Lecture I: Dani correspondence: Diophantine approximation and flows.

Diophantine approximation.

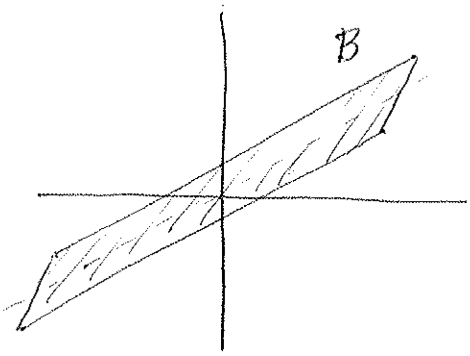
Given a vector $x \in \mathbb{R}^d$, we would like to find a rational approximation $x \simeq \frac{p}{q} \in \mathbb{Q}^d$ with $q \le R$.

Thm (Dirichlet) $\forall x \in \mathbb{R}^d$ $\forall R > 1$ $\exists p \in \mathbb{Z}^d$, $q \in \mathbb{N}$.

$$\begin{cases} \|x - \frac{p}{q}\| \le \frac{R^{-1/d}}{q}, \\ q \le R. \end{cases}$$

(here and below, $\|\cdot\|$ means the maximum norm.)

Proof:



Consider the region:

$$B = \left\{ y \in \mathbb{R}^{d+1} : \begin{array}{l} |x_i \cdot y_{d+1} - y_i| \le R^{-1/d}, \ i = 1, \dots d \\ |y_{d+1}| \le R \end{array} \right\}.$$

Note that B is closed, convex, symmetric w.r.t. the origin, and $vol(B) = 2^{d+1}$.

(see Bilu's lecture)

By Minkowski's Thm, $\exists y \in \mathbb{Z}^{d+1}$, $y \neq 0$, such that $y \in B$.

Since $R > 1$, we have $y_{d+1} \neq 0$.

Hence, we get
$$\begin{cases} |x_i - \frac{y_i}{y_{d+1}}| \le \frac{R^{-\frac{1}{d}}}{|y_{d+1}|}, \quad i = 1, \dots d, \\ |y_{d+1}| \le R. \end{cases}$$

$\lrcorner$

Can one improve the Dirichlet Thm?

example: We know that $\left|\sqrt{2}-\frac{p}{q}\right|\le\frac{1}{q^2}$ has inf. many solutions. Suppose that for some $c\in(0,1)$,
$\left|\sqrt{2}-\frac{p}{q}\right|\le\frac{c}{q^2}$ also has inf. many solutions
$p\in\mathbb{Z}$ and $q\in\mathbb{N}$. Then

$$\frac{1}{q^2}\le\frac{|2q^2-p^2|}{q^2}=\left|2-\left(\frac{p}{q}\right)^2\right|\le\frac{c}{q^2}\cdot\left|\sqrt{2}+\frac{p}{q}\right|\le\frac{c}{q^2}\left(\left|\sqrt{2}-\frac{p}{q}\right|+2\sqrt{2}\right)$$
$$\le\frac{c}{q^2}\left(\frac{c}{q^2}+2\sqrt{2}\right).$$

Hence, $1\le c\cdot\left(\frac{c}{q^2}+2\sqrt{2}\right)$ and taking $q\to\infty$, we conclude that $c\ge\frac{1}{2\sqrt{2}}$.

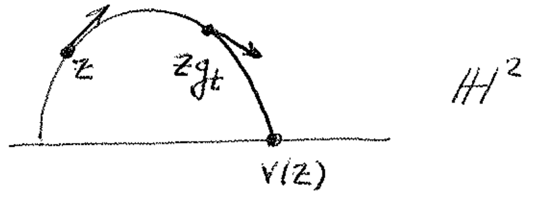
Def. 1) $x\in\mathbb{R}^d$ is called badly approximable if $\exists c>0:\ \forall p\in\mathbb{Z}^d\ \forall q\in\mathbb{N}:\ \left\|x-\frac{p}{q}\right\|>\frac{c}{q^{1+\frac{1}{d}}}$.

2) $x\in\mathbb{R}^d$ is called singular if $\forall\varepsilon>0\ \forall R\ge R_\varepsilon$ the system of inequalities
$$\begin{cases}\left\|x-\frac{p}{q}\right\|\le\frac{\varepsilon\cdot R^{-1/d}}{q}\\ q\le R\end{cases}$$
has inf. many solutions $p\in\mathbb{Z}^d$ and $q\in\mathbb{N}$.

