Lecture 5: Mostows rigidity.

Thm (Mostow) Let Γ_1 & Γ_2 be cocompact lattices in $\underbrace{SO(n_1,1)}_{G}$ and $SO(n_2,1)$; $n_1, n_2 \geq 3$.

Let $\alpha : \Gamma_1 \to \Gamma_2$ be an isomorphism.

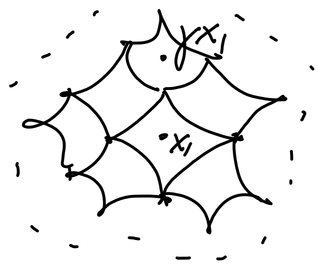
Then $n_1 = n_2$ and $\alpha(\gamma) = g\gamma g^{-1}$ for some $g \in G$.

In the proof, we use that

$$SO(n,1) \simeq Isom(\mathbb{H}^n) \simeq Conf(\underset{\underset{S^{n-1}}{\shortparallel}}{\partial \mathbb{H}^n})$$

We fix $x_i \in \mathbb{H}^{n_i}$ with $Stab_{\Gamma_i}(x_i) = 1$, and denote by D_i the Dirichlet fundamental domain.

Then $\Gamma_i \approx \mathbb{H}^{n_i} = \bigcup_{\gamma \in \Gamma_i} \gamma D_i$



Let $\pi_i : \Gamma_i \to \mathbb{H}^{n_i} : \gamma \mapsto \gamma x_i$.

$$\begin{tikzcd}
\Gamma_1 \arrow[r, "\alpha"] \arrow[d, "\pi_1"'] & \Gamma_2 \arrow[d, "\pi_2"] \\
\mathbb{H}^{n_1} \arrow[r, "F"] \arrow[u, bend left, "\sigma"] & \mathbb{H}^{n_2}
\end{tikzcd}$$

Pick $\sigma : \mathbb{H}^{n_1} \to \Gamma_1$ such that $\sigma \circ \pi_1 = Id$.

Define $F = \pi_2 \circ \alpha \circ \sigma : \mathbb{H}^{n_1} \to \mathbb{H}^{n_2}$.

Def. $f: X_1 \to X_2$ (X_1, X_2 – metric spaces) is a <u>quasi-isometry</u> if $\exists\, \lambda \geq 1$ and $C, \varepsilon > 0$:

1) $\lambda^{-1} d(x,y) - \varepsilon \leq d(F(x), F(y)) \leq \lambda d(x,y) + \varepsilon$
2) $F(X_1)$ is C-dense in X_2.

Prop. $F: \mathbb{H}^{n_1} \to \mathbb{H}^{n_2}$ is a quasi-isometry.

Γ Recall that Γ is finitely generated and fix a finite symmetric generating set S.

word metric on Γ:

$$d_S(\gamma_1, \gamma_2) = \left(\begin{array}{l}\text{min. length of } \gamma_1^{-1}\gamma_2 \text{ written as} \\ \text{a word in } S\end{array}\right).$$

Note that if S_1 & S_2 are two symmetric generating sets, then (Γ, d_{S_1}) and (Γ, d_{S_2}) are quasi-isometric. In particular, $\alpha: \Gamma_1 \to \Gamma_2$ is a quasi-isometry.

We claim that π_i (and hence σ) are also quasi-isometries.

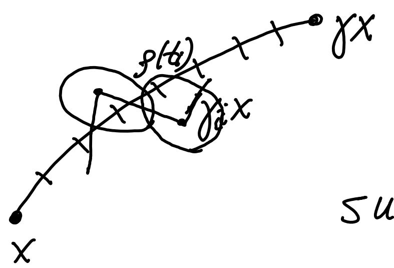
1) $d(\gamma_1 x, \gamma_2 x) = d(\underbrace{\gamma_2^{-1}\gamma_1}_{s_1 \dots s_k,\ s_i \in S} x, x)$

$$\le \sum_{i=1}^{k} d(s_1 \dots s_i x,\ s_1 \dots s_{i-1} x)$$

$$\le \underbrace{\left(\max_{s\in S} d(sx, x)\right)}_{\lambda} \cdot \underbrace{k}_{d_S(\gamma_1,\gamma_2)}.$$

2) Fix $R>0$: - Γx is R-dense,
- $S = \{\gamma \in \Gamma : d(\gamma x, x) \le 3R\}$ generates Γ.

