Lecture IV: Diophantine approximation on quadratic surfaces & Ramanujan Conjecture.

Let $X \subset \mathbb{R}^d$ be the set of solutions of a system of polynomial equations, and the set $X(\mathbb{Q})$ of rational points is dense in X.

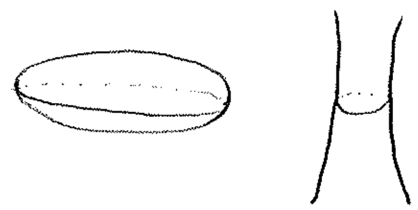
Diophantine approximation??

Waldschmidt considered elliptic curves/abelean varieties.

Let $Q(x) = \sum_{i,j=1}^{3} a_{ij} x_i x_j$ be nondegenerate quadratic form.

$a_{ij}, b \in \mathbb{Q}$

$X = \{ x \in \mathbb{R}^3 : Q(x) = b \}$



Fix a prime p, $\overline{X(\mathbb{Z}[\frac{1}{p}])} = X$

(this equivalent to Q being isotropic over $\mathbb{Q}_p$) and $X(\mathbb{Z}[\frac{1}{p}]) \neq \emptyset$.

Question: Given $x \in X$ and $\alpha > 0$,

can we solve
$$\begin{cases} \|x - r\| \leq R^{-\alpha}, \\ \mathrm{den}(r) \leq R, \\ r \in X(\mathbb{Z}[\frac{1}{p}]) \end{cases}$$

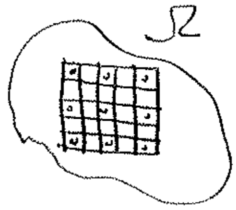
for sufficiently large R?

A lower bound.

Fix $\Omega \subset X$ such that $\overline{\Omega}$ is compact and has nonempty interior. Suppose that $\forall x \in \Omega$, one can solve:

$$\begin{cases} \|x - r\| \leq \varepsilon \\ \mathrm{den}(r) \leq R \\ r \in X(\mathbb{Z}[\tfrac{1}{p}]) \end{cases} \qquad \text{as } \varepsilon \to 0^+.$$

$\overline{\Omega}$



We have

$\#\{x \in X(\mathbb{Z}[\frac{1}{p}]) : x \in \overline{\Omega},\ \mathrm{den}(r) \leq R\} \asymp R$,

and since $\overline{\Omega}$ has nonempty interior,

$\#\{2\varepsilon\text{-seperated points in } \Omega\} \asymp \varepsilon^{-2}$.

By the pigeonhole principle,

$$R >> \varepsilon^{-2} \Rightarrow \boxed{\alpha \leq \tfrac{1}{2}}$$

Thm. Assume that either:

- X is compact, or
- Ramanujan Conj. holds.

Then for a.e. $x \in X$, every $\delta > 0$, and $R \geq R_0(x, \delta)$:

$$\begin{cases} \|x - r\| \leq R^{-\frac{1}{2}+\delta}, \\ \mathrm{den}(r) \leq R, \\ r \in X(\mathbb{Z}[\tfrac{1}{p}]), \end{cases}$$

has a solution.

A dynamical system.

Let $G = \{g \in SL_3 : Q(g\cdot x) = Q(x) \text{ for } x \in \mathbb{R}^3\}$.

For simplicity, we assume that $G \simeq SL_2$.

Let $G_\infty = SL_2(\mathbb{R})$, $G_p = SL_2(\mathbb{Q}_p)$, $\Gamma = SL_2(\mathbb{Z}[\frac{1}{p}])$.

We consider Γ as a subgroup of $G_\infty \times G_p$.

$$\Gamma \ni \gamma \longmapsto (\gamma, \gamma) \in G_\infty \times G_p.$$

<u>Lem.</u> Γ is a lattice in $G_\infty \times G_p$.

<u>Proof</u>: 1) Γ is discrete:

Suppose that for $\gamma \in \Gamma$: $\|\gamma - Id\|_\infty < 1$ and $\|\gamma - Id\|_p < 1$.

Then $\|\gamma - Id\|_p < 1 \Rightarrow \|\gamma\|_p \le 1 \Rightarrow \gamma \in G(\mathbb{Z}_p)$.