A connection with dynamics

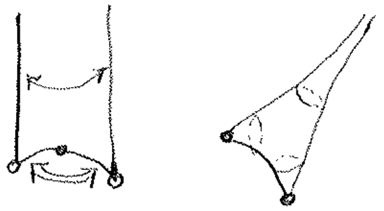
Let $T^1(\mathbb{H}^2)$ denote the unit tangent bundle of the upper-half plane (equipped with the hyperbolic metric $\frac{dx^2+dy^2}{y^2}$).

Let $g_t: T^1(\mathbb{H}^2)\to T^1(\mathbb{H}^2)$ be the geodesic flow.



Let $X = PSL_2(\mathbb{Z})\backslash T^1(\mathbb{H}^2)$ and $\pi : T^1(\mathbb{H}^2) \to X$ be the factor map.

Note that X is a unit tangent bundle of noncompact hyperbolic surface (with 2 "singular" points),



and the geodesic flow on X is the projection of the geodesic flow on $T^1(\mathbb{H}^2)$.

We define the visual map $v : T^1(\mathbb{H}^2) \to \mathbb{R}\cup\{\infty\}$

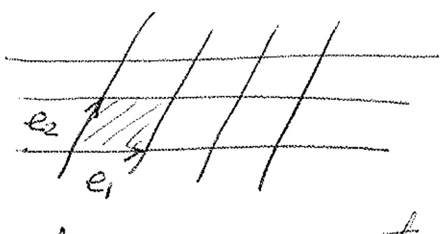
$$v(z) = \lim_{t\to+\infty} z\cdot g_t .$$

Thm 1. For every $z \in T^1(\mathbb{H}^2)$,
the semiorbit $\{\pi(z)g_t\}_{t\geq 0}$ is bounded in X. $\Leftrightarrow$ $v(x)^{-1}$ is badly approximable.

Space of lattices

$\mathcal{L}_d = \{$ the set of all lattices in $\mathbb{R}^d$ with covol $= 1\}$.

A lattice Λ in $\mathbb{R}^d$ is a subgroup $\Lambda = \mathbb{Z}e_1 + \dots + \mathbb{Z}e_d$ where $\{e_i\}$ is a basis of $\mathbb{R}^d$.

