Kazhdan property T.

G - locally compact group

~~Def.~~ G has property T if $\exists$ compact $K \subset G$:
$\varepsilon > 0$:

for every continuous unitary representation
$\pi: G \to \mathcal{U}(\mathcal{H})$, $\mathcal{H}$ - Hilbert space, without fixed vectors

$$\forall v \in \mathcal{H}: \|v\| = 1: \quad \sup_{g \in K} \|\pi(g)v - v\| \geq \varepsilon.$$

(no almost invariant vectors)

Thm. Suppose that G is discrete/countable.
If G is amenable and has property T,
then G is finite.

Consider the regular representation $G \curvearrowright L^2(G)$:

$$\rho(g): f \longmapsto f(g^{-1}x), \quad f \in L^2(G).$$

Since G is amenable, $\forall$ finite $K \subset G$: $\exists f_n \in L^2(G)$: $\|f_n\| = 1$:

$$\max_{g \in K} \|\rho(g) f_n - f_n\| \longrightarrow 0.$$

Then by property T, $L^2(G) \ni$ G-fixed vector.

Hence, $1 \in L^2(G)$ and G is finite. $\square$

Thm. $SL_n(\mathbb{R})$, $n \geq 3$, has property T.

In the proof, we use:

Spectral Theorem:

$\{U_t\}_{t\in\mathbb{R}}$ – one-par. subgroup of unitary operators on $\mathcal{H}$.

$\dim(\mathcal{H})<\infty$: $U_t = \sum_{i=1}^{s} e^{itu_i}\cdot P_i$, $\sum_{i=1}^{s} P_i = id$

where $u_i \in \mathbb{R}$ and P_i's are orthogonal projections on the eigenspaces.

$\dim(\mathcal{H})=\infty$: possibly no eigenvectors,

$\exists! P$: {Borel subsets of $\mathbb{R}$} $\to$ {orthogonal projection on $\mathcal{H}$}

(σ-additive map, $P_{\mathbb{R}} = id$.

$$\boxed{U_t = \int_{\mathbb{R}} e^{itu}\, dP(u)}$$

Proof of Thm.

Consider $\underbrace{SL_2(\mathbb{R})}_{G} \ltimes \underbrace{\mathbb{R}^2}_{A} = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & 1 \end{array}\right) \hookrightarrow SL_n(\mathbb{R})$.

Let $\pi: SL_n(\mathbb{R}) \to \mathcal{U}(\mathcal{H}_\pi)$ be a unitary representation without fixed vectors.

By the spectral theorem for $\rho(A)$,

$$\pi(a) = \int_{\mathbb{R}^2} e^{i\langle a,u\rangle} dP_\pi(u), \quad a \in A,$$

where P_π is a projection-valued $\underbrace{}_{\text{Borel}}$ measure on $\mathbb{R}^2$.

For $g \in G$ and $a \in A$,

$$\pi(g)^{-1}\pi(a)\pi(g) = \pi(g^{-1}ag) = \pi(g^{-1}(a))$$

$$\int_{\mathbb{R}^2} e^{i\langle a,u\rangle} d\left(\pi(g)^{-1} P_\pi(u)\pi(g)\right) \qquad \int_{\mathbb{R}^2} e^{i\langle g^{-1}(a),u\rangle} dP_\pi(u)$$

$$= \int_{\mathbb{R}^2} e^{i\langle a,({}^t g)^{-1}(u)\rangle} dP_\pi(u).$$

Hence, $\pi(g)^{-1} P_\pi(u)\pi(g) = P_\pi(({}^t g)u)$.

Let K be a compact generating set of G.

Suppose that for some representations π_n, without fixed vectors, and $v_n \in \mathcal{H}_{\pi_n}$: $\|v_n\| = 1$: $\sup_{g\in K} \|\pi_n(g)v_n - v_n\| \longrightarrow 0$.

