In this minicourse, we discuss lattices in $SL_n(\mathbb{R})$.

Def A lattice in $SL_n(\mathbb{R})$ is a discrete subgroup such that $\mathrm{vol}(SL_n(\mathbb{R})/\Gamma) < \infty$.

example: $SL_n(\mathbb{Z})$.

We shall prove the following 3 main results:

1) Borel density thm.
2) (Kazhdan) $\Gamma < SL_n(\mathbb{R})$, $n \geq 3$, is finitely generated.
3) (Margulis) $\Gamma < SL_n(\mathbb{R})$, $n \geq 3$, and $N \trianglelefteq \Gamma$, infinite. Then Γ/N is finite.

Borel density thm.

Thm. Γ - lattice in $G = SL_n(\mathbb{R})$
$\rho: G \to GL_N(\mathbb{R})$ - polynomial homomorphism.
Then $\forall v \in \mathbb{R}^N$: $\rho(\Gamma) v = v \Rightarrow \rho(G) \cdot v = v$.

Idea: Recurrence (ergodic theory) $\longleftrightarrow$ Transience (algebraic actions)

Lem. 1 (Poincare recurrence)

X - compact metric space

$T: X \to X$ - homeomorphism

μ - invariant Borel probability measure on X

Then for μ-a.e. $x \in X$,

$$T^{n_i} x \to x \text{ along a subsequence } n_i \to \infty.$$

Lem. 2. $T \in GL_N(\mathbb{R})$ - unipotent,

$T \circlearrowright \mathbb{P}^{N-1}$ - projective space,

$[v] \in \mathbb{P}^{N-1}$, $n_i \to \infty$.

Then $T^{n_i}[v] \xrightarrow[i\to\infty]{} [v] \implies Tv = v$.

$\ulcorner$ Let $T = I + S$, where S is nilpotent.

Pick k such that $S^k v \neq 0$ and $S^{k+1} v = 0$.

$$T^n[v] = [(I+S)^n v] = \left[\sum_{i=0}^{k} \binom{n}{i} S^i v\right] \xrightarrow[n\to\infty]{} [S^k v].$$

Since S is nilpotent, $[S^k v] = [v] \Rightarrow k = 0$. $\lrcorner$

Proof of Thm. Consider the map

$$\pi: G/\Gamma \to \mathbb{P}^{N-1}: g \longmapsto [\rho(g) v],$$

and define measure ν on $\mathbb{P}^{N-1}$:

$$\nu(B) = \mu(\pi^{-1}(B)), \quad B \subset \mathbb{P}^{N-1},$$

where μ is the inv. prob. measure on G/Γ.

Then $\nu(\mathbb{P}^{N-1}) = 1$, and ν is $\rho(G)$-inv.

Take unipotent $g \in G$, $g \neq I$.
Then $T = \rho(g)$ is unipotent.
Indeed, if $Tw = \lambda w$, then

$$\underbrace{\rho(g^n) w}_{\text{polynomial in } n} = \rho(g)^n w = \lambda^n w,$$

so that $\lambda = 1$.

By Lem. 1 & Lem. 2, for ν-a.e. $[w] \in \mathbb{P}^{N-1}$: $T \cdot w = w$.

Equivalently, for a.e. $h \in G$, $T\rho(h)v = \rho(h)v$.
Then $\rho(h^{-1}gh) \cdot v = v$ for all $h \in G$,
and $\rho(G) \cdot v = v$ because $SL_n(\mathbb{R})$ has no nontrivial infinite normal subgroups.

Thm. (Margulis) Γ - a lattice in $SL_n(\mathbb{R})$, $n \geq 3$.
$N \trianglelefteq \Gamma$ - infinite.
Then $|\Gamma/N| < \infty$.

Strategy:
- Γ/N is amenable
- Γ/N has property (T)

$\Rightarrow \Gamma/N$ - finite.