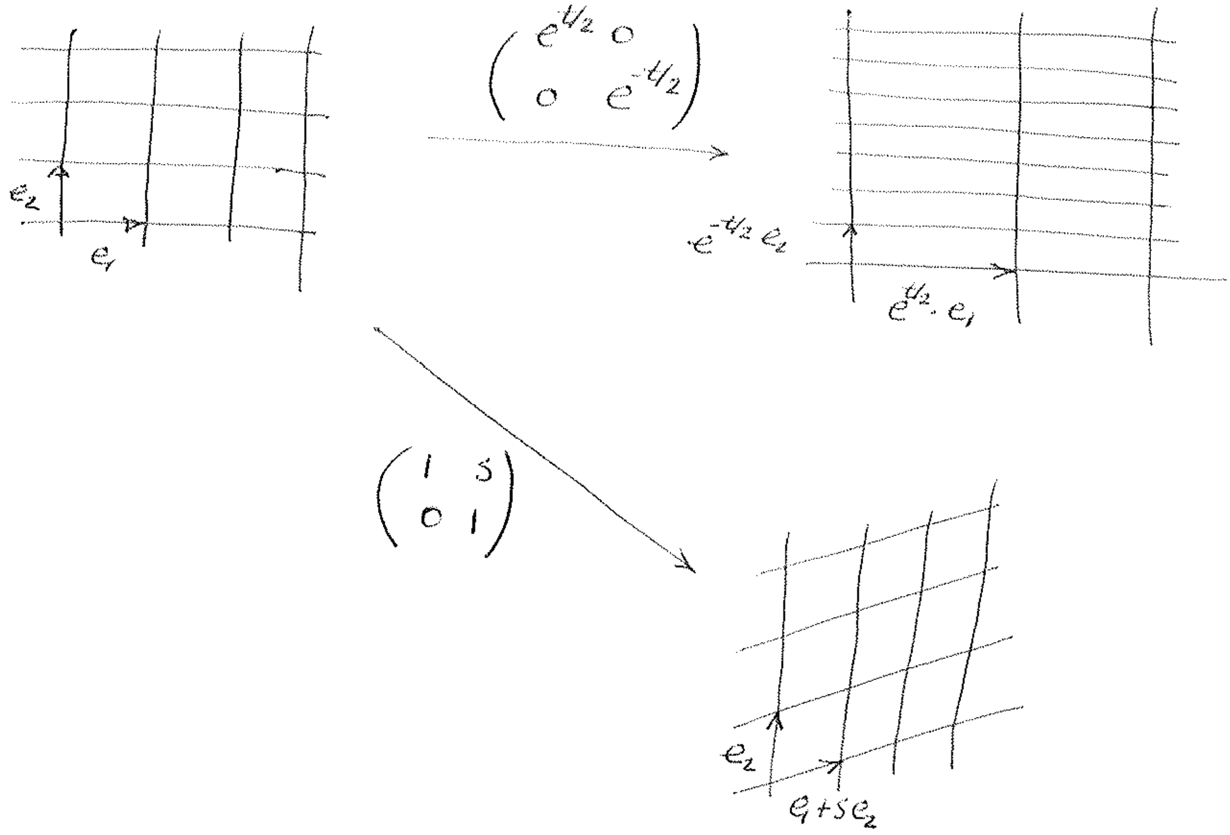


covol(Λ) = the volume of the parallelepiped spaned by the basis $\{e_i\}$.

There is a natural action of $SLd(\mathbb{R})$ on $\mathcal{L}_d$.

$\mathcal{L}_d \circlearrowleft SLd(\mathbb{R})$: For $g \in SLd(\mathbb{R})$ and $\Lambda \in \mathcal{L}_d$,

$$\Lambda \longmapsto \Lambda\cdot g .$$



This action is transitive and $\mathbb{Z}^d g = \mathbb{Z}^d \Leftrightarrow g \in SL_d(\mathbb{Z})$.

Hence, $\boxed{\mathcal{L}_d \simeq SL_d(\mathbb{Z}) \backslash SL_d(\mathbb{R})}$.

Topology on $\mathcal{L}_d$: We say that $\Lambda_n \to \Lambda$ if for some (equivalently, every) basic $\{e_i^{(n)}\}$ of Λ_n, $\{e_i^{(n)}\}$ converges to a basis of Λ (w.r.t. the topology on $\mathbb{R}^d$).

Rmk. This topology is the same as the quotient topology given by the identification $\mathcal{L}_d \simeq SL_d(\mathbb{Z}) \backslash SL_d(\mathbb{R})$.

Note that the space $\mathcal{L}_d$ is not compact: the sequence of lattces $\Lambda_n = \langle \frac{1}{n} e_1, n e_2, e_3, \dots, e_d \rangle$, $n \geq 1$, has no convergent subsequences because $\frac{1}{n} e_1 \to 0$, as $n \to \infty$.

Exercises. 1) $\exists \varepsilon_0 > 0 : \forall \Lambda \in \mathcal{L}_2 : \Lambda \cap B_{\varepsilon_0}(0)$ contains at most one primitive vector up to sign. (a vector $v \in \Lambda$ is called primitive if $v \neq n \cdot w$ for some $n \geq 2$ and $w \in \Lambda$).

1*) What is the analogue of 1) for $\mathcal{L}_d$, $d > 2$?

2) If $v_1 \in \Lambda \in \mathcal{L}_d$, then $\exists v_2, \dots v_d \in \Lambda$ such that

$$\Lambda = \langle v_1, v_2, \dots, v_d \rangle.$$

Mahler compactness criterion.

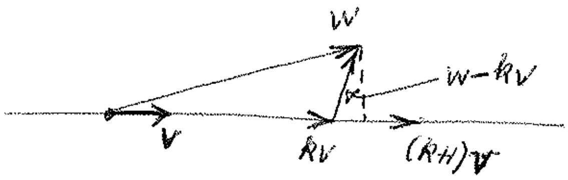
$\Omega \subset \mathcal{L}_d$ is bounded $\iff \exists \delta > 0 : \forall \Lambda \in \Omega : \forall \underset{\neq 0}{v} \in \Lambda : \|v\| \geq \delta$.

Proof (for $d = 2$)

$\Rightarrow$ Suppose not. Then $\exists \Lambda_n \in \Omega$ and $v_n \in \Lambda_n : v_n \neq 0, v_n \to 0$. Hence, the sequence $\{\Lambda_n\}$ has no convergent subsequence, which is a contradiction.