(Recall that $|\frac{m}{p^k}|_p = p^k$ for $(m,p)=1$)

Now $\|\gamma - Id\|_\infty < 1 \Rightarrow \gamma = Id$.

2) Γ has finite covolume

Note that Γ is dense in G_p

(since $\mathbb{Z}[\frac{1}{p}]$ is dense in $\mathbb{Q}_p$ and G_p is generated by $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$).

Recall that $\boxed{G_\infty = G(\mathbb{Z})\cdot F}$ where $F =$  $\subset \mathbb{H}^2$.

Since Γ is dense in G_p and $G(\mathbb{Z}_p)$ is open G_p,

$$\boxed{G_p = G(\mathbb{Z}[\tfrac{1}{p}])\cdot G(\mathbb{Z}_p)}$$

Now for $(g_\infty, g_p) \in G_\infty \times G_p$,

$$(g_\infty, g_p) = (\underset{\Gamma}{\underset{\cap}{\gamma_1}}, \gamma_1)(g'_\infty, \underset{G(\mathbb{Z}_p)}{\underset{\cap}{g'_p}}) = (\underset{G(\mathbb{Z})}{\underset{\cap}{\gamma_1\gamma_2}}, \gamma_1\gamma_2)(\underset{F}{\underset{\cap}{g''_\infty}}, \underset{G(\mathbb{Z}_p)}{\underset{\cap}{g''_p}}).$$

We conclude that $G_\infty \times G_p = \Gamma \cdot (F \times G(\mathbb{Z}_p))$. $\lrcorner$

Consider the "dynamical system":

$$\boxed{Z = \Gamma \backslash (G_\infty \times G_p) \circlearrowleft G_p .}$$

For a compact $B \subset G_p$ with $\mathrm{vol}(B) > 0$, consider the averaging operator:

$$\begin{array}{cccc} A_B : & L^2(Z) & \longrightarrow & L^2(Z) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & f & \longmapsto & \frac{1}{\mathrm{vol}(B)} \int_B f(z \cdot (1,b))\, db . \end{array}$$

[<u>Mean Ergodic Thm</u> (assuming that G_∞ is compact or Ramanujan conj. holds)

$\forall f \in L^2(Z) \quad \forall \delta > 0$:

$$\| A_B(f) - \int_Z f \|_2 \leq c(\delta) \mathrm{vol}(B)^{-\frac{1}{2}+\delta} \|f\|_2 .$$]

This Thm will be proved in the second part of the lecture.

Duality principle

Fix $x^\circ \in X(\mathbb{Z}[\frac{1}{p}])$ and set $G_\infty^\circ = \mathrm{Stab}_{G_\infty}(x^\circ)$.

$$\begin{tikzcd}[column sep=small]
& G_\infty \times G_p \arrow[dl] \arrow[dr] & \\
\Gamma \circlearrowright X \simeq (G_\infty \times G_p)/(G_\infty^\circ \times G_p) & & \Gamma\backslash(G_\infty \times G_p) = Z \circlearrowleft G_p
\end{tikzcd}$$

For $g \in G_\infty$, set $x_g = g x^\circ \in X$ and $z_g = \Gamma(g,1) \in Z$

Let $\mathcal{O}_\varepsilon = \{x \in X : \|x - x^\circ\| < \varepsilon\}$. Then $\mathrm{vol}(\mathcal{O}_\varepsilon) \asymp \varepsilon^2$.

Let $\tilde{\mathcal{O}}_\varepsilon = \mathcal{O}_\varepsilon^\infty \times \mathcal{O}^p$ where $\mathcal{O}_\varepsilon^\infty$ be a nbhd of identity in G_∞ such that $\mathcal{O}_\varepsilon^\infty \cdot x^\circ = \mathcal{O}_\varepsilon$ and $\mathrm{vol}(\mathcal{O}_\varepsilon^\infty) \asymp \varepsilon^2$, and $\mathcal{O}^p = G(\mathbb{Z}_p)$.

Let $\mathcal{O}_\varepsilon' = \Gamma \cdot \tilde{\mathcal{O}}_\varepsilon \subset Z$. Then $\mathrm{vol}(\mathcal{O}_\varepsilon') \asymp \varepsilon^2$.