Consider the sequence of prob. measures on $\mathbb{R}^2$:

$$\mu_n(B) = \langle P_{\pi_n}(B)v_n, v_n\rangle \quad \text{for Borel } B \subset \mathbb{R}^2.$$

For $g \in K$,

$$|\mu_n({}^t g B) - \mu_n(B)| = \underbrace{|\langle P_{\pi_n}(B)\pi_n(g)v_n, \pi_n(g)v_n\rangle - \langle P_{\pi_n}(B)v_n, v_n\rangle|}_{\longrightarrow 0.}$$

If $\mu_n(\{0\}) \neq 0$, then $P_{\pi_n}(\{0\}) \neq 0$ and $\mathcal{H}$ contains a $\pi(A)$-fixed vector, but this is impossible by Moore ergodicity thm. (see Furman's lectures)

Hence, μ_n are prob. measures on $\mathbb{R}^2 \setminus \{0\}$ and projecting $\mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1$, we obtain a sequence of prob. measures $\overline{\mu_n}$ on $\mathbb{P}^1$.

Let $\overline{\mu}$ be a weak$*$ limit point $\overline{\mu_n}$.

Then $\overline{\mu}$ is $SL_2(\mathbb{R})$-invariant.

This is impossible. Hence, $\exists \varepsilon > 0: \sup_{g \in K} \|\pi(g)v - v\| \geq \varepsilon$ for all π's without invariant vectors and v's with $\|v\| = 1$.

<u>Thm.</u> If Γ is a lattice in G and G has property T, then Γ has property T.

In the proof we use:

Induced representation:

$\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ - unitary representation of Γ.

Define $\hat{\mathcal{H}} = \left\{ f : G \to \mathcal{H} : \begin{array}{l} f(\gamma g) = \pi(\gamma) f(g),\ g \in G, \gamma \in \Gamma \\ \int_{\Gamma \backslash G} \|f(g)\|^2 \, dm_{\Gamma \backslash G}(g) < \infty \end{array} \right\}$

$$\hat{\pi}(g) : \hat{\mathcal{H}} \to \hat{\mathcal{H}} : f \mapsto f(x \cdot g).$$

Then $\hat{\pi} : G \to \mathcal{U}(\hat{\mathcal{H}})$ is a unitary representation.

<u>Proof of Thm:</u>

For simplicity, let's assume that G/Γ is compact.

Then $\exists$ relatively compact Borel $C \subset G$:

$$G = \bigsqcup_{\gamma \in \Gamma} \gamma C \quad (\text{disjoint union})$$

$$\underset{g}{\Psi} = \gamma(g) c(g)$$

Suppose that $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a representation such that $\forall$ finite $S \subset \Gamma$: $\exists v_n \in \mathcal{H} : \|v_n\| = 1$:

$$\max_{\gamma \in S} \|\pi(\gamma) v_n - v_n\| \to 0.$$

Consider the induced representation $\hat{\pi} : G \to \mathcal{U}(\hat{\mathcal{H}})$ and $f_n(\gamma c) = \pi(\gamma) v_n \in \hat{\mathcal{H}}$, $\|f_n\| = 1$.

Given compact $K \subset G$, $\exists$ finite $S \subset \Gamma$: $C \cdot K \subset S \cdot C$
(since Γ is discrete)

Then for $g \in K$, $\|\hat{\pi}(g) f_n - f_n\|^2 = \int_C \|\pi(\underbrace{\gamma(cg)}_{\in S}) v_n - v_n\|^2 dc \to 0.$

Since G has property T, $\hat{\mathcal{H}} \ni G$-inv. vector: $f(g) = v_0 \in \mathcal{H}$ for a.e. g. Since $f(\gamma g) = \pi(\gamma) f(g)$, v_0 is Γ-inv. Hence, Γ has property T. $\square$

<u>Thm.</u> If Γ is discrete and has property T, then Γ is finitely generated.

<u>Proof.</u> Consider $\mathcal{H} = \widehat{\bigoplus_{\Delta}} L^2(\Gamma/\Delta)$ where Δ runs over finitely generated subgroups.
For every finite $S \subset \Gamma$, $\delta_{e\langle S\rangle}$ is S-invariant.
Since Γ has property T, $\mathcal{H} \ni \Gamma$- inv. vector.
Then $L^2(\Gamma/\Delta) \ni \Gamma$-inv. vector for some Δ,
and Γ/Δ is finite. $\square$

<u>Cor.</u> If Γ is a lattice in $SL_n(\mathbb{R})$, $n \geq 3$, then Γ is finitely generated.