RATIONAL POINTS ON HOMOGENEOUS VARIETIES AND EQUIDISTRIBUTION OF ADELIC PERIODS

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(WITH APPENDIX BY MIKHAIL BOROVOI)

Abstract. Let K be a number field and $\mathbf{U} := \mathbf{L}\backslash\mathbf{G}$ be a homogeneous K-variety where $\mathbf{G}$ is a connected semisimple K-group, and $\mathbf{L}$ is a semisimple maximal connected K-subgroup of $\mathbf{G}$. Assuming that $\mathbf{G}(K_v)$ acts transitively on $\mathbf{U}(K_v)$ for almost all places v of K, we obtain the asymptotic of the number of rational points in $\mathbf{U}(K)$ of height at most T, up to bounded constants. As a corollary, we settle Manin's conjecture for wonderful compactifications of some $\mathbf{U}$.

Contents

1. Introduction

Let K be a number field and $\mathbf{X}$ a projective variety defined over K. Understanding the set $\mathbf{X}(K)$ of K-rational points in $\mathbf{X}$ is a fundamental problem in arithmetic geometry. In this paper we study the asymptotic number (as $T \to \infty$) of the points in $\mathbf{X}(K)$ of height less than T for compactifications of affine homogeneous varieties $\mathbf{U} = \mathbf{L}\backslash\mathbf{G}$ of a connected semisimple algebraic K-group $\mathbf{G}$ when $\mathbf{L}$ is a semisimple maximal connected K-subgroup. Our

A.G. is partially supported by NSF grants 0400631, 0654413 and RCUK Fellowship, Oh is partially supported by NSF grants 0333397 and 0629322, M.B. is partially supported by the Hermann Minkowski Center for Geometry and by the ISF grant 807/07.

results solve new cases of Manin's conjecture [1] on rational points of Fano varieties.

Manin's conjecture has been proved for equivariant compactifications of homogeneous spaces: flag varieties ([33], [57]), toric varieties ([2], [3]), horospherical varieties [62], equivariant compactifications of unipotent groups (see [20], [62], [63]), and for the wonderful compactification of a semisimple adjoint group defined over a number field ([65], [36]). We refer to survey papers by Tschinkel ([68], [69]) for a more precise background on this conjecture.

1.1. **Counting rational points of bounded height.** We begin by recalling the notion of a height function on the K-rational points $\mathbb{P}^d(K)$ of the projective d-space $\mathbb{P}^d$. Denote by R the set of all normalized absolute values $x \mapsto |x|_v$ of K, and by K_v the completion of K with respect to $|\cdot|_v$.

For each $v \in R$, choose a norm H_v on K_v^{d+1} which is simply the max norm $\mathrm{H}_v(x_0, \cdots, x_d) = \max_{i=0}^d |x_i|_v$ for almost all v. Then the height function $\mathrm{H} : \mathbb{P}^d(K) \to \mathbb{R}_{>0}$ associated to $\mathcal{O}_{\mathbb{P}^d}(1)$ is given by

$$\mathrm{H}(x) := \prod_{v \in R} \mathrm{H}_v(x_0, \cdots, x_d)$$

for $x = (x_0 : \cdots : x_d) \in \mathbb{P}^d(K)$. Since $\mathrm{H}_v(x_0, \cdots, x_d) = 1$ for almost all $v \in R$, we have $0 < \mathrm{H}(x) < \infty$ and by the product formula, H is well defined, i.e., independent of the choice of representative for x.

For instance, for $K = \mathbb{Q}$, if we choose H_p to be the maximum norm of $\mathbb{Q}_p^{d+1}$ for each prime p and set $\mathrm{H}_\infty(x_0, \cdots, x_d) = \left(x_0^2 + \cdots + x_d^2\right)^{1/2}$ for $x_i \in \mathbb{R}$, we have

$$\mathrm{H}(x_0 : \cdots : x_d) = \left(x_0^2 + \cdots + x_d^2\right)^{1/2}$$

where $x_0, \ldots, x_d \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_d) = 1$.

Let $\mathbf{G}$ be a linear algebraic group defined over K, with a given K-representation $\iota : \mathbf{G} \to \mathrm{GL}_{d+1}$. Then $\mathbf{G}$ acts on $\mathbb{P}^d$ via the canonical map $\mathrm{GL}_{d+1} \to \mathrm{PGL}_{d+1}$. Consider $\mathbf{U} := u_0\mathbf{G} \subset \mathbb{P}^d$ for $u_0 \in \mathbb{P}^d(K)$. Fixing a height function H on $\mathbb{P}^d(K)$, we study the asymptotic of the following number (as $T \to \infty$):

$$N_T(\mathbf{U}) := \#\{x \in \mathbf{U}(K) : \mathrm{H}(x) < T\}.$$

Our main results are proved under the following assumption:

(i) $\mathbf{G}$ is a connected semisimple K-group.
(ii) $\mathbf{L} = \mathrm{Stab}_{\mathbf{G}}(u_0)$ is a semisimple maximal connected K-subgroup of $\mathbf{G}$.
(iii) For almost all $v \in R$, $\mathbf{G}(K_v)$ acts transitively on $\mathbf{U}(K_v)$.

If $\mathbf{L}$ is the fixed points of an involution of $\mathbf{G}$, $\mathbf{U}$ is called a symmetric space. A symmetric space $\mathbf{U} = \mathbf{L}\backslash\mathbf{G}$ satisfies (ii) if $\mathbf{L}$ is connected and semisimple, since $\mathbf{L}$ is then a maximal connected K-subgroup [7].

Borovoi gave a classification of symmetric spaces $\mathbf{U} = \mathbf{L}\backslash\mathbf{G}$ satisfying (i)-(iii) with $\mathbf{G}$ absolutely almost simple (see Appendix A).

