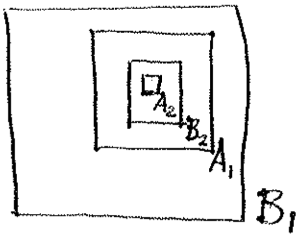
Lecture III. Schmidt games, badly approximable numbers, and bounded geodesics.

Schmidt game.

Fix a subset S of $\mathbb{R}^d$ and parameters $\alpha, \beta \in (0,1)$.

2 players: Alice and Bob.



- Bob picks a (closed) cube $B_1 \subset \mathbb{R}^d$,
- Alice picks a cube $A_1 \subset B_1$ of size $\alpha \cdot \text{size}(B_1)$,
- Bob picks a cube $B_2 \subset A_1$ of size $\alpha\beta \cdot \text{size}(B_1)$,
- Alice picks a cube $A_2 \subset B_2$ of size $\alpha^2\beta \cdot \text{size}(B_1)$,

. . . .

At the end, $\bigcap_{n \geq 1} A_n = \bigcap_{n \geq 1} B_n = \{x\}$.

Alice wins if $x \in S$.

Def. 1) $S \subset \mathbb{R}^d$ is called (α,β)-winning if Alice can design a strategy so that she can always win, regardless of moves Bob chooses.

2) S is called α-winning if it is (α,β)-winning for every $\beta \in (0, \beta_0)$.

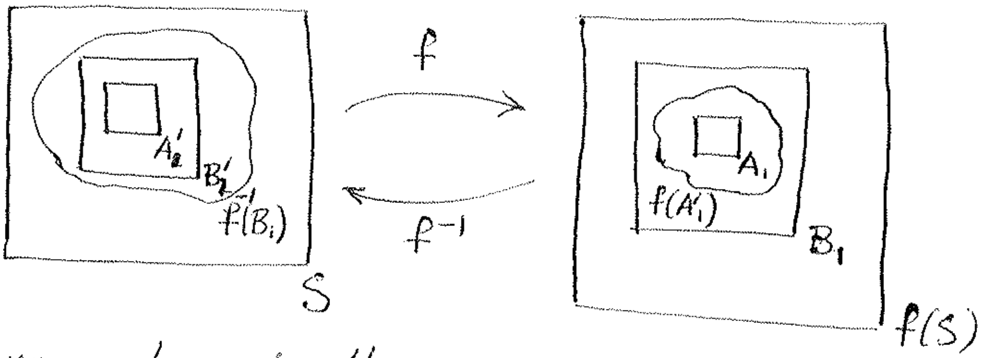
Winning sets are BIG.

Def. $f: \mathbb{R}^d \to \mathbb{R}^d$ is called c-bi-Lipschitz if

$$c^{-1}\|x_1 - x_2\| \leq \|f(x_1) - f(x_2)\| \leq c\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in \mathbb{R}^d.$$

Lem. 1. If $S \subset \mathbb{R}^d$ is α-winning, then $f(S)$ is $(\bar{c}^{2}\alpha)$-winning.

Proof.



At every step of the game $f(S)$, Alice can exploit the winning strategy of the game S as follows. As Bob picks a cube B_1 of size b, $f^{-1}(B_1)$ contains a cube of size $c^{-1}b$, and Alice can pick a cube $A_1' \subset f^{-1}(B_1)$ of size $c^{-1}b \cdot \alpha$ which realizes the winning strategy for S. Then $f(A_1')$ contains a cube A_1 of size $c^{-2}b\,\alpha$.

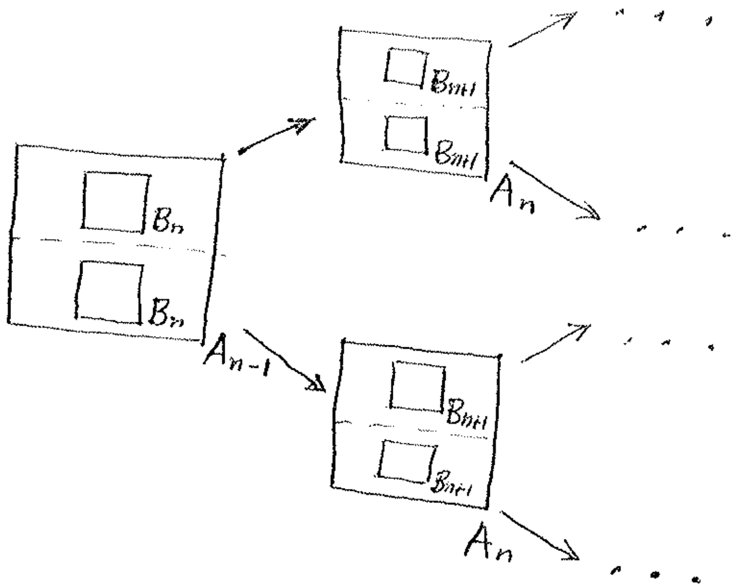
If we continue this way, $\bigcap_{n\geq 1} A_n' \subset S$. Hence $\bigcap_{n\geq 1} A_n \subset f(S)$. $\square$

