Lecture II: Khinchin Thm and Exponential Mixing.

Fix $\psi : [1,\infty) \to (0,\infty)$ – a continuous nonincreasing function.
(for example, $\psi(r) = r^{-a}(\log r)^{-b}$, $a,b>0$).

<u>Def</u>. A vector $x \in \mathbb{R}^d$ is called <u>ψ-approximable</u>
if the inequalities
$$\begin{cases} \|x - \frac{p}{q}\| \le \frac{\psi(R)}{q} \\ q \le R \\ p \in \mathbb{Z}^d,\ q \in \mathbb{N} \end{cases}$$
have solutions for unbounded set R's.

<u>example</u>: By Dirichlet thm, every $\underset{\mathbb{R}^d}{\underset{\cap}{x}}$ is $R^{-1/d}$-approximable.

$W_d(\psi)$ = the set of ψ-approximable vectors in $\mathbb{R}^d$.

<u>Thm (Khinchin – Groshev)</u>

(1) $W_d(\psi)$ has measure zero $\iff \sum_{q\ge 1} \psi(q)^d < \infty$.

(2) $W_d(\psi)$ has full measure $\iff \sum_{q\ge 1} \psi(q)^d = \infty$.

<u>example</u>: a.e. $x \in \mathbb{R}^d$ is $(R\log R)^{-1/d}$-approximable,
but not $(R(\log R)^{1+\varepsilon})^{-1/d}$-approximable for $\varepsilon>0$.
However, there are vectors in $\mathbb{R}^d$ which are <u>not</u> $(R\cdot\log R)^{-1/d}$-approximable
(for example, badly approximable vectors).

Borel-Cantelli Lemma.

Let $A_n, n\geq 1$, be measurable subsets of a measure space (X,μ) such that $\sum_{n\geq 1}\mu(A_n)<\infty$.
Then $\overline{\lim} A_n = \{x\in X : x \text{ belongs to inf. many } A_n\text{'s}\}$ has measure zero.

Proof. Note that $\overline{\lim} A_n = \bigcap_{n\geq 1}\bigcup_{k\geq n} A_k$.

$$\mu(\overline{\lim} A_n) = \lim_{n\to\infty} \mu\Big(\bigcup_{k\geq n} A_k\Big) \leq \lim_{n\to\infty}\sum_{k\geq n}\mu(A_k) = 0$$

Proof of Khinchin-Groshev Thm (1)

It is sufficient to show that $W_d(\psi)\cap(0,1)^d$ has measure zero.
Let $A_q = \{x\in(0,1)^d : \|x-\frac{p}{q}\|\leq\frac{\psi(q)}{q} \text{ for some } p\in\mathbb{Z}^d\}$.
Note that $W_d(\psi)$ consists of vectors x such that $\|x-\frac{p}{q}\|\leq\frac{\psi(q)}{q}$ has solutions for inf. many q's.
Therefore, $W_d(\psi)\cap(0,1)^d \subset \overline{\lim_q} A_q$.

We have $\text{vol}(A_q) \leq \sum_{p:\,\|p\|\leq q} \text{vol}\left(B_{\psi(q)/q}\left(\frac{p}{q}\right)\right) = (2q+1)^d\cdot\left(\frac{2\psi(q)}{q}\right)^d$

$$\leq 6^d\cdot\psi(q)^d.$$

Since $\sum_{q\geq 1}\psi(q)^d<\infty$, we have $\sum_{q\geq 1}\text{vol}(A_q)<\infty$, and by the Borel-Cantelli Lemma,

$$\text{vol}(\overline{\lim_q} A_q) = 0 \Rightarrow \text{vol}(W_d(\psi)\cap(0,1)^d) = 0.$$

Exercise: Show that the converse of Borel-Cantelli lemma is false without imposing any extra conditions.

The converse of Borel-Cantelli lemma requires some "independence" assumption on A_n's.

Converse of Borel–Cantelli Lemma.

Let A_n, $n \geq 1$, be measurable subsets of a measure space (X, μ) such that

(i) $\sum_{n \geq 1} \mu(A_n) = \infty$

(ii) $\sum_{n,m=1}^{N} \mu(A_n \cap A_m) \leq \left(\sum_{n=1}^{N} \mu(A_n) \right)^2 + C \cdot \left(\sum_{n=1}^{N} \mu(A_n) \right)$.

