Lecture 1: Lattices – basic examples.

G – a connected Lie group,
$\cup$
Γ – a discrete subgroup.

Def. 1) Γ is a cocompact lattice if G/Γ is cocompact.
2) Γ is a lattice if $\mathrm{vol}(G/\Gamma) < \infty$.

example: $\mathbb{Z}^d \subset \mathbb{R}^d$

$$\text{Properties of } G \xrightarrow{?} \text{Properties of } \Gamma$$

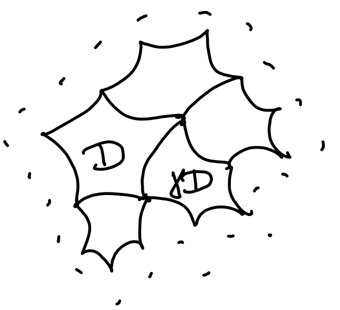
1) Geometry (fundamental groups)
2) Number Theory (arithmetic groups).

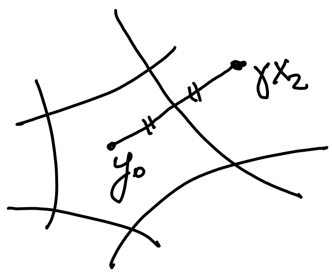
X – a Riemannian manifold with finite volume
Then $\exists$ isometric covering map $\pi : Y \to X$
(Y is simply connected)
$\exists \Gamma$ – a lattice in $\mathrm{Isom}(Y)$:
$$X \simeq \Gamma \backslash Y.$$

Suppose that $\Gamma \curvearrowright Y$ by isometries
(Y is a complete simply connected Riemannian manifold.)
Orbits of Γ are discrete.

~~Def~~ $D \subset Y$ is a <u>fundamental domain</u> for Γ if

1) $Y = \bigcup_{\gamma \in \Gamma} \gamma D$,
2) $\gamma_1 D^\circ \cap \gamma_2 D^\circ \neq \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma \Rightarrow \gamma_1 = \gamma_2$.





<u>Dirichlet fundamental domain:</u> Fix $y_0 \in Y$: $Stab_\Gamma(y_0) = 1$.

Let $\underbrace{D}_{D(y_0)} = \{ y \in Y : d(y, y_0) \leq d(y, \delta y_0), \delta \in \Gamma \}$.

<u>Prop.</u> D is a fundamental domain for Γ.

For $\gamma \in \Gamma$, $\gamma \cdot D(y_0) = \{ y : d(\gamma^{-1} y, y_0) \leq d(\gamma^{-1} y, \delta y_0), \delta \in \Gamma \}$
$= \{ y : d(y, \gamma y_0) \leq d(y, \gamma \delta \gamma^{-1} \cdot \gamma y_0), \delta \in \Gamma \}$
$= D(\gamma y_0)$.

Hence, $\gamma \cdot D(y_0) = \{ y : \gamma y_0 \text{ is a closest point to } y \text{ from the set } \Gamma \cdot y_0 \}$.

This implies that $Y = \bigcup_{\gamma \in \Gamma} \gamma D$.

Also, $D^\circ = \{y : d(y,y_0) < d(y,\delta y_0),\ \delta\in\Gamma\}$, and

$\gamma D^\circ = \{y : \gamma y_0$ is the <u>unique</u> closest point to y from the set $\Gamma\cdot y_0\}$

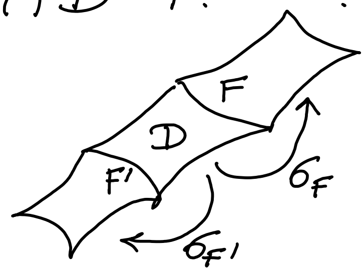
Hence, $\gamma_1 D^\circ \cap \gamma_2 D^\circ \neq \emptyset \Rightarrow \gamma_1 y_0 = \gamma_2 y_0 \Rightarrow \gamma_1 = \gamma_2$.

<u>Prop.</u> $\Gamma\backslash Y$ is compact $\Rightarrow$ Γ is finitely generated.

Since $\Gamma\backslash Y$ is compact, $\exists R>0 : \Gamma\cdot B(y_0,R) = Y$. Then
$D = \{y : d(y,y_0) \le d(\delta^{-1}y, y_0),\ \delta\in\Gamma\} \subset B(y_0,R)$.
Since $\Gamma\cdot y_0$ is discrete, D is defined by <u>finitely many</u> conditions. Let $S = \{\gamma\in\Gamma : D$ and γD have a common face$\}$.
Then $\Gamma = \langle S\rangle$.

<u>Poincare construction.</u> (assume that $\dim(Y)=2$)

Let $D\subset Y$ be a compact domain with finitely many sides.
Assume that:
- for each side F, $\exists$ isometry $\sigma_F : \sigma_F(D)\cap D = F$.
- setting $F' = \sigma_F^{-1}(F)$, $\boxed{\sigma_F\cdot\sigma_{F'} = 1}$ (1)



- For every vertex v of D,
  $\exists$ isometries $\sigma_{F_1},\dots \sigma_{F_\ell}$: $\boxed{\sigma_{F_1}\dots\sigma_{F_\ell} = 1}$ (2)

  $\sigma_{F_1}(D),\dots \sigma_{F_1}\dots\sigma_{F_\ell}(D) = D$ match perfectly around v.

Prop. $\Gamma = \langle \sigma_F : F\text{-side of } D \rangle$ is a discrete cocompact subgroup.