Let $\rho : [0,T] \to \mathbb{H}^n$ $(T = d(x, \gamma x))$ be the geodesic from x to γx.



Take a partition:
$0 = t_0 < t_1 < \dots < t_n = T$
such that $|t_{i+1} - t_i| \le R$ and $n \approx \frac{d(x,\gamma x)}{R}$.

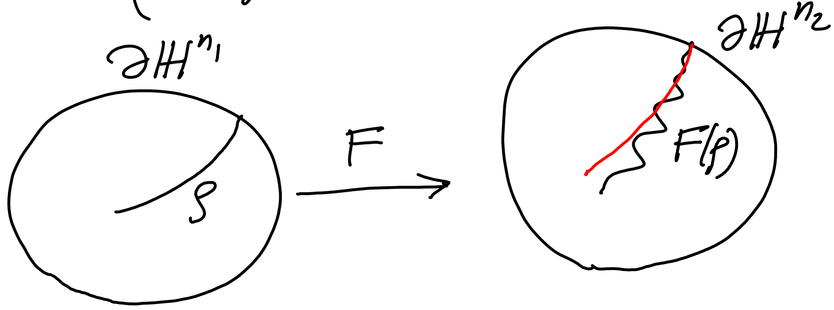
Choose $\gamma_i \in \Gamma$: $d(\gamma_i x, \rho(t_i)) \le R$, $\gamma_0 = e$, $\gamma_n = \gamma$.

Then by triangle inequality,
$d(\gamma_{i-1} x, \gamma_i x) \le 3R \Rightarrow \gamma_{i-1}^{-1}\gamma_i \in S$.

Since $\gamma = (\gamma_0^{-1}\gamma_1) \dots (\gamma_{n-1}^{-1}\gamma_n)$, $d_S(\gamma, e) \le n \approx \frac{d(x,\gamma x)}{R}$.

Quasi-isometry $F: \mathbb{H}^{n_1} \to \mathbb{H}^{n_2}$ $\rightsquigarrow$ $F: \partial\mathbb{H}^{n_1} \to \partial\mathbb{H}^{n_2}$

$$\partial\mathbb{H}^n = \{\text{geodesic rays in } \mathbb{H}^n\}/\sim$$
$$(\rho_1 \sim \rho_2 \text{ if } d(\rho_1,\rho_2) < \infty.)$$

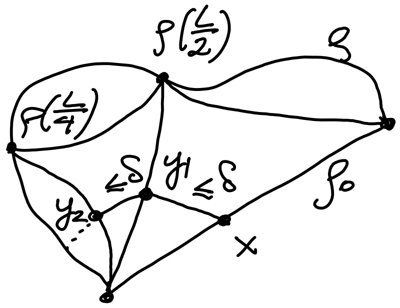


$F(\rho)$ is bounded distance from a geodesic ray.

<u>Prop.</u> Let $\rho: [0,T] \to \mathbb{H}^n$ be a (λ,ε)-quasi-geodesic.
Then $\exists$ geodesic ρ_0 (with the same end points):
$$d(\rho,\rho_0) \leq C(\lambda,\varepsilon).$$

Thin Triangle Property: $\exists\, \delta > 0$: $\forall$ triangle $T = S_1 \cup S_2 \cup S_3$:
$$S_1 \subset B_\delta(S_2 \cup S_3).$$

Lem. Let γ be a curve and γ_0 the geodesic with the same end points.
Then $\forall x \in \gamma_0$: $d(x,\gamma) \leq \delta \cdot |\log_2 L(\gamma)| + 1$.



