Lecture 3: Unipotent elements in lattices.

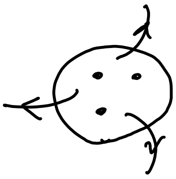
Thm (Kazhdan–Margulis) Let Γ be a lattice in $G = SL_n(\mathbb{R})$.
$\Gamma \ni$ nontrivial unipotent element $\Leftrightarrow$ G/Γ is not compact.

We note that $\overline{g^G} \ni e \Leftrightarrow g$ is unipotent.
Hence, $\Rightarrow$ follows from Compactness Criterion.
Conversely, G/Γ is noncompact $\Rightarrow \exists g_n \in G, \gamma_n \in \Gamma \setminus \{e\}$
$$g_n \gamma_n g_n^{-1} \to e.$$

So we need to study how discrete groups $g_n \Gamma g_n^{-1}$ look in a nbhd of identity.

For $g \approx e$, define $|g| = \|\exp^{-1}(g)\|$, $\|X\| = \sqrt{\sum_{i,j} X_{ij}^2}$.
Let $U_r = \{g \in G : |g| < r\}$.

Prop. $\forall c > 1 : \exists r \approx 0$ and compact $E \subset G$:
$\forall$ discrete subgroup $\Gamma < G$: $\exists g \in E$:
$$|g\gamma g^{-1}| \geq c|\gamma|, \quad \gamma \in U_r \cap \Gamma.$$



$U_r \cap \Gamma$

Sketch of the proof.

1) (Zassenhaus) $\exists$ a nbhd of e in G: $\forall$ discrete $\Gamma < G$:
$\langle U \cap \Gamma \rangle \subset$ connected nilpotent subgroup.

$\ulcorner$ Let $g = I + \varepsilon X$, $h = I + \delta Y$ with $\|X\|, \|Y\| = 1$, $\varepsilon, \delta \approx 0$.
Then $[g, h] = \varepsilon\delta[X, Y] + \ldots$

Hence, for sufficiently small nbhd $\mathcal{U}$,

$$\mathcal{U},\ [\mathcal{U},\mathcal{U}],\ [\mathcal{U},[\mathcal{U},\mathcal{U}]], \dots\ \mathcal{U}^{(n)}, \dots$$

shrinks to identity. By discreteness, $(\mathcal{U}\cap\Gamma)^{(n)} = 1$.

This implies that $\langle \mathcal{U}\cap\Gamma\rangle$ is nilpotent.

2) $$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^+ \qquad \mathfrak{g} \xrightarrow{\pi_2} \mathfrak{n}^+, \quad \mathfrak{g} \xrightarrow{\pi_1} \mathfrak{b}$$

$$\mathfrak{b} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \qquad \mathfrak{n}^+ = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

For every nilpotent subalgebra $\mathfrak{h} \subset \mathfrak{g}$,
$\exists g \in G$: $Ad(g)\mathfrak{h} \cap \mathfrak{b} = 0$.

3) $\forall$ nilpotent subalgebra $\mathfrak{h} \subset \mathfrak{g}$ $\forall c > 0$
$\exists g \in G$: $\|Ad(g)X\| \geq c\|X\|$, $X \in \mathfrak{h}$.

Without loss of generality, $\mathfrak{h} \cap \mathfrak{b} = 0$.
Then $\pi_2 : \mathfrak{h} \to \mathfrak{n}^+$ is injective, and $\|\pi_2(X)\| \geq c_0\|X\|$, $X \in \mathfrak{h}$, for some $c_0 > 0$.

Take (diagonal) $g \in G$: $\|gYg^{-1}\| \geq c_0^{-1}c \cdot \|Y\|$, $Y \in \mathfrak{n}^+$.

Then
$$\|Ad(g)X\| = \|Ad(g)\pi_1(X) + Ad(g)\pi_2(X)\| \geq \|Ad(g)\pi_2(X)\| \geq c_0^{-1}c\,\|\pi_2(X)\| \geq c \cdot \|X\|.$$

4) $\forall c > 1$ $\exists$ compact $E \subset G$: $\forall$ nilpotent $\mathfrak{h} \subset \mathfrak{g}$:
$\exists g \in E$: $\|Ad(g)X\| \geq c \cdot \|X\|$, $X \in \mathfrak{h}$.

Compactness of the space of nilpotent subalgebras.

Cor. $\exists$ nbhd $\mathcal{W}$ of e in G: $\forall$ discrete $\Gamma < G$:
$\exists g \in G$: $g\Gamma g^{-1} \cap \mathcal{W} = \{e\}$.