When both $\mathbf{G}$ and $\mathbf{L}$ are connected, the property (iii) is equivalent to the finiteness of the set of $\mathbf{G}(K)$-orbits in $\mathbf{U}(K)$ (Theorem A.1.2). We remark that (iii) always holds for $\mathbf{L}$ simply connected, by Corollary A.2.1.

Denote by $\mathbf{X} \subset \mathbb{P}^d$ the Zariski closure of $\mathbf{U}$. Then $\mathbf{X}$ is a $\mathbf{G}$-equivariant compactification of $\mathbf{U}$, and the pull back L to $\mathbf{X}$ of the line bundle $\mathcal{O}_{\mathbb{P}^d}(1)$ is a $\mathbf{G}$-linearized very ample line bundle of $\mathbf{X}$ defined over K. We assume that there is a global section s of L such that $\mathbf{U} = \{\mathrm{s} \neq 0\}$. This last condition is automatic in many cases, for instance, it holds in the setting of Example 1.3 and Corollary 1.5 below.

Theorem 1.1. *There exist $a \in \mathbb{Q}_{>0}$ and $b \in \mathbb{N}$ such that* [1]

$$N_T(\mathbf{U}) \asymp T^a (\log T)^{b-1}.$$

Moreover, if $\mathbf{G}$ is simply connected, there exists $c > 0$ such that

$$N_T(\mathbf{U}) \sim c \cdot T^a (\log T)^{b-1}.$$

The exponents a and b are given as follows: First, we assume that $\mathbf{X}$ is smooth and $\mathbf{X}\backslash\mathbf{U}$ is a divisor of normal crossings with smooth irreducible components D_α, $\alpha \in \mathcal{A}$, defined over a finite field extension of K. Let ω be a differential form of $\mathbf{X}$ of top degree, which is nowhere zero on U, and choose a global section s of L with $\mathbf{U} = \{\mathrm{s} \neq 0\}$. Then for $m_\alpha \in \mathbb{N}$ and $n_\alpha \in \mathbb{Z}$,

$$\mathrm{div}(\mathrm{s}) = \sum_{\alpha \in \mathcal{A}} m_\alpha D_\alpha \quad \text{and} \quad -\mathrm{div}(\omega) = \sum_{\alpha \in \mathcal{A}} n_\alpha D_\alpha.$$

The Galois group $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$ acts on $\mathcal{A}$. We denote by $\mathcal{A}/\Gamma_K$ the set of Γ_K-orbits. Then

$$a = \max_{\alpha \in \mathcal{A}} \left\{ \frac{n_\alpha}{m_\alpha} \right\} \quad \text{and} \quad b = \# \left\{ \alpha \in \mathcal{A}/\Gamma_K : \frac{n_\alpha}{m_\alpha} = a \right\}. \tag{1.2}$$

We note that a and b are independent of the choices of s and ω, since there are unique choices of them up to multiplication by constants as a consequence of Rosenlicht theorem.

For a general projective variety $\mathbf{X}$, we take an equivariant resolution of singularities $\pi : \tilde{\mathbf{X}} \to \mathbf{X}$ such that $\tilde{\mathbf{X}}$ is smooth and $\pi^{-1}(\mathbf{X}\backslash\mathbf{U})$ is a divisor with normal crossings. Then the constants a and b are defined as above with respect to the pull-backs $\pi^*(\mathrm{s})$ and $\pi^*(\omega)$.

Example 1.3 (Rational points on affine varieties)**.** Let $\iota_0 : \mathbf{G} \to \mathrm{SL}_d$ be a $\mathbb{Q}$-rational representation, and $\mathbf{V} = v_0\mathbf{G} \subset \mathbb{A}^d$ be Zariski closed for some non-zero $v_0 \in \mathbb{Q}^d$. We write an element of $\mathbf{V}(\mathbb{Q})$ as $\left(\frac{x_1}{x_0}, \cdots, \frac{x_d}{x_0}\right)$ where $x_0, \cdots, x_d \in \mathbb{Z}$, $x_0 > 0$ and $\mathrm{g.c.d}(x_0, \cdots, x_d) = 1$. If $\mathbf{G}$ and $\mathbf{L} := \mathrm{stab}_{\mathbf{G}}(v_0)$ satisfy the assumptions (i)-(iii), Theorem 1.1 implies

$$\#\{\left(\frac{x_1}{x_0}, \cdots, \frac{x_d}{x_0}\right) \in \mathbf{V}(\mathbb{Q}) : \sqrt{x_0^2 + \cdots + x_d^2} < T\} \asymp T^a (\log T)^{b-1};$$

[1] $A(T) \asymp B(T)$ means that for some $c > 1$, $c^{-1} B(T) \leq A(T) \leq c\, B(T)$ holds for all sufficiently large $T > 0$

$$\#\{\left(\frac{x_1}{x_0},\cdots,\frac{x_d}{x_0}\right)\in \mathbf{V}(\mathbb{Q}) : \max\{|x_0|,\cdots,|x_d|\}<T\}\asymp T^a(\log T)^{b-1}.$$

To deduce this from Theorem 1.1, consider the embedding of SL_d into PGL_{d+1} by $A\mapsto \mathrm{diag}(A,1)$, and of $\mathbb{A}^d$ into $\mathbb{P}^d$ by $(x_1,\cdots,x_d)\mapsto(x_1:\cdots:x_d:1)$. This identifies $\mathbf{V}$ with the orbit $\mathbf{U}:=(v_0:1)\mathbf{G}$ in $\mathbb{P}^d$, and $\mathrm{s}=x_{d+1}$ is an invariant section of the line bundle L obtained by pulling back $\mathcal{O}_{\mathbb{P}^d}(1)$, satisfying $\mathbf{U}=\{\mathrm{s}\neq 0\}$. Finally, $\mathrm{H}\left(\frac{x_1}{x_0}:\cdots:\frac{x_d}{x_0}:1\right)=\mathrm{H}(x_1:\cdots:x_d:x_0)$, and hence the claim follows.