Then $\overline{\lim} A_n$ has full measure.

Proof. Let $S_N(x) = \sum_{n=1}^{N} \chi_{A_n}(x)$ and $E_N = \sum_{n=1}^{N} \mu(A_n) \to \infty$.

Note that $x \in \overline{\lim} A_n$ iff $S_N(x) \to \infty$,

so we need to show that $S_N(x) \to \infty$ for a.e. x.

We have

$$\| S_N - E_N \|_2^2 = \left\langle \sum_{n=1}^{N} (\chi_{A_n} - \mu(A_n)), \sum_{m=1}^{N} (\chi_{A_m} - \mu(A_m)) \right\rangle$$

$$= \sum_{n,m=1}^{N} \langle \chi_{A_n} - \mu(A_n), \chi_{A_m} - \mu(A_m) \rangle$$

$$= \sum_{n,m=1}^{N} \left(\mu(A_n \cap A_m) - \mu(A_n)\mu(A_m) \right)$$

$$\overset{(ii)}{\leq} C \cdot E_N.$$

Hence, $\left\| \frac{S_N}{E_N} - 1 \right\|_2 \leq \frac{C}{E_N} \xrightarrow[N \to \infty]{} 0$.

Since $\frac{S_N}{E_N} \xrightarrow{L^2} 1$, $\frac{S_{N_i}(x)}{E_{N_i}} \xrightarrow{a.e.} 1$ along a subsequence.

Then $S_{N_i}(x) \xrightarrow{a.e.} \infty$ and $S_N(x) \xrightarrow{a.e.} \infty$

because $S_N(x)$ is monotone in N.

$\square$

A connection with dynamics

Lem. 1. Given a continuous nonincreasing function $\psi : [1,\infty) \to (0,\infty)$, there exists unique function $r : [t_0, \infty) \to (0,1)$ such that:

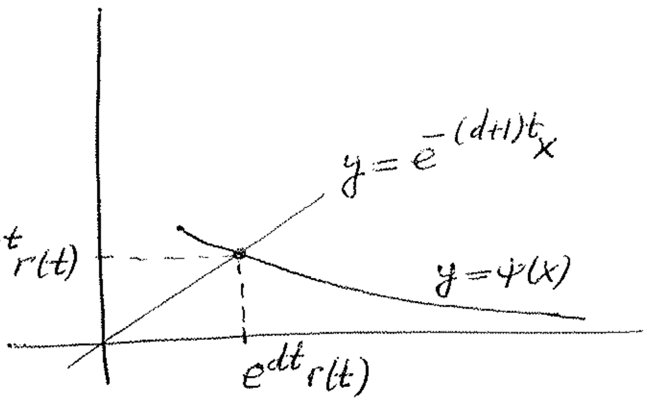
(i) $t \longmapsto e^{dt} r(t)$ is strictly increasing,

(ii) $t \longmapsto e^{-t} r(t)$ is nonincreasing,

(iii) $\psi(e^{dt} r(t)) = e^{-t} r(t)$ for $t \geq t_0$.

(iv) $\sum_{q \geq 1} \psi(q)^d = \infty \Leftrightarrow \sum_{n \geq t_0} r(n)^{d+1} = \infty$.

Proof. The function r is constructed using the picture:

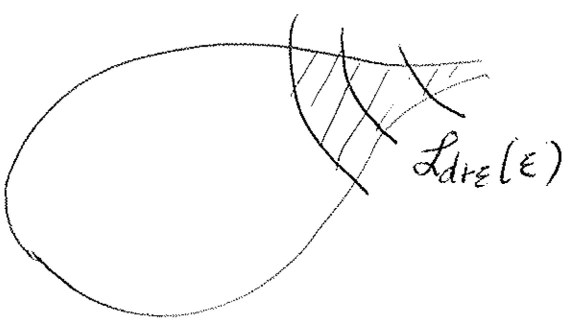


Let $\Delta : \mathcal{L}_{d+1} \to (0,\infty) : \Delta(\Lambda) = \min_{0 \neq v \in \Lambda} \|v\|$.

