Lecture 1: Uniform spectral gap & prime number theorem.

Prime Number Thm: $\#\{p \le T : p\text{-prime}\} \sim \frac{T}{\log T}$ as $T \to \infty$.

Question: Given a polynomial $f \in \mathbb{Z}[x_1, \dots, x_d]$, does $f(\mathbb{Z}^d)$ contain infinitely many prime numbers?
(here we assume that f is irreducible and $\gcd(f(\mathbb{Z}^d)) = 1$).

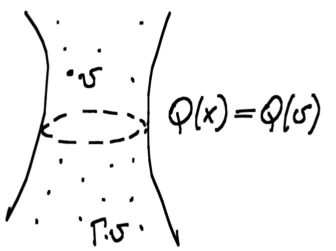
example (Iwaniec '78) $\#\{n \le T : \underbrace{n^2+1 \text{ is } 2\text{-prime}}_{= p_1 \cdot p_2 \text{ with } p_1 \& p_2 \text{ primes.}}\} \ge \text{const} \cdot \frac{T}{\log T}$

It is not known weither there are infinitely many primes of the form n^2+1.

Sarnak's Programme:

Γ - a "large" subgroup of $GL_d(\mathbb{Z})$
$v \in \mathbb{Z}^d$
$\mathcal{O} = \Gamma \cdot v$
$f \in \mathbb{Z}[x_1, \dots x_d]$
Does $f(\mathcal{O})$ contain infinitely many primes/$\underbrace{r\text{-primes}}_{p_1 \dots p_r}$?

example $\left(\begin{array}{l}\text{Liu-Sarnak}, d=3 \\ \text{G.-Nevo}, d>3\end{array}\right)$ $Q(x) = \sum_{i,j=1}^{d} a_{ij} x_i x_j$ - nondegenerate indefinite integral quadratic form.



$v \in \mathbb{Z}^d$ such that $Q(v) \neq 0$.
$\Gamma = SO_Q(\mathbb{Z})$
$f \in \mathbb{Z}[x_1, \dots x_d]$ such that
$f|_{\{Q(x)=Q(v)\}}$ is irreducible/$\mathbb{C}$.

Then $\exists r \ge 1$: $\#\left\{x \in \Gamma \cdot v : \begin{array}{l}\|x\| \le T \\ f(x) \text{ is } r\text{-prime}\end{array}\right\} \ge \text{const} \cdot \frac{\#\{x \in \Gamma \cdot v : \|x\| \le T\}}{\log T}$.

Sieving for primes.

Let $\mathcal{O}(T) = \{x \in \mathcal{O} : \|x\| \leq T\}$ and $\mathcal{O}(T, \ell) = \{x \in \mathcal{O} : \|x\| \leq T,\ f(x) = 0 \ (\mathrm{mod}\ \ell)\}$.

We need to estimate:

$$S(T, z) = \#\{x \in \mathcal{O}(T) : f(x) \text{ is coprime to primes } p \leq z\}.$$

Note that if $z = T^\alpha$, then $f(x)$ has at most $r \leq \frac{\deg(f)}{\alpha}$ prime factors for sufficiently large T.

We set-up an inclusion/exclusion formula using the Mobius function:

$$\mu(\ell) = \begin{cases} (-1)^r, & \ell = \text{product of } r \text{ distict prime factors} \\ 0, & \text{——}\,\shortparallel\,\text{——}. \end{cases}$$

Then $\displaystyle\sum_{\ell | n} \mu(\ell) = \begin{cases} 1, & n = 1, \\ 0, & \text{—}\shortparallel\text{—}. \end{cases}$

Setting $\displaystyle P_z = \prod_{\text{prime } p \leq z} p$, we have

$$S(T, z) = \sum_{k : (k, P_z) = 1} |\mathcal{O}(T) \cap \{f = k\}|$$

$$= \sum_k \Big(\sum_{\ell | (k, P_z)} \mu(\ell) \Big) \cdot |\mathcal{O}(T) \cap \{f = k\}|$$

$$= \sum_{\ell | P_z} \mu(\ell) \cdot \underbrace{\sum_{k : \ell | k} |\mathcal{O}(T) \cap \{f = k\}|}_{|\mathcal{O}(T, \ell)|}$$

We hope to have an estimate:

$$(*) \quad \boxed{|\mathcal{O}(T, \ell)| = \rho(\ell) \cdot |\mathcal{O}(T)| + \mathrm{ERROR}(T, \ell),}$$

where $\displaystyle \rho(\ell) = \frac{|\{\mathcal{O} \bmod \ell\} \cap \{f = 0 \bmod \ell\}|}{|\{\mathcal{O} \bmod \ell\}|}$, and

the error $E(T,\ell)$ is sufficiently uniform in ℓ.
Then suming over $\ell \mid P_z$, we might be able to estimate $S(T,z)$.