Since $\mathbf{U}=\{X\in\mathrm{SL}_{2n}:X^t=-X\}$ is a homogeneous variety $\mathrm{Sp}_{2n}\setminus\mathrm{SL}_{2n}$ for the action $v.g=g^tvg$ and $\mathrm{SL}_{2n}(\mathbb{Q}_p)$ acts transitively on $\mathbf{U}(\mathbb{Q}_p)$ for all p, we have:

Example 1.4. *Let $n\geq 2$. For some $a\in\mathbb{Q}^+$, $b\in\mathbb{N}$ and $c>0$, as $T\to\infty$,*

$$\#\{X\in\mathrm{SL}_{2n}(\mathbb{Q}):X^t=-X,\ \max_{1\leq i,j\leq 2n}\{|x_{ij}|,|x_0|\}<T\}\sim c\cdot T^a(\log T)^{b-1}.$$

where $X=\left(\frac{x_{ij}}{x_0}\right)$, $x_{ij}\in\mathbb{Z}$, $x_0\in\mathbb{N}$ and $\mathrm{g.c.d}\{x_{ij},x_0:1\leq i,j\leq 2n\}=1$.

Theorem 1.1 settles new cases of Manin's conjecture on rational points of some wonderful varieties, which we recall. Let $\mathbf{X}$ be a Fano K-variety, i.e., a smooth projective K-variety with its anticanonical class $-K_{\mathbf{X}}$ being ample. Let $\mathrm{Pic}(\mathbf{X})$ denote the Picard group of $\mathbf{X}$ and $\Lambda_{\mathrm{eff}}(\mathbf{X})\subset\mathrm{Pic}(\mathbf{X})\otimes\mathbb{R}$ the cone of effective divisors. Given a line bundle L on $\mathbf{X}$, there exists an associated height function H_L on $\mathbf{X}(K)$, unique up to the multiplication by bounded functions, via Weil's height function. For instance if L is very ample with a K-embedding $\psi:\mathbf{X}\to\mathbb{P}^d$, then a height function H_L is simply the pull-back of a height function of $\mathbb{P}^d(K)$ to $\mathbf{X}(K)$ via ψ. Note this depends on the choice of ψ. For an ample line bundle L, $\mathrm{H}_L=\mathrm{H}_{L^k}^{1/k}$ for $k\in\mathbb{N}$ such that L^k is very ample.

The conjecture of Manin [1], generalized by Batyrev and Manin, predicts that there exist a Zariski open subset $\mathbf{U}\subset\mathbf{X}$ and a finite field extension K' of K such that

$$\#\{x\in\mathbf{U}(K'):\ \mathrm{H}_L(x)<T\}\sim c\cdot T^{a_L}(\log T)^{b_L-1},$$

where $c>0$ and

$a_L:=\inf\{a:\ a[L]+[K_{\mathbf{X}}]\in\Lambda_{\mathrm{eff}}(\mathbf{X})\}$,

$b_L:=$ the maximal codimension of the face of $\Lambda_{\mathrm{eff}}(\mathbf{X})$ containing $a_L[L]+[K_{\mathbf{X}}]$.

A smooth connected projective $\mathbf{G}$-variety $\mathbf{X}$ defined over K is said to be *wonderful* (of rank l), as introduced by Luna [44], if

(1) $\mathbf{X}$ contains l irreducible $\mathbf{G}$-invariant divisors with strict normal crossings.
(2) $\mathbf{G}$ has exactly 2^l orbits in $\mathbf{X}$.

For a $\mathbf{G}$-homogeneous variety $\mathbf{U}$, a wonderful variety $\mathbf{X}$ is called the wonderful compactification of $\mathbf{U}$ if it is a $\mathbf{G}$-equivariant compactification of $\mathbf{U}$. Luna showed in [44] that every wonderful variety is spherical; in particular a wonderful compactification of a homogeneous space $\mathbf{U} = \mathbf{L}\backslash\mathbf{G}$ exists only when $\mathbf{L}$ is a spherical subgroup, that is, a Borel subgroup of $\mathbf{G}$ has an open orbit in $\mathbf{U}$.

The following can be deduced from Theorem 1.1:

Corollary 1.5. *Let $\mathbf{U}$ be as in Thm. 1.1 and $\mathbf{X}$ the wonderful compactification of $\mathbf{U}$. Then for any ample line bundle L on $\mathbf{X}$ over K and an associated height function H_L, we have*

$$\#\{x \in \mathbf{U}(K) : \mathrm{H}_L(x) < T\} \asymp T^{a_L}(\log T)^{b_L - 1}.$$

Moreover, if $\mathbf{G}$ is simply connected, there exists $c = c(\mathrm{H}_L) > 0$ such that

$$\#\{x \in \mathbf{U}(K) : \mathrm{H}_L(x) < T\} \sim c \cdot T^{a_L}(\log T)^{b_L - 1}.$$

De Concini and Procesi [21] constructed the wonderful compactification of a symmetric variety $\mathbf{L}\backslash\mathbf{G}$ for $\mathbf{G}$ semisimple adjoint. In these cases, a_L and b_L can also be interpreted in terms of the representation theoretical data of $\mathbf{G}$ (see 3.2). A concrete example for the above corollary holds unconditionally is the wonderful compactification of the space of sympletic forms; see 3.4.