Note that by Mahler compactness criterion,

$$\Lambda_n \to \infty \text{ in } \mathcal{L}_{d+1} \Leftrightarrow \Delta(\Lambda_n) \to 0.$$

Hence, $\mathcal{L}_{d+1}(\varepsilon) = \{\Lambda : \Delta(\Lambda) \leq \varepsilon\}$, $\varepsilon > 0$, defines a basis of nbhds of ∞.



Recall that $\Lambda_x = \mathbb{Z}^{d+1}\left(\begin{array}{c|c} Id & 0 \\ \hline x & 1 \end{array}\right)$ for $x \in \mathbb{R}^d$,

$$a_t = \left(\begin{array}{c|c} e^t \cdot Id & 0 \\ \hline 0 & e^{-dt} \end{array}\right).$$

<u>Lem 2.</u> $x \in \mathbb{R}^d$ is ψ-approximable $\Longleftrightarrow$ $\Lambda_x a_{t_n} \in \mathcal{L}_{d+1}(r(t_n))$ for a sequence $t_n \to \infty$.

<u>Proof.</u>

$\Rightarrow$ Suppose that the system of inequalities

$$\begin{cases} \|x - \frac{p}{q}\| \leq \frac{\psi(R_n)}{q} \\ q \leq R_n \\ p \in \mathbb{Z}^d,\ q \in \mathbb{N} \end{cases}$$

has solutions for a sequence $R_n \to \infty$.

We pick t_n such that $R_n = e^{dt_n} r(t_n)$. Then $t_n \to \infty$, and $\psi(R_n) = e^{-t_n} r(t_n)$, so the inequality

$$\max\{e^{t_n}\|x \cdot q - p\|, e^{-dt_n}|q|\} \leq r(t_n)$$

has a solution $p \in \mathbb{Z}^d$, $q \in \mathbb{N}$.

Hence $\Delta(\Lambda_x \cdot a_{t_n}) \leq r(t_n)$ and $\Lambda_x a_{t_n} \in \mathcal{L}_{d+1}(r(t_n))$.

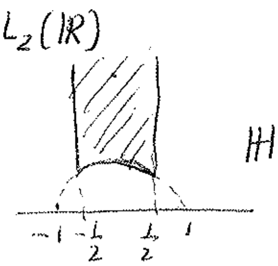
The proof of $\Leftarrow$ is similar. $\lrcorner$

<u>Lem 3.</u> $\mathrm{Vol}(\mathcal{L}_{d+1}(\varepsilon)) \asymp \varepsilon^{d+1}$ as $\varepsilon \to 0^+$.

<u>Proof.</u> (for $d=1$)

We use the identifications: $\mathcal{L}_2 \simeq SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$

$$SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO(2) \simeq$$



$\mathbb{H}^2$.

Every $g \in SL_2(\mathbb{R})$ can be written in the form: $g = v_x \cdot b_y \cdot k$ for $v_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $b_y = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$, $k \in SO(2)$.

The second indentification is given by the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{a\cdot i+b}{c\cdot i+d}.$$

$$v_x \cdot b_y \cdot k \longmapsto x+iy.$$

Hence, $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ can be "parametrised" by

$$\{v_x \cdot b_y \cdot k : |x|\le \tfrac{1}{2},\ y \ge \tfrac{\sqrt{3}}{2},\ k\in SO(2)\},$$

and $\mathcal{L}_2 = \{\mathbb{Z}^2 v_x b_y k : |x|\le\frac{1}{2},\ y\ge\frac{\sqrt{3}}{2},\ k\in SO(2)\}$.

$$\Delta(\mathbb{Z}^2 v_x b_y k) = \Delta(\mathbb{Z}^2 b_y \cdot \underbrace{v_{y^{-1}x}\, k}_{\substack{\text{uniformly}\\ \text{bounded}}}) \asymp \Delta(\mathbb{Z}^2 b_y) = y^{-\frac{1}{2}}.$$

Hence, $\Delta \ll \varepsilon \iff y^{-\frac{1}{2}} \ll \varepsilon \iff y \gg \varepsilon^{-2}$.

