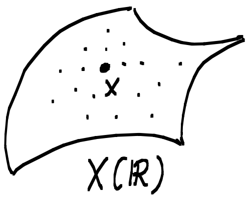
Lecture 2: Quantitative ergodic theory & Diophantine approximation.

(joint with A. Ghosh & A. Nevo)

Diophantine approximation: $x \in \mathbb{R}^d$, $x \approx r \in \mathbb{Q}^d$ with $\mathrm{den}(r)$ small.

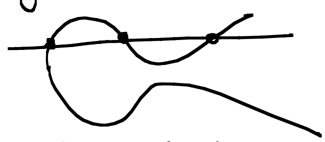
Question: Given an algebraic set $X = \{x : f_1(x) = \ldots = f_s(x) = 0\}$ and a point $x \in X(\mathbb{R})$, when a system of inequalities:



$$\begin{cases} \|x - r\| \le \varepsilon \\ \mathrm{den}(r) \le \varepsilon^{-\varkappa} \end{cases}$$

has a solution $r \in X(\mathbb{Q})$ (or $r \in X(\mathbb{Z}[\frac{1}{p}])$)?

M. Waldschmidt: $X: y^2 = x^3 + ax + b$ – an elliptic curve



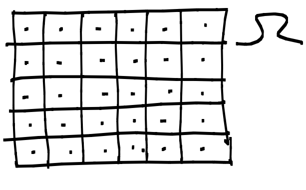
What about other algebraic sets with "group structure"?

$$\boxed{\text{We consider } X\text{'s equipped with transitive action of simple algebraic group } G \text{ defined over } \mathbb{Q}.}$$

Lower bound on $\varkappa$:

Let $a_p(X) = \sup_{\mathrm{cpt}\ \Omega \subset X(\mathbb{R})} \overline{\lim_{R \to \infty}} \dfrac{\log |\{r \in \Omega \cap X(\mathbb{Z}[\frac{1}{p}]) : \mathrm{den}(r) \le R\}|}{\log R}$

(measures growth rate of $\mathbb{Z}[\frac{1}{p}]$-points)

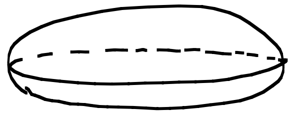


$\#(\mathbb{Z}[\frac{1}{p}]\text{-points with } \mathrm{den}(r) \le R) \ll R^{a_p(X) + \delta}$, $\delta > 0$.

$\#(\text{disjont } \varepsilon\text{-boxes}) \gg \varepsilon^{-\dim(X)}$

Hence, $\boxed{\varkappa \le \dfrac{\dim(X)}{a_p(X)}}$

example: Let $X=\{Q(x)=\sum_{i,j=1}^{3} a_{ij}x_ix_j=b\}$ be a rational 2-dim. ellipsoid. We assume that $X(\mathbb{Z}[\frac{1}{p}])$ is not discrete in $X(\mathbb{R})$.



Then: — for a.e. $x\in X(\mathbb{R})$, $\forall\delta>0$, $\forall\varepsilon\in(0,\varepsilon_0(x,\delta))$: best possible $a_p(X)=1$, $\dim(X)=2$.

$$\begin{cases}\|x-r\|\le\varepsilon\\ \mathrm{den}(r)\le\varepsilon^{-2-\delta}\end{cases}$$

has a solution $r\in X(\mathbb{Z}[\frac{1}{p}])$.

— for every $x\in X(\mathbb{R})$, $\forall\delta>0$, $\forall\varepsilon\in(0,\varepsilon_0(\delta))$:

$$\begin{cases}\|x-r\|\le\varepsilon\\ \mathrm{den}(r)\le\varepsilon^{-4-\delta}\end{cases}$$

has a solution $r\in X(\mathbb{Z}[\frac{1}{p}])$.

Dynamical encoding.

$X(\mathbb{Z}[\frac{1}{p}])\subset$ "classifying space"? (e.g. $\mathbb{Z}^d\subset\mathbb{R}^d$)

$X(\mathbb{Z}[\frac{1}{p}])\underset{\text{diagonal}}{\subset} X(\mathbb{R})\times X(\mathbb{Q}_p)$, $\mathbb{Q}_p=$ p-adic numbers

$X(\mathbb{Q})\subset X(\mathbb{A})$, $\mathbb{A}=$ adeles.