Let $B_n = \{g \in G_p : \|g\|_p \le p^n\}$.

Duality principle:

Suppose that $\varepsilon \in (0,1)$ and $n \ge 0$:

$$z_g \cdot B_n \cap \mathcal{O}_\varepsilon' \ne \emptyset.$$

Then the system of inequalities

$$\begin{cases} \|x_g - r\| < c(g)\cdot \varepsilon, \\ \mathrm{den}(r) \le p^n, \\ r \in X(\mathbb{Z}[\frac{1}{p}]) \end{cases}$$

has a solution.

<u>Proof.</u> The assumption implies that

$\exists \gamma \in \Gamma : \exists b \in B_n : (\gamma^{-1}, \gamma^{-1})(g, b) \in \tilde{\Theta}_\varepsilon = \Theta_\varepsilon^\infty \times \Theta^p.$

Then

$$\begin{cases} \gamma^{-1} g \in \Theta_\varepsilon^\infty \\ \gamma^{-1} b \in \Theta^p \end{cases} \Longleftrightarrow \begin{cases} \gamma \in g(\Theta_\varepsilon^\infty)^{-1} \\ \gamma \in b(\Theta^p)^{-1} = b \cdot G(\mathbb{Z}_p) \end{cases}$$

This implies that

$$\|\gamma\|_\infty \asymp \|g\|_\infty$$

$$\|\gamma\|_p = \|b\|_p \le p^n \Rightarrow \|\gamma \cdot x^\circ\|_p << \|\gamma\|_p \le p^n.$$

We take $r = \gamma \cdot x^\circ$.

$$\| x_g - r \| = \| g \cdot x^\circ - \gamma \cdot x^\circ \| << \| \underbrace{\gamma^{-1} g}_{\in \Theta_\varepsilon} \cdot x^\circ - x^\circ \| < \varepsilon .$$

$\square$

<u>Proof of Main Thm.</u> (Assuming the Mean Ergodic Thm).

Let $\varepsilon_n = p^{(-\frac{1}{2} + \delta)n}$ with $\delta > 0$.

We shall show that for a.e. $z \in Z$ and $n \ge n_0(z, \delta)$: $\quad z \cdot B_n \cap \Theta'_{\varepsilon_n} \ne \emptyset$.

The the Thm follows from Duality Principle.

Let $f_n = \chi_{\Theta'_{\varepsilon_n}}$ (the characteristic function of Θ'_{ε_n})

and $\quad Z_n = \{ z \in Z : z \cdot B_n \cap \Theta'_{\varepsilon_n} = \emptyset \}$.

By the Mean Ergodic Thm,

$$\int_Z \left| A_{B_n}(f_n) - \int_Z f_n \right|^2 \ll_\delta \mathrm{vol}(B_n)^{-1+\delta} \cdot \|f_n\|_2^2$$

Since for $z \in Z_n$, $A_{B_n}(f_n)(z) = 0$, we have:

$$\mathrm{vol}(Z_n) \cdot \left(\int_Z f_n \right)^2 \ll \mathrm{vol}(B_n)^{-1+\delta} \cdot \|f_n\|_2^2, \text{ and}$$

$$\mathrm{vol}(Z_n) \ll \mathrm{vol}(\mathcal{O}'_{\varepsilon_n})^{-1} \cdot \mathrm{vol}(B_n)^{-1+\delta}.$$

Note that $\mathrm{vol}(B_n) \asymp p^n$ (a computation)

$$\mathrm{vol}(\mathcal{O}'_{\varepsilon_n}) \asymp \varepsilon_n^2 = p^{(-1+2\delta)n}.$$

Hence, $\mathrm{vol}(Z_n) \ll p^{(1-2\delta)n} \cdot p^{(-1+\delta)n} = p^{-\delta n}$.

This implies that $\sum_{n \geq 0} \mathrm{vol}(Z_n) < \infty$, and by Borel-Cantelli Lemma,

$$\mathrm{vol}(\overline{\lim}\, Z_n) = 0.$$

Then for a.e. $z \in Z$: $z \in Z_n$ for only finitely many n's, and $z \cdot B_n \cap \mathcal{O}'_{\varepsilon_n} \neq \emptyset$ for $n \geq n_0(z, \delta)$, as required.