$\Leftarrow$ We claim that $\exists R > 0$ such that $\forall \Lambda \in \Omega$ has a basis $\{e_1, e_2\}$ satisfying $\delta \leq \|e_1\|, \|e_2\| \leq R$.
Then given a sequence $\Lambda_n \in \Omega$, the corresponding basises $\{e_1^{(n)}, e_2^{(n)}\}$ have a convergent subsequence which converges to, say, $\{e_1, e_2\}$. By continuity of covol function, $\mathrm{covol}(\langle e_1, e_2 \rangle) = 1$.
Hence, $\{\Lambda_n\}$ has a subsequence which converges to $\langle e_1, e_2 \rangle \in \mathcal{L}_2$, so that Ω is bounded.

In order to prove the claim, we observe that by Minkowski Thm, $\exists \underset{\neq 0}{v} \in \Lambda : \|v\| \leq 1$.



Moreover, we can pick v to be primitive.

By Exercise 2, $\exists w \in \Lambda : \Lambda = \langle v, w \rangle$.

Since the volume of the parallelogram spaned by v and w is 1, we have $\operatorname{dist}(w, \mathbb{R}\cdot v) \leq \frac{1}{\|v\|} \leq \frac{1}{\delta}$.

By the triangle inequality (see pic.),

$$\|w - kv\| \leq \|v\| + \frac{1}{\delta} \leq 1 + \frac{1}{\delta}.$$

This proves the claim. $\blacksquare$

Rmk. The proof of $\Leftarrow$ for $d>2$ is significantly more complicated and requires use of 2nd Minkowski Thm or Siegel sets.

Proof of Thm 1.

We recall that $T^1(\mathbb{H}^2)$ can be indentified with $PSL_2(\mathbb{R})$.

With respect to this identification,

$$X = PSL_2(\mathbb{Z})\backslash T^1(\mathbb{H}^2) \simeq PSL_2(\mathbb{Z})\backslash PSL_2(\mathbb{R}) \simeq SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}) \simeq \mathcal{L}_2,$$

and the geodesic flow is given by

$$\Lambda \longmapsto \Lambda \cdot g_t \quad \text{where } g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Let us compute the visual map $v : PSL_2(\mathbb{R}) \to \mathbb{R}\cup\{\infty\}$.

$\forall g \in PSL_2(\mathbb{R})$ can be written as

$$g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b' \\ ax & c' \end{pmatrix}, \quad a \neq 0.$$

or

$$g = \begin{pmatrix} 0 & * \\ x & * \end{pmatrix}.$$

We leave the 2nd case as an exercise.

We have

$$v(g) = \lim_{t\to+\infty} (gg_t \cdot i) = \lim_{t\to+\infty} \frac{ae^t \cdot i + b'e^{-t}}{axe^t \cdot i + c'e^{-t}} = \frac{1}{x}.$$

We observe that

$$gg_t = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} g_t \cdot \underbrace{\begin{pmatrix} a & e^{-t}b \\ 0 & c \end{pmatrix}}_{\text{uniformly bounded for } t \geq 0}.$$

Therefore,

$$\{\mathbb{Z}^2 g \cdot g_t\}_{t\geq 0} \text{ is bounded in } \mathcal{L}_2 \iff \left\{\mathbb{Z}^2 \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} g_t\right\}_{t\geq 0} \text{ is bounded in } \mathcal{L}_2.$$

We set $\Lambda_x = \mathbb{Z}^2 \cdot \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \{(p+xq, q) : (p,q) \in \mathbb{Z}^2\}$.

By the Mahler compactness criterion,

$$\{\Lambda_x g_t\}_{t\geq 0} \text{ is bounded in } \mathcal{L}_2 \iff \exists \delta \in (0,1) : \max\{e^{t/2}|p+xq|, e^{-t/2}|q|\} \geq \delta$$
for all $(p,q) \in \mathbb{Z}^2 \setminus \{0\}$ and $t \geq 0$.

$$\Updownarrow$$

$$\exists \delta \in (0,1) : \max\{e^{t/2}|p+xq|, e^{-t/2}|q|\} \geq \delta$$
for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $t \geq 0$.