Recall that invariant volume on $\mathbb{H}^2$ is $\frac{dx\,dy}{y^2}$,

so that $vol(\mathcal{L}_{d+1}(\varepsilon)) \asymp \iint_{\substack{|x|\le\frac{1}{2}\\ y\gg\varepsilon^{-2}}} \frac{dx\,dy}{y^2} \asymp \varepsilon^2$.

$\square$

Rmk. The proof for $d>1$ follows the same strategy, but one needs to use "parametrization" of $\mathcal{L}_{d+1}$ by Siegel sets.

An "independence property" needed for Khinchin Thm comes from exponential mixing on $\mathcal{L}_{d+1} \simeq SL_{d+1}(\mathbb{Z})\backslash SL_{d+1}(\mathbb{R})$.
(see Einsiedler's lectures).

Thm (exponential mixing) $\forall f_1, f_2 \in C_c^\infty(\mathcal{L}_{d+1})$ such that $\int_{\mathcal{L}_{d+1}} f_1 = \int_{\mathcal{L}_{d+1}} f_2 = 0$,

$$|\langle f_1\cdot a_t, f_2\rangle| \le \text{const}\cdot e^{-\delta t} S(f_1)S(f_2),\ \delta>0,$$

where $S(f_i)$ denote suitable Sobolev norms.

Proof of Khinchin-Groshev Thm (2).

Let $r(t)$ be the function defined in Lem. 1.

Then $\sum_{n \geq t_0} r(n)^{d+1} = \infty$ and by Lemma 2, it is sufficient to show that for a.e. $x \in \mathbb{R}^d$,

$\Lambda_x a_n \in \mathcal{L}_{d+1}(r(n))$ infinitely often.

We approximate the characteristic functions of the sets $\mathcal{L}_{d+1}(r(n))$ by smooth functions f_n such that:

1) $\{f_n \neq 0\} \subset \mathcal{L}_{d+1}(r(n))$, $f_n \geq 0$,

2) $\int_{\mathcal{L}_{d+1}} f_n \asymp \operatorname{vol}(\mathcal{L}_{d+1}(r(n)))$,

3) $S(f_n) \ll \int_{\mathcal{L}_{d+1}} f_n$.

Then by 2) and Lem. 3, $\boxed{\sum_{n \geq 1} \int_{\mathcal{L}_{d+1}} f_n = \infty.}$

We also claim that $\boxed{\sum_{n,m=1}^{N} \langle f_n \cdot a_n, f_m \cdot a_m \rangle \leq \left(\sum_{n=1}^{N} \int_{\mathcal{L}_{d+1}} f_n\right)^2 + C\left(\sum_{n=1}^{N} \int_{\mathcal{L}_{d+1}} f_n\right).}$

Let $\tilde{f}_n = f_n - \int_{\mathcal{L}_{d+1}} f_n$. Then the claim is equivalent to

$$\sum_{n,m=1}^{N} \langle \tilde{f}_n \cdot a_n, \tilde{f}_m \cdot a_m \rangle \ll \sum_{n=1}^{N} \int_{\mathcal{L}_{d+1}} f_n .$$

Now we apply the Thm on exponential mixing of the flow a_t, using that

$$S(f_n) \ll \int_{\mathcal{L}_{d+1}} f_n \ll 1.$$

We have

$$\sum_{n,m=1}^{N} \langle \tilde{f}_n \cdot a_n, \tilde{f}_m \cdot a_m \rangle = \sum_{n,m=1}^{N} \langle \tilde{f}_n \cdot a_{n-m}, \tilde{f}_m \rangle$$

$$\ll \sum_{n,m=1}^{N} e^{-\delta|n-m|} S(\tilde{f}_n) S(\tilde{f}_m)$$

$$\ll \sum_{1 \le m \le n \le N} e^{-\delta(n-m)} \cdot \Big(\int_{\mathcal{L}_{d+1}} f_n \Big) \cdot 1$$

$$= \sum_{n=1}^{N} \underbrace{\Big(\sum_{m=1}^{n} e^{-\delta(n-m)} \Big)}_{\text{uniformly bounded.}} \Big(\int_{\mathcal{L}_{d+1}} f_n \Big)$$

$$\ll \sum_{n=1}^{N} \Big(\int_{\mathcal{L}_{d+1}} f_n \Big).$$

This proves the claim.

