

Uniform distribution of orbits of lattices

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Problem

Let Γ be a discrete group acting on a compact topological space X . We study distribution of orbits of Γ in X :

- Fix a set of “balls” $\Gamma_T \subset \Gamma$, $T > 0$.
- For $f \in C(X)$ and $x_0 \in X$, what is the asymptotic behaviour of

$$\sum_{\gamma \in \Gamma_T} f(\gamma \cdot x_0)?$$

Example (Bogolubov-Krylov + Birkhoff)

For a homeomorphism T of X , there exists an invariant measure μ such that for μ -a.e. $x_0 \in X$ and $f \in C(X)$,

$$\frac{1}{2N+1} \sum_{i=-N}^N f(T^i x_0) \rightarrow \int_X f d\mu$$

Essentially, the same result has been extended to the class of amenable groups.

What about nonamenable groups?

Known results about equidistribution:

1. (Arnold-Krylov) Free group with t generators acting by isometries on S^2 .
2. (Kazhdan) Free group with t generators acting by isometries on \mathbb{R}^2 .
3. (G'ivarch) Free group acting on nilmanifolds.
4. (Nevo, Nevo-Fidjuwara) Pointwise ergodic theorem for free groups and hyperbolic groups.

Let Γ be a lattice in a (noncompact) semi-simple Lie group G (e.g., $\Gamma = \mathrm{SL}(n, \mathbb{Z})$). We prove equidistribution of all dense Γ -orbits for *algebraic* measure-preserving actions.

“Balls” in Γ

1. Let d be the Cartan-Killing Riemannian metric on the symmetric space $K \backslash G$ and

$$\Gamma_T = \{\gamma \in \Gamma : d(K\gamma, K) < \log T\}.$$

2. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a proper homomorphism, $\|\cdot\|$ a norm on $\mathrm{End}(V)$, and

$$\Gamma_T = \{\gamma \in \Gamma : \|\rho(\gamma)\| < T\}.$$

If G is not simple, we also need the following condition:

Balanced condition. For a decomposition $G = G_1 \cdot G_2$, where G_1 and G_2 are closed connected normal subgroups of G , and compact $C \subset G_1$,

$$\frac{\#(\Gamma_T \cap C \cdot G_2)}{\#\Gamma_T} \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

In case 1, this condition is equivalent to nonexistence of compact factors for G ; in case 2, this condition may fail even for irreducible Γ .

Algebraic measure preserving actions

1. $\Gamma \subset G$ acts on L/Λ by left multiplication where
 - G is a closed subgroup of a Lie group L .
 - Λ is a lattice in L .
2. $\Gamma \subset G$ acts on L/Λ by automorphisms (e.g., $\text{SL}(n, \mathbb{Z})$ action on \mathbb{T}^n) where
 - L is a Lie group.
 - Λ is a lattice in L .
 - G is a closed subgroup of $\text{Aut}(L)$ such that Γ stabilizes Λ .

Main result

Theorem 1 *Consider an algebraic measure preserving action of Γ on a space X equipped with finite measure μ . Then for every $x_0 \in X$ such that $\Gamma \cdot x_0$ is dense in X and every $f \in C_c(X)$,*

$$\frac{1}{\#\Gamma_T} \sum_{\gamma \in \Gamma_T} f(\gamma \cdot x_0) \rightarrow \frac{1}{\mu(X)} \int_X f d\mu.$$

Previously, Hee Oh proved equidistribution of the action Γ by left multiplication on $\mathrm{SL}(n, \mathbb{R})/\Lambda$, where Γ and Λ are lattices in $\mathrm{SL}(n, \mathbb{R})$.

Sketch of the proof (inducing action)

Let

$$Y = (G \times X) / \sim,$$

where the equivalence relation is defined by

$$(g, x) \sim (g\gamma^{-1}, \gamma x) \quad \text{for } \gamma \in \Gamma,$$

equipped with the measure

$$\nu = \text{Vol}_{G/\Gamma} \otimes m.$$

The group G acts on Y by

$$g' \cdot [(g, x)] = [(g'g, x)].$$

We show that Theorem 1 is equivalent to equidistribution of G -orbits in Y :

For $y_0 = [(e, x_0)]$ and every $\phi \in C_c(Y)$,

$$(*) \quad \frac{1}{\text{Vol}(G_T)} \int_{G_T} \phi(g \cdot y_0) dg \rightarrow \frac{1}{\nu(Y)} \int_Y \phi d\nu.$$

Sketch of the proof (proof of (*))

We use Cartan decomposition $G = KA^+K$, where K is a maximal compact subgroup of G and A^+ is positive Weyl chamber. Then

$$dg = dk d\rho(a) dl, \quad g = kal.$$

Let $\alpha_i \in C(K)$, $i = 1, \dots, N$, be a partition of unity and $l_i \in \text{supp}(\alpha_i)$.