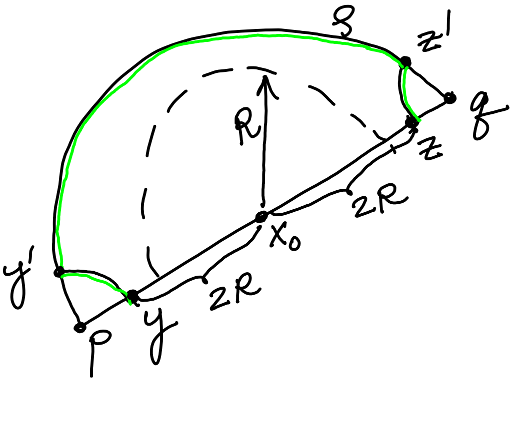
Let $L = L(\gamma)$.
By the Thin Triangle Property, we can construct $y_1, \dots y_n, \dots$ such that $d(x,y_1), \dots d(y_i, y_{i+1}) \leq \delta$,

and $d(y_n, \text{"one of vertices on } \gamma\text{"}) \leq \frac{L}{2^{n+1}}$.

Then $d(x,\gamma) \leq n \cdot \delta + \frac{L}{2^{n+1}}$. Take $n \approx \log_2 L$. $\square$

Proof of Prop. Without loss of generality,

- γ is continuous and piecewise geodesic,
- $L(\gamma|_{[t_1,t_2]}) \leq \lambda \cdot d(\gamma(t_1), \gamma(t_2)) + \varepsilon$.



Let $R = \max\{d(x,\gamma) : x \in \text{geodesic } [p,q]\}$
$= d(x_0, \gamma)$

Take $y, z \in [p,q]$: $d(y,x_0) = d(z,x_0) = 2R$ (or the end points)

$\exists\, y', z' \in \rho :\ d(y,y'),\ d(z,z') \leq R.$

The path $\sigma : y \sim y' \sim z' \sim z$ lies outside $B(x_0,R)$, so that $d(x_0,\sigma) \geq R$.

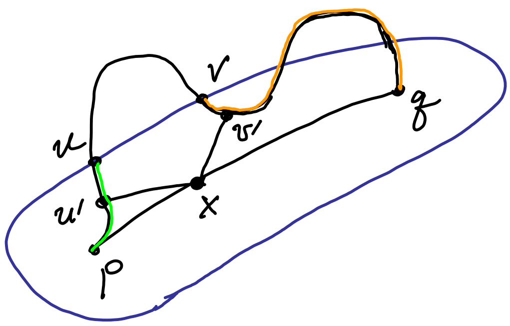
Since $d(y',z') \leq 6R$, $L(\rho|_{[y',z']}) \leq \lambda \cdot 6R + \varepsilon$.

Hence, $L(\sigma) \leq (\lambda \cdot 6R + \varepsilon) + 2R$.

By the lemma, $\underbrace{d(x_0,\sigma)}_{\substack{\rotatebox{90}{$\leq$}\\ R}} \leq \delta\,|\log_2 \underbrace{L(\sigma)}_{\leq \lambda \cdot 6R + \varepsilon + 2R}| + 1$

This implies a uniform upper bound on R.

Now we need to show that $\overbrace{\rho}^{\text{points on}}$ is within bounded of ρ_0.



Pick a maximal segment $\rho|_{[u,v]}$ which doesn't intersect $B_R([p,q])$.

Let $S_1 = \{ x \in [p,q] :\ d(x, \rho|_{[p,u]}) \leq R \}$,
$S_2 = \{ x \in [p,q] :\ d(x, \rho|_{[v,q]}) \leq R \}$.

The sets S_i are closed, and $S_1 \cup S_2 = [p,q]$.

Hence, $S_1 \cap S_2 \neq \emptyset$, and

$$\exists u' \in \rho|_{[p,u]},\ v' \in \rho|_{[v,q]},\ x \in [p,q]:$$
$$d(u',x),\ d(v',x) \leqslant R.$$

Then $d(u',v') \leqslant 2R$, and for every $z \in \rho|_{[u,v]}$,

$$d(z,x) \leqslant \lambda \cdot 2R + \varepsilon + R.$$

We define a map
$$\overline{F}: \partial \mathbb{H}^{n_1} \longrightarrow \partial \mathbb{H}^{n_2}$$
$$[\rho] \longmapsto [F(\rho)]$$

Note that:
- $\overline{F}$ is a bijection (inverse is defined by $\overline{F}_{\alpha^{-1}}$).
- $\overline{F}(\gamma x) = \alpha(\gamma)\overline{F}(x)$ (since $F|_{\Gamma \cdot x_1}$ is equivariant).