Now for functins $\varphi_n = f_n \cdot a_n$, we have

$$\sum_{n \ge 1} \int_{\mathcal{L}_{d+1}} \varphi_n = \infty,$$

$$\sum_{n,m=1}^{N} \langle \varphi_n, \varphi_m \rangle \le \Big(\sum_{n=1}^{N} \int_{\mathcal{L}_{d+1}} \varphi_n \Big)^2 + C \cdot \Big(\sum_{n=1}^{N} \int_{\mathcal{L}_{d+1}} \varphi_n \Big).$$

As in the proof of converse of Borel-Cantelli Lemma, we deduce that for a.e. $\Lambda \in \mathcal{L}_{d+1}$

$$\sum_{n \ge 1} \varphi_n(\Lambda) = \sum_{n \ge 1} f_n(\Lambda a_n) = \infty .$$

Since $\{f_n \neq 0\} \subset \mathcal{L}_{d+1}(r(n))$, this shows that for a.e. Λ, $\Lambda \cdot a_n \in \mathcal{L}_{d+1}(r(n))$ infinitely often, and for a.e. $g \in SL_{d+1}(\mathbb{R})$, $\mathbb{Z}^{d+1} g a_n \in \mathcal{L}_{d+1}(r(n))$ inf. often.

For $g = \left(\begin{array}{c|c} Id & 0 \\ \hline x & 1 \end{array}\right)\left(\begin{array}{c|c} A & b \\ \hline 0 & c \end{array}\right)$, we have

$$g \cdot a_n = \left(\begin{array}{c|c} Id & 0 \\ \hline x & 1 \end{array}\right) a_n \cdot \underbrace{\left(\begin{array}{c|c} A & e^{-(d+1)n}b \\ \hline 0 & c \end{array}\right)}_{\text{uniformly bounded.}}$$

Hence, by Fubini Thm, for a.e. $x \in \mathbb{R}^d$,

$$\underbrace{\mathbb{Z}^{d+1}\left(\begin{array}{c|c} Id & 0 \\ \hline x & 1 \end{array}\right)}_{\Lambda_x} a_n \in \mathcal{L}_{d+1}(\alpha \cdot r(n)) \text{ infinitely often,}$$

where $\alpha > 0$ is a fixed constant.

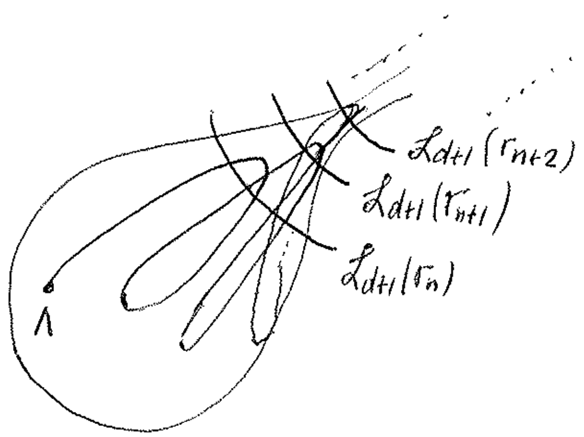
Applying the above argument to the function $\alpha^{-1} r(t)$, we conclude that $\Lambda_x a_n \in \mathcal{L}_{d+1}(r(n))$ infinitely often for a.e. $x \in \mathbb{R}^d$.

This completes the proof of the thm. $\square$

The same argument also gives:

<u>Dynamical Khinchin Thm.</u> Given a sequence $r_n \to 0^+$ such that $\sum_{n \geq 1} r_n^{d+1} = \infty$, for a.e. $\Lambda \in \mathcal{L}_{d+1}$,

$$\Lambda a_n \in \mathcal{L}_{d+1}(r_n) \text{ infinitely often.}$$



<u>Rmk.</u> Generic orbits $\{\Lambda a_t\}_{t \geq 0}$ are dense in $\mathcal{L}_{d+1}$, but at the same time they visit shrinking nbhds of infinity.