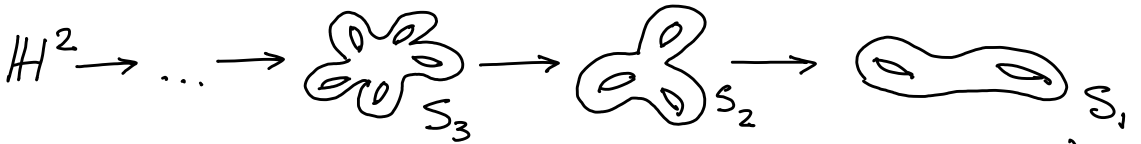
$$\begin{cases} \left| \dfrac{1}{\mathrm{vol}(B_t)} \displaystyle\int_{B_t} f(g^{-1}x)\,dg - \int_X f \right| \le c(x,f)\cdot \mathrm{vol}(B_t)^{-\delta} \\ \|c(\cdot,f)\|_2 \le C\cdot \|f\|_2 . \end{cases}$$

Let Γ be a lattice in G and $D_t = \Gamma\cap B_t$.

Thm (G.–Nevo) If $\Gamma \curvearrowright (X,\mu)$ has (SG), then $\exists\delta>0$: $\forall f\in L^2(X)$ $\forall$a.e. $x\in X$:

$$\begin{cases} \left| \dfrac{1}{|D_t|}\cdot \displaystyle\sum_{\gamma\in D_t} f(\gamma^{-1}x) - \int_X f \right| \le c(x,f)\cdot |D_t|^{-\delta} \\ \|c(\cdot,f)\|_2 \le C\cdot \|f\|_2 . \end{cases}$$

<u>Uniform spectral gap.</u>



(finite area hyperbolic surfaces)

$\lambda(S_n)$ = bottom of spectrum of the Laplace operator (excluding 0).

<u>Question:</u> $\lambda(S_n) \xrightarrow[n\to\infty]{} ?$ $\left(\lambda(\mathbb{H}^2) = \frac{1}{4}\right)$.

<u>Selberg:</u> – examples of covers with $\lambda(S_n) \to 0$,

– for $S_n = \Gamma(n)\backslash\mathbb{H}^2$, where

$\Gamma(n) = \{\gamma \in SL_2(\mathbb{Z}) : \gamma = id \ (\mathrm{mod}\ n)\}$,

$\lambda(S_n) \geq \frac{3}{16}$.

<u>Conj</u> (Selberg) $\lambda(S_n) \geq \frac{1}{4}$.

(Best known estimate: $\lambda(S_n) \geq \frac{975}{4096}$)
(Kim–Sarnak)

Conj. $\Leftrightarrow$ $q(\pi_n) = 2$, where

$\pi_n : G \circlearrowright L_0^2(SL_2(\mathbb{R})/\Gamma(n))$.

<u>Clozel:</u>

G = simply connected simple alg group $\subset GL_N$, defined over $\mathbb{Q}$.

$\Gamma(n) = \{\gamma \in G(\mathbb{Z}) : \gamma = id \ (\mathrm{mod}\ n)\}$

$\pi_n : G(\mathbb{R}) \circlearrowright L_0^2(G(\mathbb{R})/\Gamma(n))$
(unitary representations)

Then $\boxed{\sup_{n\geq 1} q(\pi_n) < \infty.}$
(property τ)

<u>Kazhdan</u>: G = simple Lie group, rank$(G) \geq 2$
(e.g., $G = SL(n,\mathbb{R})$, $n \geq 3$)

$\pi : G \to \mathcal{U}(\mathcal{H})$ – a unitary representation without G-fixed vectors.

Then $\boxed{\sup_{\pi} q(\pi) < \infty.}$
(property (T))

<u>ex.</u> For $G = SL(n,\mathbb{R})$, $n \geq 3$, $\sup_{\pi} q(\pi) = 2(n-1)$.