Generalizing the work in [21], Brion and Pauer [17] established that a spherical variety $\mathbf{L}\backslash\mathbf{G}$ possesses an equivariant compactification with exactly one closed orbit if and only if $[\mathrm{N}_{\mathbf{G}}(\mathbf{L}) : \mathbf{L}] < \infty$, where $\mathrm{N}_{\mathbf{G}}(\mathbf{L})$ denotes the normalizer of $\mathbf{L}$ in $\mathbf{G}$. Knop [40, Coro. 7.2] showed that the wonderful compactification of a spherical variety exists when $\mathrm{N}_{\mathbf{G}}(\mathbf{L}) = \mathbf{L}$. Complete classification of homogeneous spherical varieties were obtained; see [42], [15] and [47].

1.2. **On the proofs.** To explain our strategy, let $\mathbb{A}$ denote the Adele ring over K. The first key observation is that the global section s of L with $\mathbf{U} = \{\mathrm{s} \neq 0\}$ is in fact $\mathbf{G}$-invariant. And the extension of H to $\mathbf{U}(\mathbb{A})$ using s is uniformly continuous and proper for the action of compact subsets of $\mathbf{G}(\mathbb{A})$. Set

$$B_T := \{x \in \mathbf{U}(\mathbb{A}) : \mathrm{H}(x) < T\}$$

so that $N_T(\mathbf{U}) := \#B_T \cap \mathbf{U}(K)$. Under the assumption (iii), there are only finitely many $\mathbf{G}(K)$-orbits in $\mathbf{U}(K)$, and hence the counting problem reduces to each $\mathbf{G}(K)$-orbit. In general, the naive heuristic

$$\#(u_0\mathbf{G}(K) \cap B_T) \sim \mathrm{vol}(u_0\mathbf{G}(\mathbb{A}) \cap B_T)$$

is false. The reason behind this is the existence of non-trivial automorphic characters of $\mathbf{G}(\mathbb{A})$. From the dynamical point of view, this means that the translates $\mathbf{L}(K)\backslash\mathbf{L}(\mathbb{A})g_i$ of periods do not get equidistributed in the whole space $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$ as $g_i \to \infty$ in $\mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$. This requires us to pass to a suitable finite index subgroup of $\mathbf{G}(\mathbb{A})$. Denote by $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ a simply connected covering of $\mathbf{G}$ defined over K. For any compact open subgroup W of the subgroup $\mathbf{G}(\mathbb{A}_f)$ of finite adeles, we show that the product $G_W := \mathbf{G}(K)\pi(\tilde{\mathbf{G}}(\mathbb{A}))W$ is a normal subgroup of finite index in $\mathbf{G}(\mathbb{A})$, and the translates $\mathbf{L}(K)\backslash(\mathbf{L}(\mathbb{A}) \cap G_W)g_i$ become equidistributed in the space $\mathbf{G}(K)\backslash G_W$ relative to W-invariant functions. The last statement is a special case of our main ergodic theorems in adelic setting, to be detailed in the next subsection. We mention that our assumption $\mathbf{L}$ is semisimple is crucial.

In order to deduce

$$\#(u_0\mathbf{G}(K) \cap B_T) \sim \mathrm{vol}(u_0 G_W \cap B_T),$$

we prove that for any compact open subgroup W of $\mathbf{G}(\mathbb{A}_f)$ by which H is invariant, the family $\{B_T \cap u_0 G_W\}$ is well-rounded; roughly speaking, for any $\epsilon > 0$, there is a neighborhood U_ϵ of the identity in $\mathbf{G}(\mathbb{A})$ such that the volume of $(B_T \cap u_0 G_W)U_\epsilon$ is at most $(1+\epsilon)\mathrm{vol}(B_T \cap u_0 G_W)$ for all large T. Establishing this is based on the work of Chambert-Loir and Tschinkel [20] and of Benoist-Oh [10] (also [37] of Gorodnik-Nevo). Finally, we deduce the volume asymptotic $\mathrm{vol}(u_0 G_W \cap B_T)$ from [20] modulo two bounded constants. When $\mathbf{G}$ is simply connected, we have $G_W = \mathbf{G}(\mathbb{A})$ and deduce the precise volume asymptotic for $u_0\mathbf{G}(\mathbb{A}) \cap B_T$. We remark that if $\mathbf{G}(K_v)$ has no compact factors for some archimedean $v \in R$ and $\mathbf{G}(\mathbb{A}) = \mathbf{G}(K)\pi(\tilde{\mathbf{G}}(\mathbb{A}))W_{\mathrm{H}}$ where W_{H} is the subgroup of $\mathbf{G}(\mathbb{A}_f)$ consisting of elements under which H is invariant, we also have

$$\#(u_0\mathbf{G}(K) \cap B_T) \sim \mathrm{vol}(u_0\mathbf{G}(\mathbb{A}) \cap B_T) \sim c \cdot T^a \log T^{b-1}.$$

In general, replacing $\asymp$ with $\sim$ in Theorem 1.1 requires regularizing the height integrals $\int_{u_0\mathbf{G}(\mathbb{A})} \mathrm{H}^{-s}(u_0 g) \cdot \chi(g)\, d\mu$ for $\mathbf{L}(\mathbb{A})$-invariant automorphic characters χ of $\mathbf{G}(\mathbb{A})$ as in [65, Thm. 7.1]. We mention that the strategy of relating the counting problem with the equidistribution of orbits is originated in the work of Duke-Rudnick-Sarnak [25] and of Eskin-McMullen [29] (see section 5 for more details). Perhaps the most unsatisfying assumption is (iii): the finiteness of $\mathbf{G}(\mathbb{A})$-orbits in $\mathbf{U}(\mathbb{A})$. We believe this assumption should not be necessary to deduce $\#(u_0\mathbf{G}(K)\cap B_T) \sim \mathrm{vol}(u_0 G_W \cap B_T)$; however our proof of well-roundedness of $u_0 G_W \cap B_T$ relies on the finiteness assumption. With a proper use of motivic integration, it may be possible to deal with a general case. Finally we mention that there are examples where the orders of magnitude for $\#(u_0\mathbf{G}(K) \cap B_T)$ and $\#N_T(\mathbf{U})$ are not the same.