Lem. 2. If S_i, $i \geq 1$, are α-winning sets, then so is $\bigcap_{i\geq 1} S_i$.

Proof. Alice can play winning strategies for the games S_i simultaneously: at step $n = 2^{i-1} + 2^i(k-1)$, $k \geq 1$, Alice applies the winning strategy for the game $(\alpha, \beta(\alpha\beta)^{2^i})$ with the end set S_i. $\square$

<u>Lem. 3.</u> Every α-winning set has cardinality continuum.
(In fact, every α-winning set has full Hausdorff dimension.)

<u>Proof.</u> If $\beta < \frac{1}{2}$, Bob can pick at every step of the game a cube contained in the upper half or A_n or in the lower half of A_n.



Hence, Bon can pick any path on the above tree at his will. Since different paths lead to different points in $\mathbb{R}^d$, the cardinality of a winning set should be at least the cardinality of paths in the above tree. $\square$

<u>Thm</u> The set BA of badly approximable vectors in $\mathbb{R}^d$ is winning.

<u>Cor.</u> For every c-bilipschitz maps f_i,
$$\bigcap_{i \geq 1} f_i(BA) \text{ has full Hausdorff dimension.}$$

Proof of Thm. (for $d=1$).

We start with an elementary lemma:

Lem. $\exists \alpha_0, \eta \in (0,1): \forall w \in \mathbb{R}^2: \exists$ interval $A' \subset [-\frac{1}{2}, \frac{1}{2}]: |A'| = \alpha_0$ such that $\forall x \in A': |w_1 + x w_2| \geq \eta \cdot \|w\|$.

Recall that by Dani Thm (Lecture I):

$$x \in BA \iff \{\mathbb{Z}^2 u(x) a_t\}_{t \geq 0} \text{ is bounded in } \mathcal{L}_2. \quad (+)$$

Fix $\alpha \in (0, \alpha_0)$ and $\beta \in (0,1)$.

Given an interval B_1, $|B_1| = b$, which is initial pick of Bob, we need to provide a winning strategy for Alice. Note that

$$(+) \iff \{\mathbb{Z}^2 u(x) g_n\}_{n \geq 1} \text{ is bounded}$$

where $g_n = \begin{pmatrix} b^{-\frac{1}{2}} (\alpha\beta)^{-\frac{n}{2}} & 0 \\ 0 & b^{\frac{1}{2}} (\alpha\beta)^{\frac{n}{2}} \end{pmatrix}$.

We shall give a bounded set $\Omega \subset \mathcal{L}_2$ and a strategy $\{A_n\}_{n \geq 1}$ for Alice such that

$$\forall x \in A_n: \mathbb{Z}^2 u(x) g_n \in \Omega.$$

Then $\{x_\infty\} = \bigcap_{n \geq 1} A_n$ satisfies

$$\mathbb{Z}^2 u(x_\infty) g_n \in \Omega \text{ for all } n,$$

so that $x_\infty \in BA$, and Alice wins.

Recall:

Exercise 1 (Lecture I)

$\exists \varepsilon_0 > 0: \forall \Lambda \in \mathcal{L}_2: \Lambda \cap B_{\varepsilon_0}(0) \ni$ at most one (up to sign) primitive vector.

We pick $\delta \in (0, \varepsilon_0)$ such that

$(*)$ $\forall x \in B_1$: $\mathbb{Z}^2 u(x) \cap B_\delta(0) = \{0\}$,

$(**)$ $\forall x \in [-(\alpha\beta)^{-1}, (\alpha\beta)^{-1}]$: $\forall v \in \mathbb{R}^2$: $\|v \cdot u(x)\| < \delta \Rightarrow \|v\| < \varepsilon_0$.

Let $\Omega = \{\Lambda \in \mathcal{L}_2 : \forall \underset{\neq 0}{v} \in \Lambda : \|v\| \geq \eta\delta\}$.