Now suppose that $v(g)^{-1} = x$ is badly approximable. This implies that $|x - \frac{p}{q}| \geq \frac{c}{q^2}$ for some $c > 0$ and all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

Then $e^{-t/2}|q| \cdot e^{t/2}|p+xq| \geq c$ for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $t \geq 0$,
and hence $\max\{e^{-t/2}|q|, e^{t/2}|p+xq|\} \geq \sqrt{c}$.
Therefore, $\{\Lambda_x g_t\}_{t\geq 0}$ is bounded in $\mathcal{L}_2$.

Suppose that $\{\Lambda_x g_t\}_{t\geq 0}$ is bounded in $\mathcal{L}_2$.
Then for some $\delta \in (0,1)$: $\max\{e^{t/2}|p+xq|,\ e^{-t/2}|q|\} \geq \delta$
for all $p\in\mathbb{Z}$, $q\in\mathbb{N}$, $t\geq 0$.

We pick $t\geq 0$ such that $e^{-t/2}|q| = \delta/2$.

Then $\frac{2}{\delta}\cdot|q|\cdot|p+xq|\geq\delta$ for all $p\in\mathbb{Z}$ and $q\in\mathbb{N}$.

$$\Updownarrow$$

$$\left|x+\frac{p}{q}\right| \geq \frac{\delta^2/2}{q^2}.$$

Hence, $x = v(g)^{-1}$ is badly approximable.

Dani correspondence.

For $x\in\mathbb{R}^d$, set $u(x) = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline x & 1 \end{array}\right)$, and

$$\Lambda_x = \mathbb{Z}^d\cdot u(x) = \{(p+xq, q) : p\in\mathbb{Z}^d,\ q\in\mathbb{Z}\}\in\mathcal{L}_{d+1}.$$

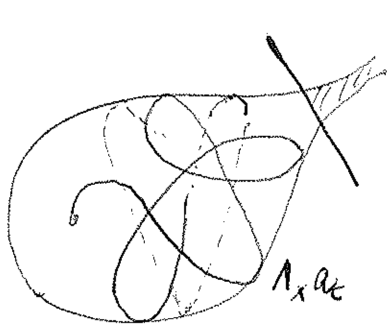
Consider the flow $a_t : \mathcal{L}_{d+1}\to\mathcal{L}_{d+1}$:

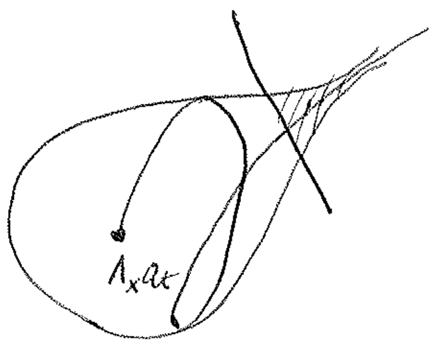
$$\Lambda \longmapsto \Lambda\cdot a_t \quad \text{where} \quad a_t = \left(\begin{array}{c|c} e^t\cdot\mathrm{Id} & 0 \\ \hline 0 & e^{-dt}\end{array}\right).$$

Thm (Dani) For $x\in\mathbb{R}^d$,

1) $\{\Lambda_x a_t\}_{t\geq 0}$ is bounded in $\mathcal{L}_{d+1}$ $\Longleftrightarrow$ x is badly approximable.

2) $\{\Lambda_x a_t\}_{t\geq 0}$ is divergent in $\mathcal{L}_{d+1}$ $\Longleftrightarrow$ x is singular.





Def. The semiorbit $\{\Lambda a_t\}_{t\geq 0}$ is called <u>divergent in $\mathcal{L}_{d+1}$</u> if $\forall$ bounded $\Omega \subset \mathcal{L}_{d+1}$: $\Lambda a_t \notin \Omega$ for all sufficiently large t.

example: $\{\mathbb{Z}^{d+1} a_t\}_{t\geq 0}$ is divergent because $e^{-dt} \cdot e_{d+1} \to 0$.

By the Mahler compactness criterion,

(*) $\{\Lambda a_t\}_{t\geq 0}$ is divergent $\Leftrightarrow$ $\forall \delta > 0$: $\Lambda a_t \cap B_\delta(0) \neq \{0\}$ for all sufficiently large t.

<u>Proof of Dani Thm.</u>

Proof 1) is very similar to the proof of Thm 1.

We give proof of 2) for $d=2$ (general case is similar).

$\Longmapsto$ Suppose that x is not singular.