Recall that G is a simple alg. group$/\mathbb{Q}$.

$G(\mathbb{Z}[\frac{1}{p}])\underset{\text{diagonal}}{\subset} G(\mathbb{R})\times G(\mathbb{Q}_p)$

discrete subgroup in the product with finite covolume

$$Y=(G(\mathbb{R})\times G(\mathbb{Q}_p))/G(\mathbb{Z}[\tfrac{1}{p}])$$

Consider the dynamical system:

$G(\mathbb{Q}_p)\circlearrowright Y$ (action by multiplication on the second coordinate).

Generalised Dani correspondence

Diophantine approximation

$X(\mathbb{Z}[\frac{1}{p}]) \subset X(\mathbb{R})$

$x \in X(\mathbb{R})$

$\|x - r\| \leq \varepsilon$

$\begin{cases} \|x-r\| \leq \varepsilon \\ \mathrm{den}(r) \leq p^n \end{cases}$ has a solution $r \in X(\mathbb{Z}[\frac{1}{p}])$

Dynamics

$G(\mathbb{Q}_p) \circlearrowright Y$

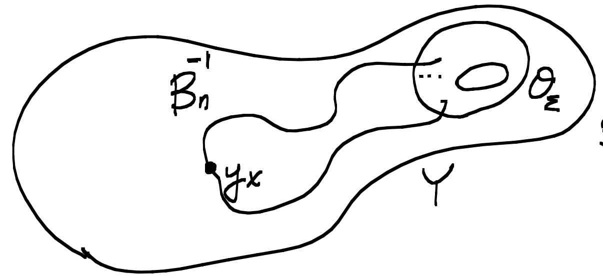
$y_x \in Y$

nbhood $\mathcal{O}_\varepsilon \subset Y$

$B_n^{-1} \cdot y_x \cap \mathcal{O}_\varepsilon \neq \emptyset$

where $B_n = \{ b \in G(\mathbb{Q}_p) : \|b^{-1}\|_p \leq p^n \}$.

This shows that the problem of Diophantine approximation amounts to the shrinking target property:



$B_n^{-1} \cdot y_x \cap \mathcal{O}_\varepsilon \neq \emptyset$

Previous works on (Diop. approximation $\longleftrightarrow$ Dyn. systems):
Dani, Kleinbock-Margulis, Hersonsky-Paulin, ...

Sketch of the proof of G.D.C. (for $X = G$):

Given $x \in G(\mathbb{R})$, we set
$y_x = (x, e)\, G(\mathbb{Z}[\frac{1}{p}]) \in Y$,
$\mathcal{O}_\varepsilon^\infty$ = ε-nbhd of identity in $G(\mathbb{R})$,
$\mathcal{O}^p$ = bounded open subset in $G(\mathbb{Q}_p)$,
$\mathcal{O} = (\mathcal{O}_\varepsilon^\infty \times \mathcal{O}^p)\, G(\mathbb{Z}[\frac{1}{p}]) \subset Y$.

Suppose that $B_n^{-1} y_x \cap \mathcal{O}_\varepsilon \neq \emptyset$. Then

$\exists\, b \in B_n,\ \omega_\infty \in \mathcal{O}_\varepsilon^\infty,\ \omega_p \in \mathcal{O}_p,\ \gamma \in G(\mathbb{Z}[\tfrac{1}{p}])$:

$$(x, b^{-1}) = (\omega_\infty \gamma, \omega_p \gamma) \Rightarrow \begin{array}{l} \gamma = \omega_\infty^{-1} x \underset{\varepsilon}{\approx} x \\ \gamma = \omega_p^{-1} b^{-1} \Rightarrow \|\gamma\|_p \ll p^n \Rightarrow den(\gamma) \ll p^n. \end{array}$$

In order to show that $B_n^{-1} y_x \cap \mathcal{O}_\varepsilon \neq \emptyset$ we need to have good control on distribution of orbits in Y, which is provided by the mean ergodic theorem with rate.

