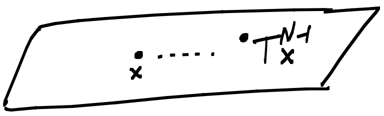
Lecture 3: Duality principle and infinite measure ergodic theory

(joint with A. Nevo)

$T: (X,\mu) \to (X,\mu)$ – measure-preserving, ergodic, conservative.

$\mu(X) = \infty$

Distribution of orbits?

X 

For $\varphi: X \to \mathbb{R}$, $\sum_{n=0}^{N-1} \varphi(T^n x) \underset{N\to\infty}{\approx} ?$

$\mu(X) = \infty \Rightarrow \quad \forall \varphi \in L^2(X): \quad \frac{1}{N} \sum_{n=0}^{N-1} \varphi(T^n x) \xrightarrow[N\to\infty]{L^2} 0.$

Is there a correct normalisation?

Aaranson: given any $a(N) \to \infty$, $\frac{a(N)}{N} \to 0$, either:

- $\frac{1}{a(N)} \sum_{n=0}^{N-1} \varphi(T^n x) \xrightarrow{a.e.} \infty$ for every $\varphi \in L^1(X)^+$,
- $\varliminf_{N\to\infty} \frac{1}{a(N)} \sum_{n=0}^{N-1} \varphi(T^n x) \overset{a.e.}{=} 0$ for every $\varphi \in L^1(X)$.

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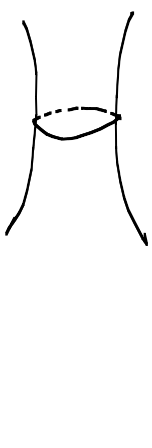
group $\Gamma \circlearrowright (X,\mu)$, $\Gamma_t \subset \Gamma$, $\varphi: X \to \mathbb{R}$, $\sum_{\gamma \in \Gamma_t} \varphi(x\gamma) \underset{t\to\infty}{\approx} ?$

"Large" groups behave better (Ledrappier, Nogueira, Ledrappier-Pollicott, Maucourant, G., G.- Weiss).

example (Arnol'd question: distribution on the de Sitter space)



$X = \{ x_1^2 + \ldots + x_d^2 - x_{d+1}^2 = 1 \}$, $d \geq 3$,

Γ – lattice in $SO(d,1)$,

$\Gamma_t = \{ \gamma \in \Gamma : \log \|\gamma\| \leq t \}$, where $\|\cdot\|$ is the Euclidean norm.

Then: - (G.-Weiss) $\forall \varphi \in C_c(X)$ $\forall v \in X$ such that $\overline{\Gamma\cdot v} = X$:

$$\lim_{t\to\infty} \frac{1}{e^{(d-2)t}} \cdot \underbrace{\sum_{\gamma\in\Gamma_t} \varphi(v\cdot\gamma)}_{\approx e^{(d-1)t} \text{ terms}} = c_\Gamma \cdot \int_X \varphi(x) \cdot \underbrace{\frac{d\,Vol(x)}{(1+\|x\|^2)^{\frac{d-2}{2}} \cdot (1+\|v\|^2)^{\frac{d-2}{2}}}}_{\substack{\cdot \text{ not invariant} \\ \cdot \text{ depends on } v}} \quad \text{with } c_\Gamma > 0.$$

with $c_\Gamma > 0$.

- (G.-Nevo) If in addition φ is piecewise analytic, then

$$\frac{1}{e^{(d-2)t}} \cdot \sum_{\gamma\in\Gamma_t} \varphi(v\cdot\gamma) \overset{a.e.}{=} c_\Gamma \cdot \int_X \varphi(x) \cdot \frac{d\,Vol(x)}{(1+\|x\|^2)^{\frac{d-2}{2}} \cdot (1+\|v\|^2)^{\frac{d-2}{2}}} + O_{v,\varphi}\left(e^{-\delta t}\right)$$

with $\delta > 0$.

General results.

G = connected semisimple alg. group $\subset GL_d(\mathbb{R})$

$\cup$

Γ = irreducible lattice

X = algebraic homogeneous space of G.

Assume that $\Gamma \curvearrowright X$ is ergodic.

$N: Mat_d(\mathbb{R}) \to [0,\infty)$ – proper homogeneous polynomial.

