Margulis normal subgroup theorem.

Thm. Let Γ be a lattice in $G = SL(n,\mathbb{R})$, $n \geq 3$, and $N \trianglelefteq \Gamma$. Then $|N| < \infty$ or $|\Gamma/N| < \infty$.

$\ulcorner$ Suppose that N is infinite normal subgroup in Γ.

We claim that:

$$\left.\begin{array}{l}(1)\ \Gamma/N \text{ has property } T \\ (2)\ \Gamma/N \text{ is amenable}\end{array}\right] \Rightarrow \Gamma/N \text{ is finite.}$$

Γ has property $(T) \Rightarrow (1)$

It remains to show that Γ/N is amenable.

Let V - locally convex top. vector space

$\cup$

Ω - nonempty, compact, convex

$\Gamma \circlearrowright \Omega$ - affine continuous action.

We need to show that $\Omega \ni \Gamma$-fixed point.

Without loss of generality, V & Ω are seperable.

Consider $L^\infty_\Gamma(G,\Omega) = \left\{ f : G \to \Omega : \begin{array}{l} f(\gamma g) = \gamma \cdot f(g) \\ \text{for } \gamma \in \Gamma,\ a.e.\ g \in G \end{array} \right\}$

equipped with weak* topology, namely, topology defined by seminorms:

$$\|f\|_{\alpha,\varphi} = \int_G \|f(g)\|_\alpha \cdot |\varphi(g)|\,dg\ , \quad \varphi \in L^1(G),$$

($\|\cdot\|_\alpha$ are the seminorms on V defining topology).

Then $L^\infty_\Gamma(G,\Omega)$ is compact.

G acts continuously on $L^\infty_\Gamma(G,\Omega)$ by

$$f \overset{g}{\longmapsto} f(x\cdot g).$$

Let $B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \subset G.$

Since B is amenable, $L^\infty(G,\Omega) \ni B$-fixed point.

Then $\boxed{\exists f : G/B \to \Omega : \ f(\gamma\cdot x) = \gamma\cdot f(x) \text{ for } \gamma\in\Gamma, \text{ a.e. } x \in G/B.}$

Margulis Factor Thm.

If f is as above, $\exists$ closed subgroup $P \supset B$:

$$\begin{tikzcd} G/B \arrow[dr, dashed, "gB \to gP"] \arrow[d, "f"'] & \\ \Omega \arrow[r, dashed, "\alpha"'] & G/P \end{tikzcd}$$

Γ-equivariant measurable isomorphism

i.e., every Γ-factor is a G-factor.

Since N acts trivially on Ω,
$N \cdot gP = gP$ for a.e. $g \in G$, and
$P \supset \langle g^{-1} N g : g \in G \rangle$ - infinite closed normal subgroup of G.

Hence, $P = G \Rightarrow f = \text{const}$ a.e.
In particular, $\Omega \ni \Gamma$-fixed point.

Proof of factor theorem.

Consider $f : G/B \to \Omega$: $f(\gamma x) = \gamma f(x)$ for $\gamma \in \Gamma$ and a.e. $x \in G/B$.

What is Ω?

$L^\infty(\Omega) \hookrightarrow L^\infty(G/B)$

Γ-invariant subalgebra

Classify Γ-inv. subalgebras of $L^\infty(G/B)$?

Γ-equivariant factors		Γ-invariant sub-σ-algebras
$f : G/B \to \Omega$	$\xleftrightarrow{\text{1-to-1}}$	$\mathcal{A} \subset \mathcal{B} = \text{Borel}(G/B)$

$f \longleftrightarrow \mathcal{A} = \{ f^{-1}(A) : A \in \text{Borel}(\Omega) \}$

Thm. Every Γ-invariant sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}=\mathrm{Borel}(G/B)$ is G-invariant.

Lem. 1. Every G-inv. sub-σ-algebra $\mathcal{A}$ of $\mathrm{Borel}(G)$ is

$$\mathcal{A}=\{\pi^{-1}(A) : A\in \mathrm{Borel}(G/P)\} \quad (*)$$

where $\pi: G\to G/P$ and P is a closed subgroup.

$\mathcal{L}=L^\infty(G,\mathcal{A})$ - the space of bounded $\mathcal{A}$-measurable functions.

We equip $L^\infty(G)$ with with topology of convergence in measure, that is, the open sets are of the form:

$$\{g : m(\{x\in C : |g(x)-f(x)|\geq \varepsilon\})<\delta\}, \quad C\subset G \text{ - finite measure}, \ f\in L^\infty(G),\ \varepsilon,\delta>0.$$

Let $\mathcal{L}_0=\mathcal{L}\cap C(G)$. Then we check that:

- $\mathcal{L}$ is closed in $L^\infty(G)$,
- $C_c(G)*\mathcal{L}\subset \mathcal{L}_0$,
- $\mathcal{L}_0$ is dence in $\mathcal{L}$.

We set $P_x=\{g\in G : f(xg)=f(x) \text{ for all } f\in\mathcal{L}_0\}$. Since $\mathcal{A}$ is G-invariant, P_x is independent of x. $\mathcal{L}_0$ can be considered as a subalgebra of $C(G/P)$, which separates points. Hence, $\mathcal{L}_0=C(G/P)$ by the Stone-Weierstrass theorem, and $(*)$ holds.

For simplicity, $G = SL(3,\mathbb{R})$.

Up to measure zero,

$$G/B = U = \left\{ \begin{pmatrix} 1 & & 0 \\ u_1 & 1 & \\ u_3 & u_2 & 1 \end{pmatrix} : u_i \in \mathbb{R} \right\}$$

Intermediate subgroups: $P_1 = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right)$, $P_2 = \left(\begin{array}{c|cc} * & * & * \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right)$.

