Amenability

G – topological group (e.g., G = a closed subgroup of $SL_n(\mathbb{R})$).

Def. V – locally convex top. vector space

$\Omega \subset V$ – nonempty, compact, convex

$G \curvearrowright \Omega$ – affine continuous action.

The group G is called <u>amenable</u> if Ω contains a G-fixed point.

Application X – compact metric space

$G \curvearrowright X$ – continuous action.

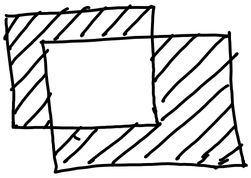
Then $G \curvearrowright \mathrm{Prob}(X)$ – convex and compact (in weak* topology).

Hence, $\exists$ G-inv. prob. measure on X.

Prop. $\mathbb{Z}^d$ and $\mathbb{R}^d$ are amenable.

$\ulcorner$ Consider $\mathbb{Z}^d \curvearrowright \Omega$.

Let $B_N = [1,N]^d$ and $\omega_N = \frac{1}{|B_N|} \cdot \sum_{z \in B_N} z \cdot \omega$ for $\omega \in \Omega$.

B_N  $z_0 + B_N$

$$\frac{|B_N \triangle (z_0 + B_N)|}{|B_N|} \xrightarrow[N \to \infty]{} 0.$$

Then $z_0 \cdot \omega_N - \omega_N = \frac{1}{|B_N|} \cdot \left(\sum_{z \in (z_0 + B_N) \setminus B_N} z \cdot \omega - \sum_{z \in B_N \setminus (z_0 + B_N)} z \cdot \omega \right)$

$$\in \frac{|B_N \triangle (z_0 + B_N)|}{|B_N|} \cdot (\pm \Omega) \longrightarrow 0.$$

By compactness, $\omega_{N_i} \to \omega_\infty \in \Omega$, and $z_0 \cdot \omega = \omega$ for all $z_0 \in \mathbb{Z}^d$!

<u>Prop.</u> Suppose that $G = G_0 \trianglerighteq G_1 \trianglerighteq \dots \trianglerighteq G_\ell \trianglerighteq G_{\ell+1} = \{e\}$ where G_i is closed and G_i / G_{i+1} is amenable. Then G is amenable.

Consider $G \curvearrowright \Omega$.
Since G_ℓ is amenable, $\Omega^{G_\ell} \neq \emptyset$.
Clearly, Ω^{G_ℓ} is closed ($\Rightarrow$ compact) and convex.
Consider $G_{\ell-1}/G_\ell \curvearrowright \Omega^{G_\ell}$ ...

<u>Prop.</u> $SL_n(\mathbb{R})$ is not amenable.

Consider $SL_n(\mathbb{R}) \curvearrowright \mathbb{P}^{n-1}$.
Suppose that $SL_n(\mathbb{R})$ is amenable.
Then $\exists$ invariant prob. measure ν on $\mathbb{P}^{n-1}$.
However, $\mathrm{supp}(\nu) \subset \mathrm{Fix}(g)$ for unipotent g.
Since $SL_n(\mathbb{R})$ is generated by unipotents,

$$\mathrm{supp}(\nu) \subset \mathrm{Fix}(G) = \emptyset$$

which is a contradiction. $\square$

Now we assume that G is discrete and coutable.

amenability $\Leftrightarrow$ invariant means $\Leftrightarrow$ almost invariant vectors

~~Def.~~ A linear map $M: L^\infty(G) \to \mathbb{C}$ is (left) invariant mean if

1) $M(1) = 1$
2) $f \geq 0 \Rightarrow M(f) \geq 0$
3) $M(g \cdot f) = M(f)$ for $g \in G$. (here: $(g\cdot f)(x) = f(g^{-1}x)$)

Note that

$$-\|f\|_\infty \leq f \leq \|f\|_\infty \Rightarrow -\|f\|_\infty \leq M(f) \leq \|f\|_\infty \Rightarrow |M(f)| \leq \|f\|_\infty,$$

so that $M \in L^\infty(G)^*$.

Thm. G is amenable $\Leftrightarrow$ $\exists$ invariant mean on $L^\infty(G)$.

$\Rightarrow$ $L^\infty(G)^* = \{$ bounded linear functionals on $L^\infty(G)^*$ with weak* topology $\}$

$\cup$

$\mathcal{M} = \{M : (1) \& 2\}$ — convex; nonempty (e.g. Dirac measures); compact (Banach-Alaoglu thm)

$G \curvearrowright L^\infty(G)^* : (g\cdot M)(f) = M(f(g^{-1}x))$

This action is continuous in weak* topology.