__Prop.__ $\overline{F}$ is continuous and quasiconformal.

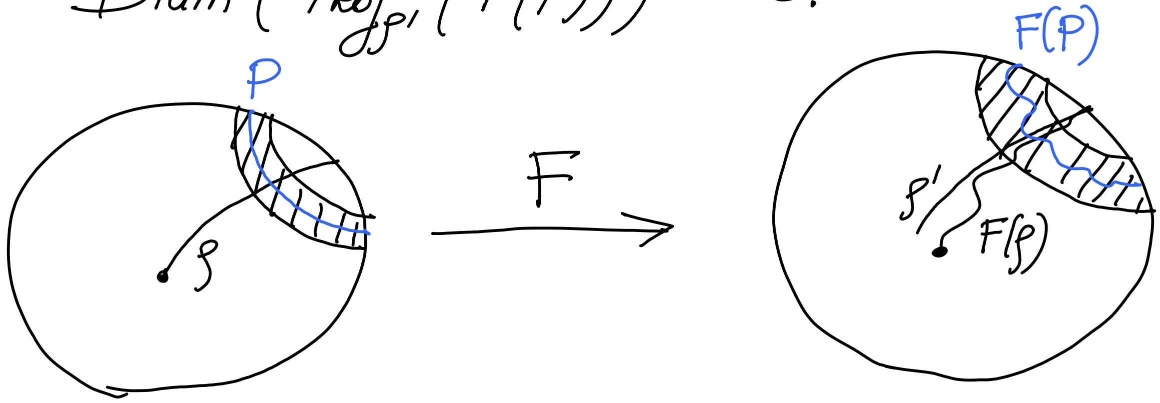
__~~Def~~__ $\varphi: S^{n-1} \to S^{m-1}$ is __K-quasi-conformal__ if

$$\forall x \in S^{n-1}: \quad \overline{\lim_{r\to 0}} \frac{\sup_{z \in S_r(x)} d(\varphi(z),\varphi(x))}{\inf_{z \in S_r(x)} d(\varphi(z),\varphi(x))} \leqslant K.$$

This follows from the following:

<u>Lem.</u> Let ρ be a geodesic ray, P a geodesic hyperplane orthogonal to ρ, and ρ' a geodisic ray such that $d(\rho', F(\rho)) < \infty$.
Then $\exists$ uniform $R > 0$:

$$\mathrm{Diam}\left(\mathrm{Proj}_{\rho'}(F(P))\right) \leq R.$$



<u>Thm</u> (Rademacher–Stepanov)
Let $\varphi : S^{n-1} \to S^{m-1}$ $(n, m \geq 3)$ be a quasi-conformal homeomorphism. Then φ is differentiable a.e., and $\exists \lambda \geq 1$: $\forall$ a.e. $x \in S^{n-1}$: $\forall v \in T_x S^{n-1}$:

$$\lambda^{-1} \|v\| \leq \|(D\varphi)_x v\| \leq \lambda \cdot \|v\|.$$

(In particular, $n = m$.)

Prop. For a.e. x, $(DF)_x$ is conformal.

Consider the map $f: T^1S^{n-1} \to \mathbb{R}$

$$(x,v) \longmapsto \frac{\|DF_x(v)\|}{\|DF_x\|}.$$

Using that $F(\gamma x) = \alpha(\gamma)F(x)$ and $\Gamma \curvearrowright S^{n-1}$ is conformal, we deduce that f is Γ-invariant.

The group $G = SO(n,1)$ acts transitively on T^1S^{n-1}, and $T^1S^{n-1} \simeq G/H$, where H is a closed noncompact subgroup.

By Howe–Moore Thm, $H \curvearrowright G/\Gamma$ is ergodic, and hence $\Gamma \curvearrowright G/H$ is ergodic.

Hence, $f \overset{a.e.}{=} \text{const}$.

Thm. If $\varphi: S^{n-1} \to S^{n-1}$ is quasi-conformal homeomorphism, and $(D\varphi)_x$ is conformal a.e. Then φ is conformal.

Now it follows that $F(x) = g \cdot x$ for some $g \in G$,
and $g \cdot \gamma x = \alpha(\gamma) \cdot gx$ for all $x \in S^{n-1}$.
Hence, $\alpha(\gamma) = g \cdot \gamma \cdot g^{-1}$.

$\lrcorner$