<u>Main Claim:</u> Alice can always pick an interval $A_n \subset B_n$: $|A_n| = \alpha |B_n|$ such that $\forall x \in A_n$: $\forall$ primitive $v \in \mathbb{Z}^2$: either:

(1) $\|v \cdot u(x) g_n\| \geq \delta$,

(2) $|(v u(x) g_n)_1| \geq \eta\delta$.

Note that this claim will finish the proof.

We use induction on n.

For $n=1$, it follows from $(*)$ that (1) holds for every $x \in B_1$, so Alice can pick any $A_1 \subset B_1$.

<u>Subclaim.</u> (1) may fail for at most one (up to sign) primitive $v \in \mathbb{Z}^2$.

Suppose that we have $v_1, v_2 \in \mathbb{Z}^2$, $v_1 \neq \pm v_2$, $x_1, x_2 \in B_n$ such that $\|v_i u(x_i) g_n\| < \delta$, $i = 1, 2$.

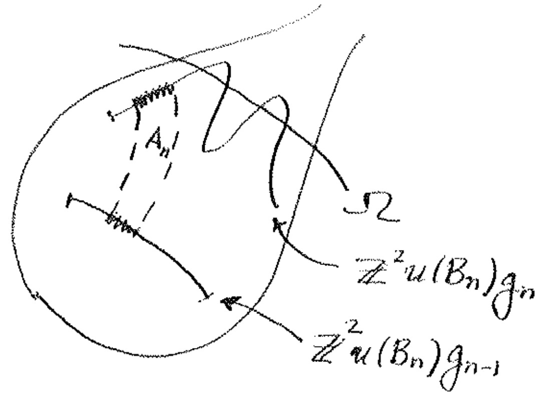
Then $v_2 u(x_1) g_n = v_2 u(x_2) g_n \cdot (g_n^{-1} u(x_1 - x_2) g_n)$

$$= v_2 u(x_2) g_n \cdot u(b^{-1}(\alpha\beta)^{-n}(x_1 - x_2))$$

Since $|B_n| = b(\alpha\beta)^{n-1}$, $b^{-1}(\alpha\beta)^{-n}(x_1 - x_2) \in [-(\alpha\beta)^{-1}, (\alpha\beta)^{-1}]$.

Hence, by $(**)$, $\|v_2 u(x_1) g_n\| < \varepsilon_0$.

Since also $\|v_1 u(x_1) g_n\| < \delta < \varepsilon_0$, we get a contradiction.



Now suppose that the claim holds for $n-1$. Then for every $x \in B_n$ and primitive $v \in \mathbb{Z}^2$, either:

(1) $\| v u(x) g_{n-1} \| \geq \delta$,

(2) $|(v u(x) g_{n-1})_1| \geq \eta \delta$.

$$(2) \Rightarrow |(v u(x) g_n)_1| = (\alpha\beta)^{-\frac{1}{2}} \cdot |(v u(x) g_{n-1})_1| \geq \eta \delta.$$

If $\| v u(x) g_n \| \geq \delta$ fails for some primitive $v \in \mathbb{Z}^2$, it can fail for at most one vector (up to sign) $- v$.

We have $B_n = b(\alpha\beta)^{n-1} [-\frac{1}{2}, \frac{1}{2}] + x_n$ for $x_n \in B_n$.

$$\underset{x}{\overset{\Psi}{}} = b(\alpha\beta)^{n-1} \cdot y + x_n \quad \text{for } y \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

$$v u(x) g_{n-1} = v u(x_n) g_{n-1} \cdot g_{n-1}^{-1} u(b(\alpha\beta)^{n-1} y) g_{n-1}$$

$$= \underbrace{v u(x_n) g_{n-1}}_{w} \cdot u(y).$$

Pick $A' \subset [-\frac{1}{2}, \frac{1}{2}]$, $|A'| = \alpha$, according to Lem., and set $A_n = b(\alpha\beta)^{n-1} A' + x_n$.

Then for every $x = b(\alpha\beta)^{n-1} y + x_n \in A_n$, we have

$$|(v u(x) g_n)_1| \geq |(v u(x) g_{n-1})_1| = |(w u(y))_1|$$

$$= |w_1 + y w_2| \underset{\text{Lem.}}{\geq} \eta \|w\| = \eta \cdot \| v u(x_n) g_{n-1} \| \underset{(1)}{\geq} \eta \delta$$

Hence, (2) holds for "bad" v.

This proves the claim. $\lrcorner$