1.3. **Equidistribution of Adelic periods.** We now describe our main ergodic results on the equidistribution of Adelic periods. Our results presented in this section are much more general than what is needed for the application on rational points.

Let $\mathbf{G}$ be a connected semisimple group defined over a number field K.

Set $X := \mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$ and $x_0 := [\mathbf{G}(K)] \in X$. For a connected semisimple K-subgroup $\mathbf{L}$ of $\mathbf{G}$, we denote by $\pi : \tilde{\mathbf{L}} \to \mathbf{L}$ a simply connected covering over K, which is unique up to K-isomorphism. Then π induces the map $\tilde{\mathbf{L}}(\mathbb{A}) \to \mathbf{L}(\mathbb{A})$ and hence $\tilde{\mathbf{L}}(\mathbb{A})$ acts on X via π, and the orbit $x_0.\tilde{\mathbf{L}}(\mathbb{A})$

is closed and carries a unique $\tilde{\mathbf{L}}(\mathbb{A})$-invariant probability measure supported in the orbit.

Let $\{\mathbf{L}_i\}$ be a sequence of connected semisimple K-subgroups of $\mathbf{G}$ and $\{g_i \in \mathbf{G}(\mathbb{A})\}$ be given. Let μ_i denote the (unique) $\tilde{\mathbf{L}}_i(\mathbb{A})$-invariant probability measure in X supported on the orbit $Y_i := x_0.\tilde{\mathbf{L}}_i(\mathbb{A})$. The translate $g_i\mu_i$ of μ_i by g_i is defined by

$$g_i\mu_i(E) := \mu_i(Eg_i^{-1})$$

for any Borel subset $E \subset X$.

Denoting by $\mathcal{P}(X)$ the space of all Borel probability measures on X, that a sequence $\nu_i \in \mathcal{P}(X)$ weakly converges to $\mu \in \mathcal{P}(X)$ means that for every $f \in C_c(X)$,

$$\lim_{i\to\infty} \int_X f(x)\, d\nu_i(x) = \int_X f\, d\mu.$$

We study the following question:

Describe the weak-limits of $g_i\mu_i$ in $\mathcal{P}(X)$.

We remark that the reason of considering the translates of $x_0\tilde{\mathbf{L}}_i(\mathbb{A})$ rather than those of the orbit $x_0\mathbf{L}_i(\mathbb{A})$, is essentially because X has more than one *connected* components and the adele group of the simply connected cover plays exactly the role of the identity component in a suitable sense.

Definition 1.6. *A valuation $v \in R$ is said to be strongly isotropic for $\mathbf{G}$ if for every connected normal K_v-subgroup $\mathbf{N}$ of $\mathbf{G}$, $\mathbf{N}(K_v)$ is non-compact. We denote by $\mathcal{I}_{\mathbf{G}}$ the set of all strongly isotropic $v \in R$ for $\mathbf{G}$.*

For a compact open subgroup W_f of the group $\mathbf{G}(\mathbb{A}_f)$ of finite adeles, we denote by $C_c(X, W_f)$ the set of all continuous W_f-invariant functions on X whose support is compact and contained in the set $x_0\pi(\tilde{\mathbf{G}}(\mathbb{A}))W_f$.

Theorem 1.7. *Suppose that $\cap_i \mathcal{I}_{\mathbf{L}_i} \neq \emptyset$, and let $g_i \in \mathbf{G}(K)\pi(\tilde{\mathbf{G}}(\mathbb{A}))$.*

1. *If the centralizer of $\mathbf{L}_i$ is K-anisotropic for each i, then the sequence $\{g_i\mu_i\}$ does not escape to infinity, that is, for any $\epsilon > 0$, there exists a compact subset $\Omega \subset X$ such that*

   $$g_i\mu_i(\Omega) > 1 - \epsilon \quad \textit{for all large } i.$$

2. *Let $\mu \in \mathcal{P}(X)$ be a weak limit of $g_i\mu_i$. Then there exists a connected K-subgroup $\mathbf{M}$ of $\mathbf{G}$ such that*
   - *for some $\delta_i \in \mathbf{G}(K)$,*

     $$\delta_i\mathbf{L}_i\delta_i^{-1} \subset \mathbf{M} \quad \textit{for all sufficiently large } i;$$

   - *for any compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$, there exist a finite index normal subgroup M_0 of $\mathbf{M}(\mathbb{A})$ containing $\mathbf{M}(K)\pi(\tilde{\mathbf{M}}(\mathbb{A}))$ and $g \in \pi(\tilde{\mathbf{G}}(\mathbb{A}))$ such that*

     $$\mu(f) = g\mu_{M_0}(f) \quad \textit{for all } f \in C_c(X, W_f)$$

where $g\mu_{M_0}$ is the invariant probability measure supported on x_0M_0g, and there exists $h_i \in \pi(\tilde{\mathbf{L}}_i(\mathbb{A}))$ such that $\delta_i h_i g_i$ converges to g as $i \to \infty$.

- *If the centralizers of $\mathbf{L}_i$ are K-anisotropic, $\mathbf{M}$ is semisimple.*

See Corollary 4.14 where we discuss a special case of the above theorem for $\mathbf{L}$ maximal semisimple. The analogous theorems in the case of a homogeneous space of a connected semisimple *real Lie* group have been studied previously in [23], [25], [29], [30], [52], [32] and [28], etc. Via the strong approximation properties of simply connected semisimple groups, our proof of Theorem 1.7 is reduced to the generalizations of the aforementioned results, especially of Dani-Margulis [23] and Mozes-Shah [52], in the S-algebraic setting (see Theorem 4.6). We make a vital use the classification theorem on ergodic measures invariant under unipotent flows in this set-up obtained by Ratner [60], Margulis-Tomanov [49], and also refined by Tomanov [67] in the arithmetic situation. Our approach is based on the linearization methods developed by Dani-Margulis [24].