$\Gamma_t = \{\gamma \in \Gamma_t : \log N(\gamma) \le t\}$

For compact $\Omega \subset X$ with nonempty interior, set

$$a = \overline{\lim_{t\to\infty}} \frac{\log |\Gamma_t \cdot x \cap \Omega|}{t}$$ (one can check that a is independent of a & x).

Thm 1 (when $a=0$) There exists $b \in \mathbb{Z}_{\geq 0}$ such that

$$a_t(\varphi)(x) \overset{\text{def}}{=} \frac{1}{t^b} \sum_{\gamma \in \Gamma_t} \varphi(x\gamma)$$

satisfies for all $p>1$ and compact $D \subset X$:

1) (strong max. inequality): $\forall \varphi \in L^p(D)$:

$$\left\| \sup_{t \geq 1} |a_t(\varphi)| \right\|_{L^p(D)} \leq C(p,D) \cdot \|\varphi\|_{L^p(D)}.$$

2) (pointwise ergodic thm) $\forall \varphi \in L^1(D)$:

$$a_t(\varphi) \xrightarrow[t\to\infty]{a.e.} c_\Gamma \cdot \int_X \varphi \cdot d\mathrm{Vol} \quad \text{with } c_\Gamma > 0.$$

G-inv. measure

Thm (when $a>0$) There exist $b \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{\geq 0}$ such that

$$a_t(\varphi)(x) = \frac{1}{e^{at} t^b} \cdot \sum_{\gamma \in \Gamma_t} \varphi(x \cdot \gamma)$$

satisfies for all $p>1$ and compact $D \subset X$,

1) $\forall \varphi \in L^p_\ell(D)$: $\quad \left\| \sup_{t \geq 1} |a_t(\varphi)| \right\|_{L^p(D)} \leq C(p,D) \cdot \|\varphi\|_{L^p_\ell(D)}.$

sobolev space of regularity ℓ

2) $\forall \varphi \in L^p_\ell(D)$: $\quad a_t(\varphi)(x) \xrightarrow[t\to\infty]{a.e.} \int_X \varphi \, d\nu_x$

where $\{\nu_x\}_{x \in X}$ is a family of absolutely continuous measures on X with full support (ν_x's are analogues of the Patersson–Sullivan measures)

3) $\forall \varphi \in C_c(X)$ – piecewise analytic:

$$a_t(\varphi)(x) \overset{a.e.}{=\!=} \int_X \varphi \, d\nu_x + \sum_{i=1}^{b} c_i(\varphi,x) \cdot t^{-i} + O_{\varphi,x}(e^{-\delta t})$$

with $\delta > 0$.

Duality principle.

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr, "\pi"] & \\ H \circlearrowright Y = G/\Gamma & & X \simeq H\backslash G \circlearrowleft \Gamma \arrow[ul, dashed, bend right] \end{tikzcd}$$

Take a section s:
$\pi \circ s = \mathrm{id}$

Let $a_t(\varphi)(x) \overset{def}{=} \sum_{\gamma \in \Gamma_t} \varphi(x\gamma)$

$B_t(\varphi)(x) \overset{def}{=} \int_{G_t} \varphi(x\gamma)\, dg$ $\left(\begin{array}{l}\text{the measure is normalised} \\ \text{so that } \mathrm{vol}(G/\Gamma) = 1\end{array}\right)$.

<u>Step 1:</u> Show that $a_t(\varphi) \approx B_t(\varphi)$.

<u>Step 2:</u> Analyse $B_t(\varphi)$.

<u>Sketch of Step 1:</u> Let $r(g) = s(\pi(g))$ and $h(g) = r(g) g^{-1} \in H$

This gives an identification $G \simeq H \times X$, $g = h(g)^{-1} r(g)$.

Given $\varphi : X \to \mathbb{R}$, we define functions:

$f : G \to \mathbb{R}$: $\quad f(g) = \chi_\varepsilon(h(g)) \cdot \varphi(\pi(g))$

bump function at e in H with $\int_H \chi_\varepsilon = 1$, $\chi_\varepsilon \geq 0$.

$F : Y \to \mathbb{R}$: $\quad F(g\Gamma) = \sum_{\gamma \in \Gamma} f(g\gamma)$.

Let $H_t[g_1, g_2] = \{ h \in H : \log N(g_1^{-1} h g_2) \leq t \}$.