$\square$

Proof of Mean Ergodic Thm.

Ramanujan Conjecture:

- estimates on eigenvalues of Hecke/Laplace operators of modular/Maas forms
- description of automorphic representations
- integrability property:

For a dense family of functions

$$f_1, f_2 \in L_0^2(Z) = \{ f \in L^2(Z) : \int_Z f = 0 \},$$

$$\langle f_1 \circ (e,b), f_2 \rangle \in L^{2+\varepsilon}(G_p) \text{ for every } \varepsilon > 0.$$
$$b \in G_p$$

Let $f : G_p \to [0,\infty)$ be a measurable bounded function such that $\overline{\{f \neq 0\}} = \Omega$ is compact, and

$$f(g) = f(g^{-1}) \text{ for } g \in G_p.$$

We consider the averaging operator:

$$A_f : L_0^2(Z) \longrightarrow L_0^2(Z)$$
$$\varphi \longmapsto \int_{G_p} f(g)\varphi(z\cdot(1,g))\,dg$$

Note that $A_B = A_f$ for $f = \chi_B/\mathrm{vol}(B)$.

Recall that convolution is defined by:

$$(f_1 * f_2)(g) = \int_{G_p} f_1(gh^{-1}) f_2(h)\,dh, \quad f_1, f_2 : G_p \to \mathbb{R}.$$

Lem. 1) $A_f^n = A_{f^{*n}}$,

2) $A_f^* = A_f$.

Proof: direct computation.

We also consider an averaging operator on $L^2(G_p)$.

$$R_f : L^2(G_p) \longrightarrow L^2(G_p)$$
$$\varphi \longmapsto \int_{G_p} f(g)\varphi(hg)\,dg.$$

Cowling - Hagerup - Howe Inequality:

$$\|A_f\| \leq \|R_f\|.$$

Proof: Since A_f is self-adjoint operator, its norm can be computed by the formula:

$$\|A_f\| = \sup_{\varphi \in \mathcal{L}} \lim_{n\to\infty} |\langle A_f^n \varphi, \varphi\rangle|^{1/n}.$$

where $\mathcal{L}$ is a dense subspace of $L_0^2(Z)$.

We have:

$$\langle A_f^n \varphi, \varphi\rangle = \langle A_{f^{*n}} \cdot \varphi, \varphi\rangle = \int_Z \left(\int_{G_p} f^{*n}(g)\varphi(zg)\overline{\varphi(z)}\,dg\right) dz$$

$$= \int_{\Omega^n} f^{*n}(g) \underbrace{\left(\int_Z \varphi(z\cdot g)\overline{\varphi(z)}\,dz \right)}_{=\psi(g)} dg.$$

The by Cauchy-Schwartz and Hölder Inequalities:

$$|\langle A_f^n \varphi, \varphi\rangle| \leq \left(\int_{\Omega^n} |f^{*n}|^2 dg\right)^{1/2} \cdot \left(\int_{\Omega^n} |\psi|^2 dg\right)^{1/2}$$

$$\leq \|f^{*n}\|_2 \cdot \left(\int_{\Omega^n} |\psi|^{2+\varepsilon} dg\right)^{\frac{1}{2+\varepsilon}} \cdot \left(\int_{\Omega^n} 1\, dg\right)^{\frac{1}{2}-\frac{1}{2+\varepsilon}}$$

$$\leq \|f^{*n}\|_2 \cdot \|\psi\|_{2+\varepsilon} \cdot \mathrm{vol}(\Omega^n)^{\frac{1}{2}-\frac{1}{2+\varepsilon}}.$$

Lem. For some $C>0$, $\mathrm{vol}(\Omega^n) \leq C^n$.