In the case of $\mathbf{G} = \mathrm{PGL}_2$, and $\mathbf{L}_i$ a K-anisotropic torus, the analogue of the above theorem can be deduced from a theorem of Venkatesh (Theorem 6.1 in [70]) using Waldspurger's formula (cf. [48, 2.5]) which relates the integral over a period with special values of L-functions. For $\mathbf{G} = \mathrm{PGL}_3$ and $\mathbf{L}_i$ a $\mathbb{Q}$-anisotropic maximal torus, it was obtained by Einsiedler, Lindenstrauss, Michelle, and Venkatesh [26]. $\mathbf{L}_i$'s being tori, the methods in [70] and [26] are very different from ours. The powerful theorems on unipotent flows ([60] and [49]) are essentially what makes our theorem 1.7 so general.

1.4. **Other applications.** Theorem 1.7 should be useful in many future arithmetic applications. For instance, an application of Theorem 1.7 in a problem of Linnik, considered in [32] and [27], is discussed in [56]. We state only two below, which are most relevant to the subject of this paper. One application of Theorem 1.7 is an ergodic theoretic proof of Adelic mixing obtained in [36] though given only in a non-effective form.

Corollary 1.8 (Adelic mixing)**.** *Let $\mathbf{G}$ be simply connected and almost K-simple. For any $f_1, f_2 \in L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ and any sequence $g_i \in \mathbf{G}(\mathbb{A})$,*

$$\int_{\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})} f_1(xg_i)f_2(x)d\mu_G \to \int f_1 d\mu_G \cdot \int f_2 d\mu_G \quad \text{as } g_i \to \infty,$$

where μ_G is the invariant probability measure on $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$.

The proof in [36] is based on the information on local harmonic analysis of the groups $\mathbf{G}(K_v)$ [54] as well as the automorphic theory of $\mathbf{G}$ [18], and gives a rate of convergence. In the methods of this paper, it suffices to know the mixing property of $\mathbf{G}_S := \prod_{v\in S}\mathbf{G}(K_v)$ for some finite S containing all archimedean valuations and containing at least one strongly isotropic v. This property can either be deduced from the classical Howe-Moore theorem [35], or from the property of unipotent flows in $\mathbf{G}_S$ modulo lattices.

In the following corollary, let $\mathbf{U}$ be an affine variety defined over $\mathbb{Z}$ such that $\mathbf{U} = v_0\mathbf{G}$ where $\mathbf{G} \subset \mathrm{GL}_N$ is a connected simply connected semisimple $\mathbb{Q}$-group and $v_0 \in \mathbb{Q}^N \setminus \{0\}$. Suppose that $\mathbf{L} := \mathrm{stab}_{\mathbf{G}}(v_0)$ is a semisimple maximal connected $\mathbb{Q}$-subgroup of $\mathbf{G}$. We let μ_p, $v \in R$ be invariant measures on $v_0\mathbf{G}(\mathbb{Q}_p)$ such that $\mu = \prod \mu_p$ is a measure on $v_0\mathbf{G}(\mathbb{A})$ compatible with the probability invariant measures μ_G and μ_L on $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$ and $\mathbf{L}(K)\backslash\mathbf{L}(\mathbb{A})$ respectively.

As another corollary, we obtain the following local-global principle, which can be seen as a higher dimensional analogue of the classical Hasse principle:

Corollary 1.9. (1) *For all sufficiently large $m \in \mathbb{N}$,*

$$\mathbf{U}\left(\frac{\mathbb{Z}}{m}\right) \neq \emptyset \quad \textit{iff} \quad \mathbf{U}\left(\frac{\mathbb{Z}_p}{m}\right) \neq \emptyset \quad \textit{for all prime } p.$$

(2) *If $\mathbf{L}$ is simply connected, then for any compact subset $\Omega \subset v_0\mathbf{G}(\mathbb{R})$ of boundary of measure zero,*

$$\#\mathbf{U}\left(\frac{\mathbb{Z}}{m}\right) \cap \Omega \sim \mu_\infty(\Omega) \prod_p \mu_p\left(\mathbf{U}\left(\frac{\mathbb{Z}_p}{m}\right)\right)$$

provided the right hand side is not zero as $m \to \infty$.

We remark that the assumption that both $\mathbf{G}$ and $\mathbf{L}$ are simply connected imply that the group $\mathbf{G}(\mathbb{A}_f)$ of finite adeles acts transitively on $\mathbf{U}(\mathbb{A}_f)$ [BR], and hence μ_p's are invariant measures on $v_0\mathbf{G}(\mathbb{Q}_p) = \mathbf{U}(\mathbb{Q}_p)$ for each finite p.

When $\mathbf{U} = \mathbf{G}$, i.e., a group variety, Corollary 1.9 was observed in [38], as an application of the Adelic mixing theorem. (2) of the above corollary was previously obtained in [32] and [55] assuming that both $\mathbf{G}(\mathbb{R})$ and $\mathbf{L}(\mathbb{R})$ has no compact factors and that $\mathbf{U}(m^{-1}\mathbb{Z}) \neq \emptyset$. See also [9] and [26] for the case when $\mathbf{L}$ is a torus.