A number of much more efficient sieving techiniques have been developed starting with the work of Brunn. In any case, one needs $(*)$ as a starting point.

Then

Let $\Gamma(\ell) = \{\gamma \in \Gamma : \gamma = id \pmod{\ell}\}$.

$$\Theta(T,\ell) = \bigsqcup_{i=1}^{N_\ell} \{x \in \Gamma(\ell)\gamma_i v : \|x\| \le T\}.$$

Hence, it is sufficient to estimate $\#\{x \in \Gamma(\ell) v_i : \|x\| \le T\}$.

Now we assume that Γ is a lattice (i.e., a discrete subgroup with finite covolume) in a closed connected subgroup $G < GL_d(\mathbb{R})$.

Here analysis on spaces $G/\Gamma(\ell)$ comes into play...
Uniformity in ℓ?

Uniform spectral gap.

example.

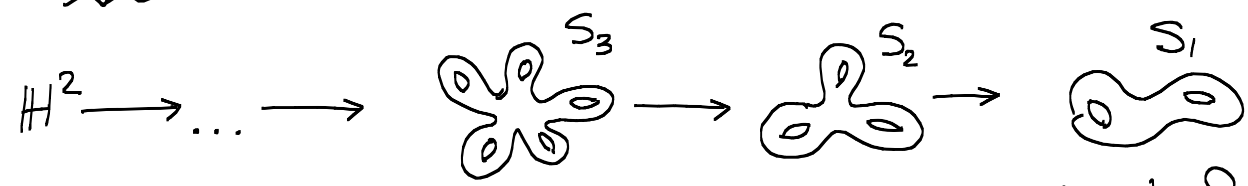
$$\begin{array}{ccccccc} & & S_k & & S_2 & & S_1 \\ & & \| & & \| & & \| \\ \mathbb{R} & \longrightarrow \cdots \longrightarrow & \mathbb{R}/2^k\mathbb{Z} & \longrightarrow \cdots \longrightarrow & \mathbb{R}/2\mathbb{Z} & \longrightarrow & \mathbb{R}/\mathbb{Z} \end{array}$$

$$\left\{e^{\frac{2\pi i j x}{2^k}}\right\}_{j\in\mathbb{Z}} \qquad \left\{e^{\frac{2\pi i j x}{2}}\right\}_{j\in\mathbb{Z}} \qquad \left\{e^{2\pi i j x}\right\}_{j\in\mathbb{Z}}$$

Let $\lambda(S_k)$ = bottom of the spectrum of the Laplace operator on S_k excluding zero.

Clearly, $\lambda(S_\kappa) \xrightarrow[\kappa\to\infty]{} 0 = \lambda(\mathbb{R})$

Hyperbolic surfaces:



Is it true that $\lambda(S_\kappa) \xrightarrow[\kappa\to\infty]{} \lambda(\mathbb{H}^2) = \frac{1}{4}$?

Selberg '65: – examples of covers with $\lambda(S_\kappa) \to 0$.
– for $S_\ell = \Gamma(\ell)\backslash\mathbb{H}^2$ where $\Gamma(\ell) = \{\gamma \in SL_2(\mathbb{Z}) : \gamma = id \ (\mathrm{mod}\ \ell)\}$

property (τ) $\to \left[\lambda(S_\ell) \geq \frac{3}{16} \right]$

(the best known estimate: $\lambda(S_\ell) \geq \frac{975}{4096}$ (Kim-Sarnak)
but conjecturally $\lambda(S_\ell) \geq \frac{1}{4}$ (Selberg))

Kazhdan '67: G = a simple noncompact Lie group with $\mathrm{Rank}(G) \geq 2$ (e.g., $SL_d(\mathbb{R})$ with $d \geq 3$)

Given a compact $B \subset G$ with nonempty interior, $B^{-1} = B$, and a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, we define an averaging operator:

$$\pi(\beta_B)v = \frac{1}{vol(B)} \int_B \pi(g) v \, dg, \quad v \in \mathcal{H}.$$

Kazhdan showed that:

property (T) $\to \left[\begin{array}{l} \exists\, \delta = \delta(B) < 1 : \quad \|\pi(\beta_B)\| \leq \delta \\ \text{for all } \pi\text{'s without } G\text{-inv. vectors.} \end{array} \right.$

Cowling, Howe-Moore '79: Given a unitary representation $\pi: G \to U(\mathcal{H})$, we define an integrability exponent:

$$q(\pi) = \inf\{q>0 : \langle \pi(g)v_1, v_2\rangle \in L^q(G) \text{ for } v_1, v_2 \in \text{dense subspace of } \mathcal{H}\}.$$

Then for higher rank semisimple Lie groups

property (T) $\to$ $\left[\ \sup_{\pi} q(\pi) < \infty\right.$

where π is a unitary representation without G-inv. vectors.