$\mathcal{U} = V \cdot W$, $V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}$, $W = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

<u>G-factors:</u>

$$\begin{tikzcd} & G/B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G/P_1 & & G/P_2 \end{tikzcd}$$

$\pi_1 : G/B \longrightarrow G/P_1$

$(v, w) \longmapsto v.$

$\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}$ – corresponding σ-algebras.

<u>Convergence in measure:</u>

$A_n \to A \Leftrightarrow |(A_n \triangle A) \cap B| \to 0$ for every ball in $\mathcal{U}$.

$\mathcal{A}$ is closed under convergence in measure.

<u>example:</u> $A \subset \mathbb{R}^d$; for a.e. $a \in A$,

$$\tau \cdot (A - a) \xrightarrow[\tau\to\infty]{} \begin{cases} \mathbb{R}^d, & a \in A, \\ \emptyset, & a \notin A. \end{cases}$$

Indeed, for a.e. $a \in A$, $|\tau(A-a) \cap B| = \dfrac{|A \cap a + \tau^{-1}B|}{\tau^{-d}} \to |B|$

by the Lebesgue density Lemma,

and the same holds for A^c.

We set: $a_\tau = \begin{pmatrix} \tau^{-1/3} & & 0 \\ & \tau^{-1/3} & \\ 0 & & \tau^{2/3} \end{pmatrix}$, $V = \begin{pmatrix} I & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ * \, * & 1 \end{pmatrix}$.

$a_\tau : G/B \to G/B : (v, w) \longmapsto (\tau \cdot v, w)$.

As in the example:

<u>Lem. 2.</u> For every Borel $A \subset G/B \simeq \mathcal{U}$ and a.e. $v \in V$,

$$a_\tau(v^{-1}A) \xrightarrow[r\to\infty]{} V \cdot A_v,$$

where $A_v = \{ w \in W : (v, w) \in A \}$.

<u>Lem. 3.</u> For a.e. $v \in V$, $\overline{\{\Gamma v a_r^{-1}\}_{r\geq 1}} = G$.

We know by Moore's ergodicity Thm, that $\{a_r\}$ act ergodically on $\Gamma\backslash G$. This implies that for a.e. $g \in G$,

$$\overline{\{\Gamma g a_r^{-1}\}_{r\geq 1}} = G.$$

a.e. $g = v \cdot p$ where $v \in V$ and $p \in P_1$.

Then $\Gamma g a_r^{-1} = \Gamma v a_r^{-1} \cdot (a_r p a_r^{-1})$, where

$$a_r p a_r^{-1} = \left(\begin{array}{cc|c} p_{11} & p_{12} & r^{-1}p_{13} \\ p_{21} & p_{22} & r^{-1}p_{23} \\ \hline 0 & 0 & p_{33} \end{array}\right) \xrightarrow[r\to\infty]{} \left(\begin{array}{c|c} * & \begin{matrix}0\\0\end{matrix} \\ \hline 0\ 0 & p_{33} \end{array}\right).$$

Hence, $\overline{\{\Gamma v \bar{a}_r^{-1}\}_{r\geq 1}} = G \iff \overline{\{\Gamma g a_r^{-1}\}_{r\geq 1}} = G.$

Proof of Thm.

1) Suppose that $\mathcal{A} \subset \mathcal{B}_1$.
Then $\exists A \in \mathcal{A}$ with nontrivial A_v (i.e., $|A_v| \neq 0$ and $|A_v^c| \neq 0$).
for $v \in$ positive measure set in V.

By Lem. 2 and Lem. 3, we can pick this v so that

$$\overline{\{\Gamma v a_r^{-1}\}_{r\geq 1}} = G$$

$$a_r(v^{-1}A) \to V \cdot A_W = \tilde{A}.$$

Note that $\tilde{A}$ is nontrivial.

Then $\forall g \in G$: $g_n = \gamma_n v \bar{a}_{r_n}^{-1} \to g$ for $\gamma_n \in \Gamma$, $r_n \geq 1$.

We have $g_n a_{r_n} v^{-1} \cdot A = \gamma_n A \in \mathcal{A}$.

Hence, $g\tilde{A} \in \mathcal{A}$ for all $g \in G$.

This shows that $\mathcal{A} \supset$ nontrivial G-inv. sub-σ-algebra $\neq \mathcal{B}_1$.

Thus, $\mathcal{B}_2 \subset \mathcal{A}$.

2) Similar argument shows that

$$\mathcal{A} \not\subset \mathcal{B}_2 \Longrightarrow \mathcal{B}_1 \subset \mathcal{A}.$$

3) Either: $\mathcal{A} \subset \mathcal{B}_1$, $\mathcal{A} \subset \mathcal{B}_2 \Rightarrow \mathcal{A} \subset \mathcal{B}_1 \cap \mathcal{B}_2 = \{\varnothing, G/P\}$

$\mathcal{A} \subset \mathcal{B}_1$, $\mathcal{A} \not\subset \mathcal{B}_2 \Rightarrow \mathcal{B}_1 \subset \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{B}_1$

$\mathcal{A} \not\subset \mathcal{B}_1$, $\mathcal{A} \subset \mathcal{B}_2 \Rightarrow \mathcal{B}_2 \subset \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{B}_2$

$\mathcal{A} \not\subset \mathcal{B}_1$, $\mathcal{A} \not\subset \mathcal{B}_2 \Rightarrow \mathcal{B}_2 \subset \mathcal{A}$, $\boxed{\mathcal{B}_1 \subset \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{B}.}$