Without loss of generality, we may assume that Ω has nonempty interior. Then by compactness,

$$\Omega^2 \subset \bigcup_{i=1}^N \omega_i \Omega, \text{ and}$$

$$\Omega^{n+1} = \bigcup_{i_1,\dots i_n=1}^N \omega_{i_1}\dots\omega_{i_n}\Omega \Rightarrow \mathrm{vol}(\Omega^{n+1}) \leq N^n \cdot \mathrm{vol}(\Omega).$$

Now using that $L^2_c(Z)$ contains a dense family of functions φ such that $\psi(g) = \langle \varphi\cdot g, \varphi\rangle$ is in $L^{2+\varepsilon}(G_p)$ for every $\varepsilon>0$, we conclude that

$$\|A_f\| \leq \lim_{n\to\infty} \|f^{*n}\|_2^{1/n} \cdot C^{\frac{1}{2}-\frac{1}{2+\varepsilon}}$$

for every $\varepsilon>0$. Hence,

$$\boxed{\|A_f\| \leq \lim_{n\to\infty} \|f^{*n}\|_2^{1/n}.}$$

We have: $R_f \cdot f = \int\limits_{G_p} f(xg)f(g)dg = \int\limits_{G_p} f(xg)f(g^{-1})dg$

$$= \int\limits_{G} f(xg^{-1})f(g)dg = f^{*2}.$$

Hence, $f^{*n} = R_f^{n-1} \cdot f$ and

$$\|f^{*n}\| \leq \|R_f\|^{n-1} \cdot \|f\|.$$

This implies that

$$\boxed{\lim_{n\to\infty} \|f^{*n}\|^{1/n} \leq \|R_f\|.}$$

Review of invariant measures.

The group G_p supports a biinvariant measure:

$$\int\limits_{G_p} f(g) = \int\limits_{G_p} f(g_1 g g_2)dg, \quad \text{for all } g_1, g_2 \in G_p.$$

We have $G_p = PK$ where $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $K = G(\mathbb{Z}_p)$.

$$\int\limits_{G_p} f(g)dg = \int\limits_{P\times K} f(pk)\,dp\,dk,$$

where dp is left-invariant measure on P, and dk is biinvariant probability measure on K.

The measure dp is <u>not</u> right-invariant, but

$$\int\limits_{P} f(pp_0)dp = \Delta(p_0)\int\limits_{P} f(p)dp,$$

where $\Delta : P \to \mathbb{R}^\times$ is explicit homomorphism.

Kunze-Stein Inequality

K.-S. Inequality: For every $\varphi \in L^2(G_p)$ and $s \in [1,2)$,

$$\|\varphi * f\|_s \leq c(s) \cdot \|\varphi\|_2 \cdot \|f\|_s.$$

Proof (assuming that f is K-biinvariant)

It is sufficient to show that $\forall \psi \in L^2(G_p)$:

$$|\langle \varphi * f, \psi \rangle| \leq c(s) \cdot \|f\|_s \cdot \|\varphi\|_2 \cdot \|\psi\|_2.$$

We have:

$$|\langle \varphi * f, \psi \rangle| = \left| \int_{G_p} \left(\int_{G_p} \varphi(xy^{-1}) f(y)\, dy \right) \psi(x)\, dx \right|$$

$$\leq \int_{G_p} f(y) \left(\int_{G_p} |\varphi(xy^{-1}) \psi(x)|\, dx \right) dy$$

$$= \int_{G_p} f(y) \left(\int_{P \times K} |\varphi(pky^{-1}) \psi(pk)|\, dp\, dk \right) dy$$

(Cauchy-Schwartz inequality)

$$\overset{\downarrow}{\leq} \int_{G_p \times K} f(y) \cdot \underbrace{\left(\int_P |\varphi(pky^{-1})|^2 dp \right)^{1/2}}_{\tilde{\varphi}(ky^{-1})} \cdot \underbrace{\left(\int_P |\psi(pk)|^2 dp \right)^{1/2}}_{\tilde{\psi}(k)} dy\, dk$$

$$= \int_{G_p \times K} f(y)\, \tilde{\varphi}(ky^{-1})\, \tilde{\psi}(k)\, dy\, dk$$