Then $\exists \varepsilon \in (0,1)$: $\exists R_i \to \infty$, $R_i \geq 1$: $|x - \frac{p}{q}| > \frac{\varepsilon R_i^{-1}}{q}$ for all $p \in \mathbb{Z}$ and $q = 1, \dots \lfloor R_i \rfloor$.

Pick t_i such that $R_i < e^{t_i} < \frac{R_i}{\varepsilon}$.

Then $t_i \to \infty$ and $e^{t_i} |x \cdot q - p| > \varepsilon$ for all $p \in \mathbb{Z}$ and $q = 1, \dots \lfloor R_i \rfloor$.

On the other hand, for $q > R_i$,

$$e^{-t_i} |q| > \varepsilon.$$

Hence, $\max\{e^{t_i}|xq - p|, e^{-t_i}|q|\} > \varepsilon$ for all $(p,q) \in \mathbb{Z}^2 \setminus \{0\}$,

and $\Lambda_x a_{t_i} \cap B_\varepsilon(0) = \{0\}$,

which contradicts (*).

$\Leftarrow$ Let $\varepsilon \in (0, \delta^2)$. Since x is singular,

$\forall R \geq R_\varepsilon \ \exists p \in \mathbb{Z}, q \in \mathbb{N}: \begin{cases} |x - \frac{p}{q}| \leq \frac{\varepsilon R^{-1}}{q} \\ q \leq R \end{cases}$

Consider the vector $v = (xq - p, q) \in \Lambda_x$.

Since $v \cdot a_t = (e^t(xq-p), e^{-t}q)$, we have

$v \cdot a_t \in B_\delta(0)$ provided that $\varepsilon R^{-1} \cdot e^t \leq \delta$ and $e^{-t} \cdot R \leq \delta$.

Let $I_R = [\log \frac{R}{\delta}, \log \frac{\delta R}{\varepsilon}]$.

Then $v \cdot a_t \in B_\delta(0)$ for every $t \in I_R$.

Note that $|I_R| = \log \frac{\delta^2}{\varepsilon} > 0$, and

for sufficiently large R, $\log \frac{R+1}{\delta} < \log \frac{\delta R}{\varepsilon}$.

This implies that $\bigcup_{R \geq R_\varepsilon} I_R$ contain

an interval $[t_0, \infty)$.

We conclude that $\forall t \geq t_0: \Lambda_x a_t \cap B_\delta(0) \neq \{0\}$.

Hence, by $(*)$, $\{\Lambda_x a_t\}_{t \geq 0}$ is divergent. $\square$

We recall that the space $\mathcal{L}_{d+1} \simeq SL_{d+1}(\mathbb{Z}) \backslash SL_{d+1}(\mathbb{R})$ is equipped with probability invariant measure and the flow a_t is ergodic (see Einsiedler's lectures).

Cor. The sets BA and Sing of badly approximable and singular vectors in $\mathbb{R}^d$ both have measure zero.

<u>Proof.</u> Since the flow a_t is ergodic, it follows from the pointwise ergodic Thm. that for a.e. $g \in SL_{d+1}(\mathbb{R})$, the semiorbit $\{\mathbb{Z}^{d+1} g\, a_t\}_{t\geq 0}$ is dense in $\mathcal{L}_{d+1}$.

Such orbits cannot be either bounded or divergent.

On the other hand, a.e. $g \in SL_{d+1}(\mathbb{R})$ can be written as:

$$g = u(x) \cdot \left(\begin{array}{c|c} A & b \\ \hline 0 & c \end{array}\right).$$

We observe that

$$g \cdot a_t = u(x) \cdot a_t \cdot \underbrace{\left(\begin{array}{c|c} A & e^{-(d+1)t} b \\ \hline 0 & c \end{array}\right)}_{\text{uniformly bounded for } t\geq 0}.$$

This implies that:

$\{\mathbb{Z}^{d+1} g\, a_t\}_{t\geq 0}$ is bounded/divergent $\iff$ $\{\mathbb{Z}^{d+1} u(x)\, a_t\}_{t\geq 0}$ is bounded/divergent.

Now it follows from Fubini Thm that for a.e. $x \in \mathbb{R}^d$, $\{\mathbb{Z}^{d+1} u(x)\, a_t\}_{t\geq 0}$ is neither bounded nor divergent.

By Dani Thm, a.e. $x \in \mathbb{R}^d$ is neither badly approximable nor singular. $\square$