$$\begin{aligned} & \int_{G_T} \phi(g \cdot y_0) dg \\ &= \int_K dk \int_K dl \int_{a: \|kal\| < T} \phi(kaly_0) d\rho(a) \\ &= \sum_{i=1}^N \int_K dk \int_K \alpha_i(l) dl \int_{a: \|kal\| < T} \phi(kaly_0) d\rho(a) \\ &\simeq \sum_{i=1}^N \int_K dk \int_K \alpha_i(l) dl \int_{a: \|kal_i\| < T} \phi(kaly_0) d\rho(a) \\ &= \sum_{i=1}^N \int_K dk \int_{a: \|kal_i\| < T} \left(\int_K \phi(kaly_0) \alpha_i(l) dl \right) d\rho(a). \end{aligned}$$

Using Ratner theory (N. Shah), one shows that for $\phi \in C_c(Y)$ and $\alpha \in C(K)$,

$$\int_K \phi(aly_0) \alpha(l) dl \rightarrow \left(\int_K \alpha dl \right) \cdot \frac{1}{\nu(Y)} \int_Y \phi d\nu$$

as $a \rightarrow \infty$ in strong sense.

Spaces with infinite volume

Example (Ledrappier, Noguira): Consider action of $\Gamma = \text{SL}(n, \mathbb{Z})$ on \mathbb{R}^n . If $x_0 \in \mathbb{R}^n$ is not a multiple of a rational vector, then $\Gamma \cdot x_0$ is dense in \mathbb{R}^n and for every $f \in C_c(\mathbb{R}^n - \{0\})$,

$$\frac{1}{T^{(n-1)^2}} \sum_{\gamma \in \Gamma: \|\gamma\| < T} f(\gamma \cdot x_0) \rightarrow c_n \int_{\mathbb{R}^n - \{0\}} f(x) \frac{dx}{\|x\|}$$

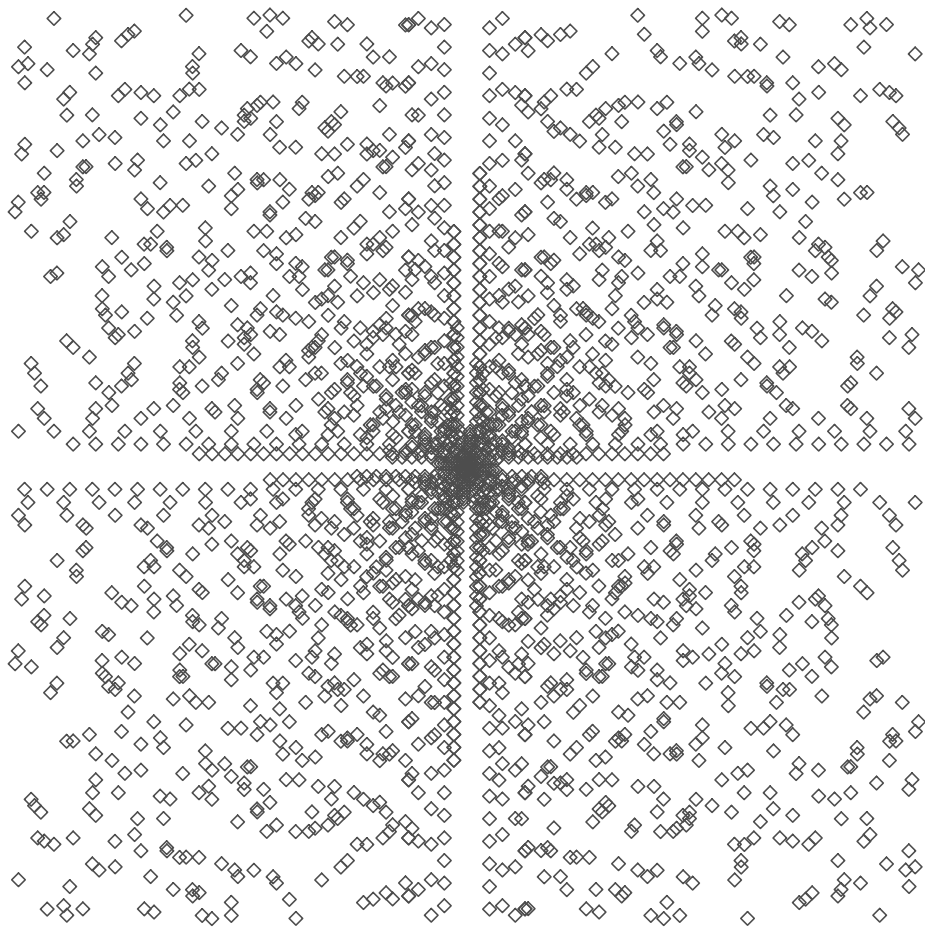
for some $c_n > 0$.

Note that

1. $\#\{\gamma \in \Gamma : \|\gamma\| < T\} \sim c \cdot T^{n^2-n}$.

2. The measure $\frac{dx}{\|x\|}$ is *not* invariant under Γ .

Example



Infinite volume homogeneous spaces

Let G be a connected Lie group, H a closed subgroup of G , and Γ a lattice in G . We investigate distribution of dense orbits for the action of Γ on G/H by left multiplication.

Asymptotics for discrete Γ -orbits in G/H (for reductive H) was obtained by Eskin, Mozes, Shah.