$\ulcorner$ Let $\tilde{\Gamma} = \langle s_F : \text{relations (1) \& (2)} \rangle$,

$\tilde{Y} = (\tilde{\Gamma} \times D)/\sim$, $(\gamma s_F, x) \sim (\gamma, \sigma_F(x))$, $x \in F$!

Then we have a natural equivariant covering map $\pi : \tilde{Y} \to Y$.

Since Y is simply connected, $\tilde{Y} \simeq Y$ and $\tilde{\Gamma} \simeq \Gamma$.

By construction, D is a fundamental domain of $\tilde{\Gamma}$. $\lrcorner$

Arithmetic groups.

Thm. $\underset{\Gamma}{\underset{\parallel}{SL_d(\mathbb{Z})}}$ is a lattice in $\underset{G}{\underset{\parallel}{SL_d(\mathbb{R})}}$.

Iwasawa decomposition: $G = KAN$,

$K = SO(d)$, $A = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix} : a_i > 0, \prod_i a_i = 1 \right\}$, $N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Siegel set: $\Sigma_{s,t} = \left\{ kan : k \in SO(d), \frac{a_i}{a_{i+1}} \leq s, |n_{ij}| \leq t \right\}$.

Prop. 1) For $s \geq \frac{2}{\sqrt{3}}$ and $t \geq \frac{1}{2}$, $G = \Sigma_{s,t} \Gamma$.

2) $\{ \gamma \in \Gamma : \gamma \cdot \Sigma_{s,t} \cap \Sigma_{s,t} \neq \emptyset \}$ is finite.

Since $\mathrm{vol}(\Sigma_{s,t}) < \infty$, this implies that Γ is a lattice.

Let $\mathcal{L}_d = \{$lattices in $\mathbb{R}^d$ with covol$=1\}$.
$G \curvearrowright \mathcal{L}_d$ - transitively and $\mathrm{Stab}_G(\mathbb{Z}^d) = \Gamma$.
Hence, $\mathcal{L}_d \simeq G/\Gamma$.

Take any $g \in G$ and consider $L = g\mathbb{Z}^d$.
A basis $(v_1, \dots v_d)$ of L is called <u>reduced</u> if:

1) v_1 has the least norm in $L \setminus \{0\}$,
2) Let $P: \mathbb{R}^d \to \langle v_1 \rangle^\perp$ be the orthogonal projection. Then $(P(v_2), \dots P(v_d))$ is a reduced basis of $P(L)$,
3) $P(v_i)$, $i \geq 2$, has minimal norm in $P^{-1}(P(v_i))$.

A reduced basis (v_i) always exists.
Write $v_i = he_i$ with $h \in G$ and (e_i) = standard basis.

<u>Claim.</u> $h \in \Sigma_{s,t}$ $(\Rightarrow g\mathbb{Z}^d = h\mathbb{Z}^d \Rightarrow g \in \Sigma_{s,t}\Gamma)$.

Let $h = kan$ $k \in K, a \in A, n \in N$.
$(w_i) = (k^{-1}v_i) = (ane_i)$ is a reduced basis of $k^{-1}L$.

$$\| \quad \begin{pmatrix} a_1 \cdot e_1 \\ a_1 n_{12} e_1 + a_2 e_2 \\ \dots \quad \dots \\ a_1 n_{1d} e_1 + \dots \quad + a_d e_d \end{pmatrix}$$

By induction, $|n_{ij}| < s$ for $i \geq 2$,
$\frac{a_i}{a_{i+1}} < t$ for $i \geq 2$.

By 3), $\|w_i\| \leq \|w_i + \ell w_1\|$ for all $\ell \in \mathbb{Z}$

$$\begin{array}{ccc} \| & & \shortparallel \\ \sqrt{a_1^2 n_{1i}^2 + \ldots + a_i^2} & & \sqrt{a_1^2 (n_{1i}+\ell)^2 + \ldots + a_i^2} \end{array}$$

Hence, $|n_{1i}| \leq \frac{1}{2}$.

By 1), $\|w_1\| \leq \|w_2\| \Rightarrow a_1^2 \leq a_1^2 n_{12}^2 + a_2^2 \leq \frac{1}{4} a_1^2 + a_2^2$

$$\Downarrow$$

$$\frac{a_1}{a_2} \leq \frac{2}{\sqrt{3}}.$$

Cor. (Mahler compactness criterion)

$\Omega \subset \mathcal{L}_d$ is relatively compact $\Leftrightarrow$ $\exists \delta > 0$: $\forall L \in \Omega$ $\forall v \in L \setminus \{0\}$ $\|v\| \geq \delta$.

$\Rightarrow$ easy

$\Leftarrow$ Write $\Omega = \Sigma \cdot \mathbb{Z}^d$ for $\Sigma \subset \Sigma_{s,t}$.

For $g = kan \in \Sigma$, $|n_{ij}| \leq t$, $\frac{a_i}{a_{i+1}} \leq t$.

$\|g e_1\| = a_1 \geq \delta \Rightarrow a_i \geq t^{-(i-1)} \delta$.

Also, $a_1 \ldots a_d = 1 \Rightarrow a$ is bounded.

Thm (Borel – Harish-Chandra)

Let $G < GL_n(\mathbb{R})$ be connected algebraic group defined over $\mathbb{Q}$.

$G(\mathbb{Z})$ is a lattice in $G \Leftrightarrow \nexists$ characters $\chi: G \to \mathbb{R}^\times$ defined over $\mathbb{Q}$.