(since f is K-invariant)

$$\overset{\downarrow}{=} \int_{G_p \times K} f(yk_1) \left(\int_K \tilde{\varphi}(ky^{-1})\, \tilde{\psi}(k)\, dk \right) dy\, dk,$$

$$= \int_{G_p} f(y) \left(\int_{K\times K} \tilde{\varphi}(kk_1 y^{-1}) \tilde{\varphi}(k)\, dk\, dk_1 \right) dy$$

$$= \int_{G_p} f(y) \left(\int_{K\times K} \tilde{\varphi}(k_1 y^{-1}) \tilde{\varphi}(k)\, dk\, dk_1 \right) dy$$

$$= \left(\int_{G_p} f(y) \left(\int_K \tilde{\varphi}(k_1 y^{-1})\, dk_1 \right) dy \right) \cdot \left(\int_K \tilde{\varphi}(k)\, dk \right).$$

$$\int_{G_p\times K} f(y)\, \tilde{\varphi}(k_1 y^{-1})\, dk_1\, dy = \int_{G_p\times K\times K} f(k_2 y)\, \tilde{\varphi}(k_1 y^{-1})\, dy\, dk_1\, dk_2$$

$$= \int_{G_p} f(y) \left(\int_{K\times K} \tilde{\varphi}(\underbrace{k_1 y^{-1}}_{} k_2)\, dk_1\, dk_2 \right) dy$$

$= p(k_1 y^{-1}) \cdot k(k_1 y^{-1})$ for $p(k_1 y^{-1}) \in P$, $k(k_1 y^{-1}) \in K$.

$$= \int_{G_p} f(y) \left(\int_{K\times K} \tilde{\varphi}(p(k_1 y^{-1}) \cdot \underbrace{k(k_1 y^{-1})}_{} k_2)\, dk_1\, dk_2 \right) dy$$

cancels by invariance of dk_2.

$$= \int_{G_p} f(y) \left(\int_{K\times K} \tilde{\varphi}(p(k_1 y^{-1}) k_2)\, dk_1\, dk_2 \right) dy$$

Now $\tilde{\varphi}(p_0 \cdot g) = \left(\int_P |\varphi(pp_0 g)|^2\, dp \right)^{1/2} = \Delta(p_0)^{1/2} \left(\int_P |\varphi(pg)|^2\, dp \right)^{1/2}.$

Hence, $\int_{K\times K} \tilde{\varphi}(p(k_1 y^{-1}) k_2)\, dk_1\, dk_2 = \left(\int_K \Delta(p(k_1 y^{-1}))^{1/2} dk_1 \right) \cdot \left(\int_K \tilde{\varphi}(k_2)\, dk_2 \right).$

__Def.__ The function $\Xi(y) = \int_K \Delta(p(k_1 y^{-1}))^{1/2} dk_1$, $y \in G_p$, is called the __Harish-Chandra function__

Rmk: In the case $G_p = PGL_2(\mathbb{Q}_p)$, Ξ can be computed explicitly:

$$\Xi\left(k_1 \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} k_2\right) = p^{-\frac{n}{2}} \cdot \left(1 + \frac{p-1}{p+1} \cdot n\right).$$

Also, one can check that

$$\Xi \in L^t(G_p) \text{ for every } t > 2.$$

We conclude that

$$|\langle \varphi * f, \psi \rangle| \le \langle f, \Xi \rangle \cdot \left(\int_K \tilde{\varphi}(k)\, dk\right) \cdot \left(\int_K \tilde{\psi}(k)\, dk\right).$$

By Hölder inequality,

$$|\langle f, \Xi \rangle| \le \|f\|_s \cdot \|\Xi\|_t \quad \text{where} \quad \frac{1}{s} + \frac{1}{t} = 1.$$

By Jensen inequality,

$$\int_K \left(\int_P |\varphi(pk)|^2 dp\right)^{1/2} dk \le \left(\int_{P \times K} |\varphi(pk)|^2 dp\, dk\right)^{1/2} = \|\varphi\|_2,$$

and similarly, $\left(\int_K \tilde{\psi}(k)\, dk\right) \le \|\psi\|_2$.

Hence,

$$|\langle \varphi * f, \psi \rangle| \le \|\Xi\|_t \cdot \|f\|_s \cdot \|\varphi\|_2 \cdot \|\psi\|_2. \quad \lrcorner$$

Proof of Mean Ergodic Thm.

Now the thm follows from Cowling-Hagerup-Howe Inequality and Kunze-Stein Inequality, applied to the function $f = \frac{\chi_{B_n}}{vol(B_n)}$. $\lrcorner$