Organization: In section 2, we discuss how to extend a height function of $\mathbf{U}(K)$ to $\mathbf{U}(\mathbb{A})$ so that the action of $\mathbf{G}(\mathbb{A})$ is uniformly continuous and proper, and obtain the asymptotic of the volume of the height balls in each M-orbit of $\mathbf{U}(\mathbb{A})$ for a finite index subgroup M of $\mathbf{G}(\mathbb{A})$. The second part uses the work of Chambert-Loir and Tschinkel. In section 3 we discuss the wonderful varieties, introduced by Luna, which are the generalization of the wonderful compactification of symmetric varieties constructed by De Concini- Procesi. They provide main examples of our theorem 1.5. In section 4, we deduce Theorem 1.7 from the corresponding theorem 4.6 in the S-arithmetic setting, which is proved in the last 2 sections of this paper. In section 5, we prove main theorems of the introduction. In section 6, we prove one part of Theorem 4.6, and the other part is proved in section 7.

Acknowledgment We thank Akshay Venkatesh for helpful conversations; in particular, for his suggestion of the use of strong approximations in the study of adelic orbits. We thank Mikhail Borovoi who kindly wrote up the appendix on our request. Oh also wants to thank the IAS where some part of this work was done during her stay in Feb-Mar, 2006. Gorodnik would like to thank Princeton University for hospitality.

2. Heights and Volume estimates

Let K be a number field and R the set of all normalized absolute values of K. By R_∞, we mean the subset of R consisting of all archimedean ones and set $R_f := R \setminus R_\infty$. For each $v \in R$, we denote by K_v the completion of K with respect to the absolute value $|\cdot|_v$, by k_v the residue field, and by $\mathcal{O}_v$ the ring of integers of K_v. The cardinality of k_v is denoted by q_v. For a finite subset S of R, the ring of S-integers is the subring of K defined by $\mathcal{O}_S := \{x \in K : |x|_v \leq 1 \text{ for all non-archimedean } v \notin S\}$.

Throughout section 2, we let $\mathbf{G}$ be a connected semisimple algebraic K-group with a given K-representation $\mathbf{G} \hookrightarrow \mathrm{GL}_{d+1}$. Fix $u_0 \in \mathbb{P}^d(K)$ such that the orbit $\mathbf{U} := u_0\mathbf{G}$ is a K-subvariety. We fix integral models $\mathcal{U}$ and $\mathcal{G}$ of $\mathbf{U}$ and $\mathbf{G}$, respectively, over the ring $\mathcal{O}_S$ for some S.

Then the adelic space $\mathbf{U}(\mathbb{A})$ is the restricted topological product of $\mathbf{U}(K_v)$'s with respect to $\mathcal{U}(\mathcal{O}_v)$'s. As well-known, this is a locally compact space.

For finite $S \subset R$, we set $\mathbf{U}_S := \prod_{v\in S} \mathbf{U}(K_v)$ and denote by $\mathbf{U}(\mathbb{A}_S)$ the the restricted topological product of $\mathbf{U}(K_v)$'s, $v \in R\setminus S$, with respect to $\mathcal{U}(\mathcal{O}_v)$'s. Then $\mathbf{U}(\mathbb{A})$ is canonically identified with the direct product $\mathbf{U}_S \times \mathbf{U}(\mathbb{A}_S)$. We set $\mathbf{U}_{\mathbb{A}_f} := \mathbf{U}_{\mathbb{A}_{R_\infty}}$ and $\mathbf{U}_\infty := \prod_{v\in R_\infty} \mathbf{U}(K_v)$. The notations $\mathbf{G}(\mathbb{A})$, $\mathbf{G}_S$ and $\mathbf{G}_\infty$ etc. are similarly defined. Note that both $\mathbf{G}_S$ and $\mathbf{G}(\mathbb{A}_S)$ can be considered as subgroups of $\mathbf{G}(\mathbb{A})$ in a canonical way.

Let $\mathbf{X} \subset \mathbb{P}^d$ be the Zariski closure of $\mathbf{U}$, which is then a $\mathbf{G}$-equivariant compactification of $\mathbf{G}$. Consider the line bundle L of $\mathbf{X}$ given by the pull-back of $\mathcal{O}_{\mathbb{P}^d}(1)$. Then L is very ample and $\mathbf{G}$-linearized; in fact any $\mathbf{G}$-linearized very ample line bundle is of this form for some embedding.

Since $\mathbf{U}(K) \neq \emptyset$, Rosenlicht's theorem implies (cf. [11, Lem. 1.5.1]):

Lemma 2.1. *There is no non-constant invertible regular function on* $\mathbf{U}$.

Using a theorem of Luna [44] and the above lemma, we obtain the following:

Theorem 2.2. *Suppose that* $\mathbf{L} := \mathrm{stab}_{\mathbf{G}}(u_0)$ *is semisimple and* $[\mathrm{N}_{\mathbf{G}}(\mathbf{L}) : \mathbf{L}] < \infty$. *Then any global section* s *of* L *such that* $\mathbf{U} = \{\mathrm{s} \neq 0\}$ *is* $\mathbf{G}$-*invariant, and unique up to a scalar multiple.*

Proof. Pick a point $y \in K^{d+1} \setminus \{0\}$ lying above u_0. Let $\mathbf{H}$ denote the stabilizer of y in $\mathbf{G}$. Since $\mathbf{H}$ is a normal co-abelian subgroup of $\mathbf{L}$ and $\mathbf{L}$ is semisimple, $\mathbf{H}$ is also semisimple and $\mathbf{H}^\circ$ is a finite index in $\mathbf{L}$. Hence the finiteness of $[\mathrm{N}_{\mathbf{G}}(\mathbf{L}) : \mathbf{L}]$ implies that $\mathbf{H}$ has finite index in its normalizer. Now a theorem of Luna [44, Corollary 3] says that the orbit of y is closed. By [53, Ch 2, §1, Prop 2.2], there exists a global $\mathbf{G}$-invariant section s_1 of L^k for some k such that $\mathrm{s}_1(u_0) \neq 0$. Hence $\mathbf{U} \subset \{\mathrm{s}_1 \neq 0\}$.