Mean ergodic Thm

Fix a good maximal compact subgroup $U_p \subset G(\mathbb{Q}_p)$.

Let $L^2_{00}(Y) = \{ f \in L^2(Y) :\ f\text{ - orthogonal to characters of } G(\mathbb{R}) \times G(\mathbb{Q}_p) \}$.

Note that $L^2_{00}(Y)$ has finite codimension in $L^2(Y)$.

We define (spherical) integrability exponent:

$$q_p(G) = \inf \left\{ q > 0 :\ \int_Y f_1(g^{-1}y) f_2(y)\, dy \in L^q(G(\mathbb{Q}_p)) \text{ for all } U_p\text{-inv. } f_1, f_2 \in L^2_{00}(Y) \right\}$$

Then $q_p(G) < \infty$ (Clozel),

$q_p(SO_3) = 2$ (Deligne + Jacquet-Langlands correspondence),

$q_p(SL_2) \leq \frac{64}{25}$ (Kim-Sarnak), but conjecturally $q_p(SL_2) = 2$.

For compact $B \subset G(\mathbb{Q}_p)$ with nonempty interior, we define an averaging operator $\pi_p(\beta_B) : L^2_{00}(Y) \to L^2_{00}(Y)$

$$\pi_p(\beta_B) f(y) = \frac{1}{\mathrm{Vol}(B)} \int_B f(b^{-1} y)\, db.$$

Mean Ergodic Thm: Assuming that B is bi-invariant under U_p, we have

$$\|\pi_\varphi(\beta_B)\| \leq \text{const}(\theta)\cdot \text{vol}(B)^{-\theta}$$

for every $\theta < q_p(G)^{-1}$.

Combining the mean ergodic thm with the generalised Dani correspondence, we obtain:

Thm Let $X \subset \mathbb{C}^d$ be an algebraic set/$\mathbb{Q}$ equipped with a transitive action of a simple algebraic group $G \subset GL_d$ defined over $\mathbb{Q}$.

Assume that $X(\mathbb{Z}[\frac{1}{p}])$ is not discrete in $X(\mathbb{R})$.

Then: - $\forall$ a.e. $x \in \overline{X(\mathbb{Z}[\frac{1}{p}])}$ $\forall \delta > 0$ $\forall \varepsilon \in (0, \varepsilon_0(x,\delta))$:

$$\begin{cases} \|x-r\| \leq \varepsilon \\ \text{den}(r) \leq \left(\varepsilon^{-\frac{\dim(X)}{a_p(G)}-\delta}\right)^{\frac{q_p(G)}{2}} \end{cases}$$

has a solution $r \in X(\mathbb{Z}[\frac{1}{p}])$.

- $\forall x \in \overline{X(\mathbb{Z}[\frac{1}{p}])}$ $\forall \delta > 0$ $\forall \varepsilon \in (0, \varepsilon_0(\delta))$:

$$\begin{cases} \|x-r\| \leq \varepsilon \\ \text{den}(r) \leq \left(\varepsilon^{-\frac{\dim(X)}{a_p(G)}-\delta}\right)^{q_p(G)} \end{cases}$$

has a solution $r \in X(\mathbb{Z}[\frac{1}{p}])$.

Rmk. Typically, $a_p(G) \geq a_p(X)$. Then the first part gives the best possible result when $q_p(G) = 2$.

Cor. $q_p(SL_d) \geq 2(d-1)$.

Proof. Consider the problem of Diophantine approximation

$$\mathbb{Z}[\tfrac{1}{p}]^d \subset \mathbb{R}^d.$$

By elementary considerations, this problem has positive solution for $\varkappa \geq 1$ only.

On the other hand, by our thm,

$$\varkappa \leq \left(\frac{d}{a_p(SL_d)} + \delta\right) \cdot \frac{q_p(SL_d)}{2} = \left(\frac{d}{d^2-d} + \delta\right) \cdot \frac{q_p(SL_d)}{2}$$

for all $\delta > 0$. This implies the estimate. $\square$