

In this minicourse, we discuss lattices in  $SL_n(\mathbb{R})$ .

Def A lattice in  $SL_n(\mathbb{R})$  is a discrete subgroup such that  $\text{vol}(SL_n(\mathbb{R})/\Gamma) < \infty$ .

example:  $SL_n(\mathbb{Z})$ .

We shall prove the following 3 main results:

1) Borel density thm.

2) (Kazhdan)  $\Gamma < SL_n(\mathbb{R})$ ,  $n \geq 3$ , is finitely generated.

3) (Margulis)  $\Gamma < SL_n(\mathbb{R})$ ,  $n \geq 3$ , and  $N \trianglelefteq \Gamma$ ,  
infinite  
Then  $\Gamma/N$  is finite.

### Borel density thm.

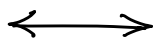
Thm.  $\Gamma$  - lattice in  $G = SL_n(\mathbb{R})$

$f: G \rightarrow GL_N(\mathbb{R})$  - polynomial homomorphism.

Then  $\forall v \in \mathbb{R}^N: f(\Gamma)v = v \Rightarrow f(G)v = v$ .

Idea:

Recurrence  
(ergodic theory)



Transience  
(algebraic actions)

Lem. 1 (Poincaré recurrence)

$X$  - compact metric space  
 $T: X \rightarrow X$  - homeomorphism  
 $\mu$  - invariant Borel probability measure on  $X$

Then for  $\mu$ -a.e.  $x \in X$ ,

$T^{n_i} x \rightarrow x$  along a subsequence  $n_i \rightarrow \infty$ .

Lem. 2.  $T \in GL_N(\mathbb{R})$  - unipotent,  
 $T \subset \mathbb{P}^{N-1}$  - projective space,  
 $[v] \in \mathbb{P}^{N-1}$ ,  $n_i \rightarrow \infty$ .

Then  $T^{n_i} [v] \xrightarrow{i \rightarrow \infty} [v] \implies T v = v$ .

Let  $T = I + S$ , where  $S$  is nilpotent.  
Pick  $k$  such that  $S^k v \neq 0$  and  $S^{k+1} v = 0$ .  
 $T^n [v] = [(I+S)^n v] = \left[ \sum_{i=0}^k \binom{n}{i} S^i v \right] \xrightarrow{n \rightarrow \infty} [S^k v]$ .

Since  $S$  is nilpotent,  $[S^k v] = [v] \implies k=0$ .

Proof of Thm. Consider the map

$$\pi: G/\Gamma \rightarrow \mathbb{P}^{N-1}: g \mapsto [P(g)v],$$

and define measure  $\nu$  on  $\mathbb{P}^{N-1}$ .

$$\nu(B) = \mu(\pi^{-1}(B)), \quad B \subset \mathbb{P}^{N-1}$$

where  $\mu$  is the inv. prob. measure on  $G/\Gamma$ .

Then  $\nu(\mathbb{P}^{N-1}) = 1$ , and  $\nu$  is  $\rho(G)$ -inv.

Take unipotent  $g \in G$ ,  $g \neq I$ .

Then  $T = \rho(g)$  is unipotent.

Indeed, if  $Tw = \lambda w$ , then

$$\rho(g^n)w = \rho(g)^n w = \lambda^n w,$$

↑ polynomial in  $n$

so that  $\lambda = 1$ .

By Lem. 1 & Lem. 2, for  $\nu$ -a.e.  $[w] \in \mathbb{P}^{N-1}$ :  
 $T \cdot w = w$ .

Equivalently, for a.e.  $h \in G$ ,  $T_{\rho(h)}v = \rho(h)v$ .

Then  $\rho(h^{-1}gh) \cdot v = v$  for all  $h \in G$ ,

and  $\rho(G) \cdot v = v$  because  $SL_n(\mathbb{R})$

has no nontrivial infinite normal subgroups.

Thm. (Margulis)  $\Gamma$  - a lattice in  $SL_n(\mathbb{R})$ ,  $n \geq 3$ .  
 $N \trianglelefteq \Gamma$  - infinite.

Then  $|\Gamma/N| < \infty$ .

Strategy:  $\left. \begin{array}{l} - \Gamma/N \text{ is amenable} \\ - \Gamma/N \text{ has property (T)} \end{array} \right\} \Rightarrow \Gamma/N \text{ - finite.}$

