Problems

(1) Let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$, and let N be a finite normal subgroup of Γ. Prove that $N < \langle \pm I \rangle$.
(2) Let H be a closed subgroup of $\mathrm{SL}(n,\mathbb{R})$ with finite covolume. Prove that H is discrete.
(3) Show that nonabelian free group is not amenable.
(4) Show that there are infinitely many different invariant means on $L^\infty(\mathbb{Z})$.
(5) (a) Show that a closed subgroup H of amenable group G is amenable.
    (b) Is it true when the subgroup is not closed?
(6) Let Γ be a lattice subgroup in a locally compact group G. Prove that if Γ is amenable, then G is also amenable.
(7) (a) Show that $\mathrm{SL}_2(\mathbb{Z})$ does not have property T.
    (b) Deduce that $\mathrm{SL}(2,\mathbb{R})$ does not have property T.

Hints

(1) Use the Borel density theorem.
(2) Show that the Lie algebra of H is an ideal in the Lie algebra of G.
(3) Construct an action on $\mathbb{P}^1$ without an invariant measure.
(4) Use construction of the means as averages over sequences of intervals I_n.
(5) Given an affine action of H on Ω, consider the space $L^\infty_H(G, \Omega)$ consisting of equivariant measurable maps.
(6) Show that $L^2(G)$ contains an almost invariant vector.
(7) Use that for property-T groups $\Gamma/[\Gamma, \Gamma]$ is finite.