$\ulcorner$ Take $s > 0$: $g\mathcal{U}_s g^{-1} \subset \mathcal{U}_r$ for all $g \in E^{-1}$.
For $g \in G$, set $r(g) = \min\{|\gamma| : \gamma \in g\Gamma g^{-1} \setminus \{e\}\} > 0$.
Suppose that $r_0 = \sup_{g \in G} r(g) < s$.
Take $g \in G$: $r(g) > c^{-1} r_0$.
We have $\mathcal{U}_s \cap g\Gamma g^{-1} \neq \{e\}$ and $\forall \underset{\neq e}{\gamma} \in g\Gamma g^{-1} \cap \mathcal{U}_s$: $|\gamma| > c^{-1} r_0$.
Take $h \in E$: $|h\gamma h^{-1}| \geq c|\gamma|$ for all $\gamma \in g\Gamma g^{-1} \cap \mathcal{U}_r$.
Now $\forall \gamma \in hg\Gamma g^{-1} h^{-1} \cap \mathcal{U}_s$: $\gamma = h\delta h^{-1}$ with $\delta \in g\Gamma g^{-1}$,
where $\delta = h^{-1}\gamma h \in \mathcal{U}_r$, and $|\gamma| = |h\delta h^{-1}| \geq c|\delta| \Rightarrow \delta \in \mathcal{U}_s$.
Then $|\gamma| \geq c|\delta| > c \cdot c^{-1} r_0 = r_0$ for all $\underset{\neq e}{\gamma} \in hg\Gamma g^{-1} h^{-1}$!
This shows that $r(gh) > r_0$, which is a contradiction.
Hence, $r_0 \geq s$, and we can take $\mathcal{W} = \mathcal{U}_s$. $\lrcorner$

Cor. $\exists c > 0$: $\forall$ lattice $\Gamma < G$: $\mathrm{vol}(G/\Gamma) \geq c$.

$\ulcorner$ Take a nbhd $\mathcal{U}$ of e in G: $\mathcal{U}^{-1} \cdot \mathcal{U} \subset \mathcal{W}$.
Then $\mathcal{U} \to G/g\Gamma g^{-1}$ is injective.
Hence, $\underset{\mathrm{Vol}(G/\Gamma)}{\underset{\parallel}{\mathrm{Vol}(G/g\Gamma g^{-1})}} \geq \mathrm{Vol}(\mathcal{U})$. $\lrcorner$

Cor. Γ^G accumulates at e $\Rightarrow$ $\Gamma \ni$ nontrivial unipotent element.

Key Property: $\forall$ discrete $\Delta < \Gamma$: $\exists h \in E$: $|h\gamma h^{-1}| \geq c|\gamma|$ for all $\gamma \in \mathcal{U}_r \cap \Gamma$. $(*)$

Take $s>0$: $g\mathcal{U}_s g^{-1} \subset \mathcal{U}_r$ for $g \in E$.

By Compactness Criterion, $\exists$ compact $K \subset G$:

$$\{g \in G:\ g\Gamma g^{-1} \cap \mathcal{U}_s = 1\} \subset K\Gamma. \quad (**)$$

Let $\Pi = \{\gamma \in \Gamma:\ K\gamma \cap E\mathcal{U}_s E^{-1}K \neq \emptyset\}$. – finite set (since Γ is discrete.)

For non-unipotent $\gamma \in \Pi$, $\gamma^G \not\ni e$.

Taking smaller $s>0$, we may assume that:

$$\gamma^G \cap \mathcal{U}_s \neq \emptyset,\ \gamma \in \Pi \Rightarrow \gamma\text{ - unipotent}.$$

Take: $g \in G$ such that $g\Gamma g^{-1} \cap \mathcal{U}_s \neq 1$.

$h_0 \in E$ such that $(*)$ holds for $\Delta_0 = g\Gamma g^{-1}$.

$h_1 \in E$ such that $(*)$ holds for $\Delta_1 = h_0 \Delta_0 h_0^{-1}$.

$\cdots\cdots\cdots$

$h_n \in E$ —— " —— $\Delta_n = h_{n-1}\Delta_{n-1}h_{n-1}^{-1}$.

Let $g_n = h_n \cdots h_0$.

If $\gamma \in \Delta_n$ and $h_n \gamma h_n^{-1} \in \mathcal{U}_s$, then $\gamma \in \mathcal{U}_r$ and by $(*)$,

$|h_n\gamma h_n^{-1}| \geq c|\gamma|$, so that $|\gamma| \leq c^{-1}s$.

Inductively, if $\gamma \in \Delta_0$ and $g_n\gamma g_n^{-1} \in \mathcal{U}_s$, then $|\gamma| \leq c^{-(n+1)}s$.

Since Δ_0 is discrete, this show that

$\Delta_n \cap \mathcal{U}_s = 1$ after finitely many steps.

Take the smallest n with this property.

Then: 1) $\underbrace{(g_{n-1}g)\Gamma(g_{n-1}g)^{-1}}_{\Delta_n} \cap \mathcal{U}_s = 1.$

Hence, by (**), $g_{n-1}g\gamma \in K$ for some $\gamma \in \Gamma$.

2) $\exists \delta \in \Gamma \setminus \{e\}: (g_{n-2}g)\delta(g_{n-2}g)^{-1} \in \mathcal{U}_s$.

Then $g\delta g^{-1} \in \mathcal{U}_s$, and

$$g_{n-1}g\delta\gamma = \underbrace{h_{n-i}}_{\in E}\underbrace{\left(g_{n-2}g\delta(g_{n-2}g)^{-1}\right)}_{\in \mathcal{U}_s}\cdot\underbrace{h_{n-i}^{-1}}_{\in E^{-1}}\cdot\underbrace{(g_{n-1}g\gamma)}_{\in K}$$

$$\| \qquad \underbrace{(g_{n-1}g\gamma^{-1})}_{\in K}\cdot\gamma^{-1}\delta\gamma$$

Hence, $\rho = \gamma^{-1}\delta\gamma \in \Pi$.

Since $\rho^G \cap \mathcal{U}_s \neq \emptyset$, ρ is unipotent. $\square$