Skew balls: For $g_1, g_2 \in G$, define

$$H_T[g_1, g_2] = g_1^{-1}G_Tg_2^{-1} \cap H.$$

Moderate volume growth condition: for any bounded $B \subset G$ and $\varepsilon > 0$, there exists a neighborhood of e in H such that for any $g_1, g_2 \in B$,

$$\text{Vol}(\mathcal{O} \cdot H_T[g_1, g_2]) \leq (1 + \varepsilon)\text{Vol}(H_T[g_1, g_2]).$$

Asymptotics

Idea:

$$\begin{array}{ccc} & G & \\ \pi \swarrow & & \searrow \tau \\ \Gamma \backslash G & & G/H \end{array}$$

Theorem 2 For $g_0H \in G/H$ such that

$$\overline{\Gamma g_0H} = G,$$

the following is equivalent:

- For any $g \in G$ and $\phi \in C_c(\Gamma \backslash G)$,

$$\frac{1}{\text{Vol}(H_T[g, g_0])} \int_{H_T[g, g_0]} \phi(\Gamma g_0 h^{-1}) dh \rightarrow \int_{\Gamma \backslash G} \phi.$$

- For every $f \in C_c(G/H)$,

$$\sum_{\gamma \in \Gamma_T} f(\gamma \cdot g_0H) \sim \int_{G_T} f(g \cdot g_0H) dg.$$

The first condition can be proved for semisimple groups H (that satisfies balanced condition) and for some unipotent groups H .

Equidistribution

Let $Y \subset G$ be measurable section of the projection map $\tau : G \rightarrow G/H$, i.e., the product map $Y \times H \rightarrow G$ is a Borel isomorphism. Then

$$dg = d\nu_Y(y) \otimes dh, \quad g = yh,$$

where dg and dh are right Haar measures on G and H , and ν_Y is a measure on Y . We assume existence of the limit

$$(+)\quad \lim_{T \rightarrow \infty} \frac{\text{Vol}(H_T[g_1, g_2])}{\text{Vol}(H_T[e, e])} \stackrel{\text{def}}{=} \alpha(g_1, g_2).$$

Let

$$\nu_{g_0}(y) = \alpha(y, g_0^{-1}) d\nu_Y(y).$$

Main result

Theorem 3 *Assuming the moderate volume growth condition, (+), and equidistribution for the H -orbit in $\Gamma \backslash G$ (as in Theorem 2), we have*

$$\frac{1}{\text{Vol}(H_T)} \sum_{\gamma \in \Gamma_T} f(\gamma \cdot g_0 H) \rightarrow \int_{G/H} f d\nu_{g_0}.$$

Proof.

$$\begin{aligned} & \int_{G_T} f(g \cdot g_0 H) dg \\ &= \int_{g: \|gg_0^{-1}\| < T} f(gH) dg \\ &= \int_Y d\nu_Y(y) \int_{h: \|yhg_0^{-1}\| < T} f(yhH) dh \\ &= \int_Y f(yH) \text{Vol}(H_T[y, g_0^{-1}]) d\nu_Y(y) \\ &\sim \text{Vol}(H_T) \cdot \int_{G/H} f d\nu_{g_0}. \end{aligned}$$

A counterexample (when H is not semisimple)

For $G = \mathrm{SL}(3, \mathbb{R})$ and Γ a lattice in G , there exist

- a connected solvable subgroup $H \subset G$,
- two *different* smooth measures μ, ν on G/H ,
- sequences $s_n, t_n \rightarrow \infty$,
- $x \in G/H$ with $\overline{\Gamma \cdot x} = G/H$

such that for every $f \in C_c(G/H)$,

$$\frac{1}{\mathrm{Vol}(H_{s_n})} \sum_{\gamma \in \Gamma_{s_n}} f(\gamma^{-1} \cdot x) \rightarrow \int_{G/H} f d\mu$$

and

$$\frac{1}{\mathrm{Vol}(H_{t_n})} \sum_{\gamma \in \Gamma_{t_n}} f(\gamma^{-1} \cdot x) \rightarrow \int_{G/H} f d\nu$$

as $n \rightarrow \infty$.

Volume asymptotics

Condition (+) is proved in the following two cases:

1. H is connected semisimple Lie group, $\rho : H \rightarrow \text{GL}(V)$ a proper representation, $\|\cdot\|$ a norm on $\text{End}(V)$, and

$$H_T = \{h \in H : \|\rho(g_1 h g_2)\| < T\}.$$

Then

$$\text{Vol}(H_T[g_1, g_2]) \sim c(g_1, g_2) \cdot (\log T)^l T^m$$

for explicit constants $c(g_1, g_2)$, l , m .

This result was also recently proved by F. Maucourant.

2. G and H are connected semisimple Lie groups, d the Cartan-Killing metric on the symmetric space $K \backslash G$, and for $g_1, g_2 \in G$,

$$H_T[g_1, g_2] = \{h \in H : d(K g_1 h g_2, K) < \log T\}.$$

Then

$$\text{Vol}(H_T[g_1, g_2]) \sim c(g_1, g_2) \cdot (\log T)^l T^m$$

for explicit constants $c(g_1, g_2)$, l , m .