Lecture 4. Deformations of discrete groups.

G = connected Lie group

Γ = discrete group

Consider a homomorphism $\rho : \Gamma \to G$.

Can ρ be deformed?

Trivial deformations: $\gamma \longmapsto g\rho(\gamma)g^{-1}$, $g \in G$.

Assume that $\Gamma = \langle \gamma_1, \dots \gamma_s \rangle \simeq F/N$.
(F = free group, N = normal subgroup).

$$\mathrm{Hom}(\Gamma, G) \cong \{ g \in G^s : w(g) = 1 \text{ for } w \in N \}$$

G by conjugation

Def $\rho \in \mathrm{Hom}(\Gamma, G)$ is locally rigid if ρ^G contains a nbhd of ρ in $\mathrm{Hom}(\Gamma, G)$.

Prop. (rigidity $\Rightarrow$ arithmeticity)
Let $G < GL_d(\mathbb{C})$ be an algebraic group defined$/\mathbb{Q}$.
Assume that $\rho : \Gamma \to G$ is locally rigid.
Then $\exists g \in G$: $g \cdot \rho(\Gamma) g^{-1} \subset GL_d(K)$
where K is a finite extension of $\mathbb{Q}$.

$X = \mathrm{Hom}(\Gamma, G)$ is an affine algebraic variety defined over $\mathbb{Q}$.

It is known that $X(\overline{\mathbb{Q}})$ is dense in X.

In particular, every nbhd of ρ contains ρ' with $\rho'(\Gamma) \subset GL_d(\overline{\mathbb{Q}})$.

Now apply local rigidity...

Deformations $\longrightarrow$ **Cocycles**

$\rho_t : \Gamma \to G$, $\rho_0 = \rho$

one-parameter smooth family

Define $c : \Gamma \to \mathrm{Lie}(G) = \mathfrak{g}$:

$$c(\gamma) = \left(\frac{d}{dt} \rho_t(\gamma)\right)\Big|_{t=0} \circ \rho_0(\gamma)^{-1}.$$

By Product Rule,

$$\boxed{c(\gamma_1\gamma_2) = c(\gamma_1) + Ad(\rho_0(\gamma_1))c(\gamma_2)} \quad (*)$$

Trivial deformations

$\rho_t(\gamma) = g_t \rho_0(\gamma) g_t^{-1}$, $g_0 = e$

$$\boxed{c(\gamma) = X - Ad(\rho_0(\gamma))X} \quad (**)$$

where $X = \frac{dg_t}{dt}\Big|_{t=0}$

Cocycles: $\underset{\cup}{Z^1} = \{c : \Gamma \to \mathfrak{g} : \text{satisfying } (*)\}$

Coboundaries: $B^1 = \{c : \Gamma \to \mathfrak{g} : \text{satisfying } (**)\}$

Cohomology: $H^1(\Gamma, Ad\circ\rho) = Z^1/B^1$.

<u>Thm</u> (Weil) If $H^1(\Gamma, Ad\circ\rho) = 0$, then $\rho : \Gamma \to G$ is locally rigid.

Recall that $\Gamma = \langle \gamma_1, \dots \gamma_s \rangle \simeq F/N$.
ρ is associated to $\bar{g} = (\rho(\gamma_1), \dots \rho(\gamma_s)) \in G^s$.

Consider:

$$G \xrightarrow{\varphi} \underset{\substack{\cup \\ Hom(\Gamma, G) = \psi^{-1}(e)}}{G^s} \xrightarrow{\psi} \prod_{w \in N} G$$

where $\varphi : g \longmapsto (g g_i g^{-1})$,
$\psi : \bar{h} \longmapsto (w(\bar{h}) : w \in N)$.

Local rigidity is equivalent to:

$$\varphi(G) \supset \text{a nbhd of } \bar{g} \text{ in } \psi^{-1}(e)$$

$$\Uparrow \longleftarrow \text{Implicit function Thm}$$

$$Im(d\varphi_e) = Ker(d\psi_{\bar{g}}). \quad (*)$$

We use the identification:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "d\varphi_e"] \arrow[dr, dashed] & T_{\bar g}(G^S) \arrow[r, "d\psi_{\bar g}"] \arrow[d, phantom, "\wr\,\times \bar g^{-1}"] & \mathfrak{g}^N \\
& \mathfrak{g}^S \arrow[ur, dashed] &
\end{tikzcd}$$

Then $d\varphi_e(X) = (X - Ad(g_i)X)$,

and $Im(d\varphi_e) \simeq B^1$.

Every $\bar X \in \mathfrak{g}^S$ defines a cocycle

$$c_{\bar X} : F \to \mathfrak{g}.$$

One can check that $c_{\bar X}(w) = (dw)_{\bar g}(\bar X \cdot \bar g)$.

Hence, $Ker(d\psi_g) \simeq Z^1$, and

$(*) \Leftrightarrow H^1(\Gamma, Ad\circ\rho) = 0.$

Thm. (Weil, Garland, Raghunathan, Margulis, ...)
Let Γ be a lattice in a noncompact simple Lie group G.
Then $H^1(\Gamma, Ad) = 0$ unless:

- $G \sim SL_2(\mathbb{R})$,
- $G \sim SL_2(\mathbb{C})$ and Γ is nonuniform.