<u>Lem.</u>: $\quad a_t(\varphi)(gH) \approx \sum_{\gamma \in \Gamma} \int_{H_{t \pm \varepsilon}[g, r(g\gamma)]} f(h^{-1} g\gamma)\, dh$.

Now if $\mathrm{supp}(\varphi)$ is "small" (this can be achieved by a partition of unity argument),
then $\mathrm{supp}(f)$ is also small, and
$r(g\gamma) = r(h^{-1}g\gamma) \approx r$ is approximately constant

Then
$$a_t(\varphi)(gH) \approx \int_{H_{t\pm\varepsilon}[g,r]} F(h^{-1}g\Gamma)\,dh.$$

Similarly,
$$B_t(\varphi)(gH) \approx \int_G \int_{H_{t\pm\varepsilon}[g,r(g\gamma)]} f(h^{-1}gu)\,dh\,du \approx \mathrm{vol}(H_{t\pm\varepsilon}[g,r]) \int_G f \doteq \int_Y F$$

Hence, to show that $a_t(\varphi) \approx B_t(\varphi)$, we need to prove the following equidistribution result on Y:
$$\frac{1}{\mathrm{vol}(H_t[g,r])} \cdot \int_{H_t[g,r]} F(h^{-1}g\Gamma)\,dh \xrightarrow[t\to\infty]{} \int_Y F.$$
This can be done in some cases using the theory of unipotent flows (as in G.-Weiss) or using harmonic analysis (as in G.-Nevo).

<u>Proof of Lem.:</u> For $\gamma \in \Gamma_t$: $g\gamma = h(g\gamma)^{-1} r(g\gamma) \in gG_t$,
$$h(g\gamma)^{-1}\mathrm{supp}(\chi_\varepsilon) \subset g G_t\, r(g\gamma)^{-1}\mathrm{supp}(\chi_\varepsilon) \subset g\, G_{t+\varepsilon}\, r(g\gamma)^{-1},$$
$$\mathrm{supp}(\chi_\varepsilon) \subset h(g\gamma) \cdot H_{t+\varepsilon}[g, r(g\gamma)].$$

Now since $\int_H \chi_\varepsilon\, dh = 1$,

$$\varphi(Hg\gamma) = \varphi(Hg\gamma) \int_{h(g\gamma) H_{t+\varepsilon}[g, r(g\gamma)]} \chi_\varepsilon(h)\, dh$$

$$= \varphi(Hg\gamma) \int_{H_{t+\varepsilon}[g, r(g\gamma)]} \chi_\varepsilon(h(g\gamma)h)\, dh$$

$$= \varphi(H \cdot h^{-1} g\gamma) \int_{H_{t+\varepsilon}[g, r(g\gamma)]} \chi_\varepsilon(h(h^{-1}g\gamma))\, dh$$

$$= \int_{H_{t+\varepsilon}[g, r(g\gamma)]} f(h^{-1}g\gamma)\, dh .$$

Now assuming that $\varphi \geq 0$,

$$\sum_{\gamma \in \Gamma_t} \varphi(Hg\gamma) \leq \sum_{\gamma \in \Gamma} \int_{H_{t+\varepsilon}[g, r(g\gamma)]} f(h^{-1}g\gamma)\, dh.$$

This proves the upper bound.

Sketch of Step 2: We have identification:

$$G \simeq H \times X$$

$$g = s(x_0)^{-1} \cdot h \cdot s(x)$$

$$dg \longleftrightarrow dh \cdot dx$$

$$(\mathcal{B}_t \varphi)(x_0) = \int_{G_t} \varphi(x_0 g)\, dg = \int_{s(x_0)^{-1} h s(x) \in G_t} \varphi(x_0\, s(x_0)^{-1} h \cdot s(x))\, dh\, dx$$

$$= \int_{s(x_0)^{-1} h s(x) \in G_t} \varphi(x)\, dh\, dx = \int_X \varphi(x) \cdot \mathrm{vol}(H_t[s(x_0), s(x)])\, dx.$$

We show that for a suitable choice of a & b

$$d\nu_{x_0}(x) = \left(\lim_{t\to\infty} \frac{\mathrm{vol}(H_t[s(x_0), s(x)])}{e^{at} t^b} \right) dx$$

exists and has positive density.