Lecture 2: Diagonalisable subgroups in lattices.

Compactness criterion.

Prop. Let Γ be a lattice in G and $\Omega \subset G$. Then $\Omega\Gamma$ is relatively compact in $G/\Gamma \iff e$ is an isolated point in $\Gamma^\Omega = \{g\gamma g^{-1} : g \in \Omega, \gamma \in \Gamma\}$.

$\Rightarrow$ Suppose that $g_n \gamma_n g_n^{-1} \to e$ for $g_n \in \Omega$, $\gamma_n \in \Gamma \setminus \{e\}$.
Since $\Omega\Gamma$ is rel. compact, $g_n = h_n \delta_n$ for bounded $h_n \in G$ and $\delta_n \in \Gamma$. Hence, passing to a subsequence $h_n \to h$.
Then $\delta_n \gamma_n \delta_n^{-1} \to e$, and $\gamma_n = e$ for large n. Contradiction.

$\Leftarrow$ Take any nbhd $\mathcal{U}$ of e in G, and a bounded nbhd V such that $V^{-1} V \subset \mathcal{U}$.
Suppose that $\exists g_n \in \Omega : g_n\Gamma \to \infty$ in G/Γ.
Take a compact $K \subset G/\Gamma$: $\mathrm{vol}(K) \geq \mathrm{vol}(G/\Gamma) - \mathrm{vol}(V)$.

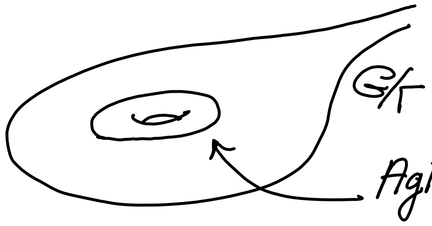
For sufficiently large n, $Vg_n\Gamma \subset K^c$.
Then $\mathrm{vol}(Vg_n\Gamma) < \mathrm{vol}(V)$, and the map $V \to Vg_n\Gamma$ is not injective,
i.e., $Vg_n \cap Vg_n\gamma \neq \emptyset$ for some $\gamma \in \Gamma \setminus \{e\}$.
Then $V^{-1} V \cap g_n \Gamma g_n^{-1} \neq \{e\}$.
This shows that e is an accumulation point of Γ^Ω.
Contradiction.

Maximal abelian subgroups of Γ.

Thm. (Selberg, Wolf, Prasad–Raghunathan) Let Γ be a lattice in $G = SL_n(\mathbb{R})$. The Γ contains diagonalisable subgroups $\simeq \mathbb{Z}^{n-1}$.

Cor. If $\Gamma_i < SL_{n_i}(\mathbb{R})$ are lattices and $n_1 \neq n_2$, then $\Gamma_1 \not\simeq \Gamma_2$.

$\boxed{\text{Diagonalisable subgroups}} \longleftrightarrow \boxed{\text{Compact orbits of } A = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \text{ in } G/\Gamma}$



$Ag\Gamma/\Gamma \simeq A/(A \cap g\Gamma g^{-1})$

Lem. Let $\gamma \in \Gamma$. Then $Z_G(\gamma)\Gamma$ is closed in G/Γ.

$\ulcorner$ Consider a map $\varphi: G \to G : h \mapsto g^{-1}hg$. Since $\varphi(\Gamma) \subset \Gamma$, $\varphi(\Gamma)$ is closed. Then $\varphi^{-1}(\varphi(\Gamma)) = Z_G(\gamma)\Gamma$ is closed. $\lrcorner$

Def. $g \in SL_n(\mathbb{R})$ is <u>$\mathbb{R}$-regular</u> if eigenvalues of g have distinct absolute values.

Lem. A lattice contains $\mathbb{R}$-regular elements.

$\ulcorner$ Take an $\mathbb{R}$-regular $g \in G$. Then g^n $(n \geq 1)$ is also $\mathbb{R}$-regular. Moreover, $\exists$ a nbhd $\mathcal{U}$ of e in G: $\mathcal{U}^{-1} g^n \mathcal{U}$ consists of $\mathbb{R}$-regular elements. Consider the sets $g^n \mathcal{U}\Gamma \subset G/\Gamma$. Since $\mathrm{vol}(G/\Gamma) < \infty$, $g^{n_1}\mathcal{U}\Gamma \cap g^{n_2}\mathcal{U} \neq \emptyset$ for some $n_1 > n_2$. Then $\mathcal{U}^{-1} g^{n_1 - n_2} \mathcal{U} \cap \Gamma \neq \emptyset$. $\lrcorner$

Proof of Thm (when Γ is cocompact)

Take an $\mathbb{R}$-regular $\gamma \in \Gamma$. Then $Z_G(\gamma)$ is congugate to A.

By the lemma, $Z_G(\gamma)\Gamma/\Gamma \simeq Z_G(\gamma)/(Z_G(\gamma)\cap\Gamma)$ is compact.

Hence, $Z_G(\gamma)\cap\Gamma \supset$ a copy of $\mathbb{Z}^{n-1}$. $\blacksquare$

ex. ($\mathbb{R}$-regular is not enough in general)

Take $\mathbb{R}$-regular $\gamma = \left(\begin{array}{cc|c} * & * & 0 \\ * & * & \\ \hline 0 & & \begin{matrix} * & * \\ * & * \end{matrix} \end{array}\right) \in \Gamma = SL_4(\mathbb{Z})$.

Then $g = \left(\begin{array}{c|c} \begin{matrix} a & 0 \\ 0 & a \end{matrix} & 0 \\ \hline 0 & \begin{matrix} a^{-1} & 0 \\ 0 & a^{-1} \end{matrix} \end{array}\right) \in Z_G(\gamma)$, and

for $\delta = \left(\begin{array}{c|c} I & * \\ \hline 0 & I \end{array}\right) \in \Gamma\setminus\{e\}$, $g^{-1}\delta g \to e$ as $a \to +\infty$.

Hence, by compactness criterion $Z_G(\gamma)\Gamma$ is not compact.

Def. $g \in G$ is $\mathbb{R}$-hyper-regular if for $Ad(g) \curvearrowright \bigwedge Lie(G)$ (exterior algebra) has the least multiplicity of eigenvalues λ with $|\lambda| = 1$.

Rmk. γ in the example is $\mathbb{R}$-regular, but not $\mathbb{R}$-hyper-regular.

Prasad–Raghunathan:

1) $\exists$ $\mathbb{R}$-hyper-regular $\gamma \in \Gamma$

2) $\gamma \in \Gamma$ - $\mathbb{R}$-hyperregular $\Rightarrow Z_G(\gamma)\Gamma/\Gamma$ is compact.