Clozel '03: G = a simply connected simple algebraic group$/\mathbb{Q}$ (e.g. $G = SL_d$)

$$\Gamma(\ell) = \{\gamma \in G(\mathbb{Z}) : \gamma = id \pmod{\ell}\}.$$

Consider the family of unitary representations:

$$\pi_\ell : G(\mathbb{R}) \curvearrowright L^2_0(G(\mathbb{R})/\Gamma(\ell)).$$

Then

property (τ) $\to$ $\left[\ q(G) = \sup_{\ell} q(\pi_\ell) < \infty.\right.$

Property (τ) leads to:

Quantitative Mean Ergodic Thm:

G = a simply connected simple algebraic group$/\mathbb{Q}$ such that $G(\mathbb{R})$ is not compact

$\Gamma(\ell) = \{\gamma \in G(\mathbb{Z}) : \gamma = id \pmod{\ell}\}$

B = a compact subset of $G(\mathbb{R})$ with nonempty interior

$\pi_\ell : G(\mathbb{R}) \curvearrowright L^2_0(G(\mathbb{R})/\Gamma(\ell))$.

Then $\quad \|\pi_\ell(\beta_B)\| \le \text{const}(\theta)\cdot \text{vol}(B)^{-\theta},$

where $\Theta < (2n_e(\mathfrak{g}(G)))^{-1}$,
$n_e(q)$ = least even integer $\geq p/2$.

Counting for congruence subgroups.

Combining the above ergodic theorem with the argument for counting lattice points (from Nevo's lectures), we obtain:

<u>Thm</u>: Let $B_T \subset G(\mathbb{R})$ be a well rounded family of sets. Then for every T, $\gamma \in \Gamma(1)$, $\ell \geq 1$,

$$(**) \quad \#(\Gamma(\ell)\gamma \cap B_T) = \frac{vol(B_T)}{vol(G(\mathbb{R})/\Gamma(\ell))} + O_\theta\left(vol(B_T)^{1-\frac{\theta}{1+\dim(G)}}\right).$$

<u>Rmk</u>: It is crucial for the application to almost primes that the error term is uniform in γ and ℓ.

Once we established $(**)$, we can adopt the classical sieving arguments to deduce:

<u>Thm</u> (Liu-Sarnak, Nevo-Sarnak, G.-Nevo) Given $v \in \mathbb{Z}^d$ such that $Stab_G(v)$ is symmetric, and $f \in \mathbb{Z}[x_1, \dots x_d]$ such that $g \mapsto f(gv)$ is irreducible$/\mathbb{C}$ and $\gcd(f(\mathcal{O})) = 1$ where $\mathcal{O} = G(\mathbb{Z})v$; there exists (explicit) $r = r(G, v, deg(f)) \geq 1$ such that

$$\#\{x \in \mathcal{O}(T) : f(x) \text{ is } r\text{-prime}\} \geq const \cdot \frac{|\mathcal{O}(T)|}{T}.$$

In conclusion, we explain how to deduce estimate on

$$\#\{x \in \Gamma(\ell)\gamma\cdot v : \|x\| \leq T\}$$

from (**). For this we construct well-rounded $B_T \subset G(\mathbb{R})$:

$$\begin{tikzcd}
g \arrow[d, mapsto] & \in & G(\mathbb{R}) \arrow[d] & \supset & B_T \arrow[d, two heads] & \supset & \Gamma(1)\cap B_T \arrow[d, two heads] \\
g\cdot v & \in & G(\mathbb{R})\cdot v & \supset & \{x \in G(\mathbb{R})v : \|x\| \leq T\} & \supset & \mathcal{O}(T)
\end{tikzcd}$$

such that B_T surjects onto $G(\mathbb{R})v \cap \{\|x\| \leq T\}$ and the map $\Gamma(1)\cap B_T \to \mathcal{O}(T)$ is onto and has fibers with uniformly bounded cardinalities.

This is the place where we use that $\mathrm{Stab}_G(v)$ is symmetric.