Hence, $\mathcal{M} \ni G$-fixed point.

$\Leftarrow$ is not used below.

Thm. G is amenable $\Leftrightarrow$ $\left[\begin{array}{l}\forall \text{finite } K \subset G : \exists \varphi_n \in L^2(G) : \|\varphi_n\| = 1 : \\ \|g\cdot\varphi_n - \varphi_n\| \to 0 \text{ for } g \in K.\end{array}\right]$

(almost invariant vector)

$\Rightarrow$ Let M be an invariant mean on $L^\infty(G)$.

For $\varphi \in L^1(G)$, we define $L_\varphi \in L^\infty(G)^*$ by

$$L_\varphi(f) = \langle \varphi, f\rangle = \sum_{x\in G} \varphi(x) f(x).$$

For a finite partition $\sigma = \{G_1, \dots G_e\}$ of G, we define $\varphi_\sigma(g) = \sum_{i=1}^{e} M(\chi_{G_i})\, \chi_{\{g_i\}} \in L^1(G)$,

where $g_i \in G_i$.

Then $\|\varphi_\sigma\|_1 = \sum_{i=1}^{e} M(\chi_{G_i}) = 1$.

If σ is a refinement of the partition $\{A, G\setminus A\}$, then $\langle \varphi_\sigma, \chi_A\rangle = M(\chi_A)$.

Recall: – weak topology on $L^1(G)$:

$$\varphi_n \to \varphi \Leftrightarrow \langle \varphi_n, f\rangle \to \langle \varphi, f\rangle \text{ for all } f \in L^\infty(G).$$

– If $\Omega \subset L^1(G)$ is convex,

$$\text{weak-closure}(\Omega) = \text{norm-closure}(\Omega).$$

Approximating functions in $L^\infty(G)$ by linear combinations of characteristic functions, we deduce that

$\forall f_1, \dots f_s \in L^\infty(G)$: $\exists \sigma_n$: $\langle \varphi_{\sigma_n}, f_i\rangle \xrightarrow[n\to\infty]{} M(f)_i$

Take $g \in G$, $f \in L^\infty(G)$. Then $\exists \sigma_n = \sigma_n(f, g)$: $\langle \varphi_{\sigma_n}, f\rangle \to M(f)$

$$\langle \varphi_{\sigma_n}, g^{-1} f\rangle \to M(g^{-1} f) = M(f).$$

Then $\langle g\cdot\varphi_{\sigma_n} - \varphi_{\sigma_n}, f\rangle = \langle \varphi_{\sigma_n}, g^{-1} f - f\rangle \xrightarrow[n\to\infty]{} 0.$

This shows that $0 \in \text{weak-closure}(\{g\cdot\varphi_\sigma - \varphi_\sigma\})$.

Let $\Omega = \text{convex-closure}(\{g\cdot\varphi_\sigma - \varphi_\sigma\})$.

Since Ω is convex,

$$\text{norm-closure}(\Omega) = \text{weak-closure}(\Omega) \ni 0.$$

Hence, $\exists \varphi_n = \text{convex-closure}(\{\varphi_\sigma\})$: $\|g\cdot\varphi_n - \varphi_n\|_1 \to 0$.

$\Downarrow$

$\|\varphi_n\|_1 = 1$.

Given finite $K \subset G$, we apply the same argument to $L^1(G)^{|K|}$ and $(g\cdot\varphi_\sigma - \varphi_\sigma : g \in K)$.

We deduce that $\exists \varphi_n : \|\varphi_n\| = 1$:

$$\|g\cdot\varphi_n - \varphi_n\|_1 \to 0 \quad \text{for } g \in K.$$

Finally, let $\psi_n = \varphi_n^{1/2}$. Then

$$\|g\cdot\psi_n - \psi_n\|_2^2 = \sum_x |\psi_n(g^{-1}x) - \psi_n(x)|^2 \leq \sum_x |\psi_n(g^{-1}x)^2 - \psi_n(x)^2|$$

$$= \|g\cdot\varphi_n - \varphi_n\|_1 \longrightarrow 0,$$

where we used that $|a-b|^2 \leq |a^2 - b^2|$, $a,b\geq 0$.

$\Leftarrow$ is not used below.