Since $\mathbf{U} = \{\mathrm{s}^k \neq 0\}$, the ration $\mathrm{s}_1 / \mathrm{s}^k$ is an invertible regular function on $\mathbf{U}$, which is a constant by Lemma 2.1. Hence s^k is $\mathbf{G}$-invariant. For any $g \in \mathbf{G}$, s^g/s is a constant function, say, α_g, on $\mathbf{U}$ by the above lemma. Now $\alpha : g \mapsto \alpha_g$ defines a homomorphism from $\mathbf{G}$ into the group of k-roots of unity. Since $\mathbf{G}$ is connected, α must be 1. Hence s is invariant. The uniqueness follows by a similar argument. □

2.1. **Heights.** Let $\mathrm{s}_0, \cdots, \mathrm{s}_d$ be the global sections of L obtained by pulling back the coordinate functions x_i's. We assume that there is a $\mathbf{G}$-invariant global section s of L such that $\mathbf{U} = \{\mathrm{s} \neq 0\}$.

Definition 2.3. *An adelic metrization on the* $\mathbf{G}$*-linearized line bundle* L *on* $\mathbf{X}$ *(with respect to* s*) is a collection of* v*-adic metrics on* L *for all* $v \in R$ *such that*

(1) *for each* $v \in R_f$, $\|\cdot\|_v$ *is locally constant in the* v*-adic topology.*
(2) *for almost all* $v \in R$,

$$\|\,\mathrm{s}(x)\|_v = \left(\max_{0 \le i \le d} \left| \frac{\mathrm{s}_i(x)}{\mathrm{s}(x)} \right|_v \right)^{-1} \quad \text{for all } x \in \mathbf{U}(K_v).$$

(3) *for each* $v \in R_\infty$ *and for any* $\epsilon > 0$, *there exists a neighborhood* W_ϵ *of* e *in* $\mathbf{G}(K_v)$ *such that for all* $x \in \mathbf{U}(K_v)$ *and* $g \in W_\epsilon$,

$$(1-\epsilon)\|\,\mathrm{s}(x)\|_v \le \|\,\mathrm{s}(xg)\|_v \le (1+\epsilon)\|\,\mathrm{s}(x)\|_v.$$

Recall that a v-adic metric $\|\cdot\|_v$ on L is a family $(\|\cdot\|_{x,v})_{x\in\mathbf{X}(K_v)}$ of v-adic Banach norms on the fibers L_x such that for every Zariski open $U \subset \mathbf{X}$ and every section $\mathrm{s} \in H^0(U, L)$, the map $U(K_v) \to \mathbb{R}$ given by $x \to \|\,\mathrm{s}\,\|_{x,v}$ is continuous in the v-adic topology on $U(K_v)$.

We write $(\|\cdot\|_v)_{v\in R}$ for an adelic metric on L and call a pair $\mathcal{L} = (L, \|\cdot\|_v)$ an adelically metrized line bundle. Note that an adelic metrization of L extends naturally to tensor products L^k for any $k \in \mathbb{N}$.

An adelically metrized line bundle $\mathcal{L}$ induces a family of local heights on $\mathbf{U}(K_v)$:

$$\mathrm{H}_{\mathcal{L},v}(x) := \|\,\mathrm{s}(x)\|_v^{-1}.$$

The following lemma can be proved in a standard way (see [5, Ch. 2] for a detailed discussion of heights).

Lemma 2.4. (1) *For each* $v \in R$, $\inf_{x\in\mathbf{U}(K_v)} \mathrm{H}_{\mathcal{L},v}(x) > 0$.
(2) *For almost all* v, $\inf_{x\in\mathbf{U}(K_v)} \mathrm{H}_{\mathcal{L},v}(x) = 1$.
(3) *For almost all* v, $\{x \in \mathbf{U}(K_v) : \mathrm{H}_{\mathcal{L},v}(x) = 1\} = \mathcal{U}(\mathcal{O}_v)$.
(4) *Let* $v \in R$. *If* $x \to \infty$ *in* $\mathbf{U}(K_v)$, *then* $\mathrm{H}_{\mathcal{L},v}(x) \to \infty$.

Definition 2.5. *An adelic height function* $\mathrm{H}_{\mathcal{L}} : \mathbf{U}(\mathbb{A}) \to \mathbb{R}_{>0}$ *associated to* $\mathcal{L}$ *is defined by*

$$\mathrm{H}_{\mathcal{L}}(x) := \prod_{v\in R} \mathrm{H}_{\mathcal{L},v}(x) \quad \text{for } x \in \mathbf{U}(\mathbb{A}). \tag{2.6}$$

The previous lemma implies that $\mathrm{H}_{\mathcal{L}}$ is a well-defined continuous proper function. Moreover the following holds:

Lemma 2.7. (1) *Set*

$$W_{\mathrm{H}_{\mathcal{L}}} := \{g \in \mathbf{G}(\mathbb{A}_f) : \mathrm{H}_{\mathcal{L}}(xg) = \mathrm{H}_{\mathcal{L}}(x) \text{ for all } x \in \mathbf{U}(\mathbb{A})\}.$$

Then $W_{\mathrm{H}_{\mathcal{L}}}$ is a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$.

(2) *For every compact subset $B \subset \mathbf{G}(\mathbb{A})$, there exists $c > 0$ such that for every $g \in B$ and $x \in \mathbf{U}(\mathbb{A})$,*

$$\mathrm{H}_{\mathcal{L}}(xg) < c\, \mathrm{H}_{\mathcal{L}}(x).$$

(3) *For every $\epsilon > 0$, there exists a neighborhood W of e in $\mathbf{G}(\mathbb{A})$ such that for every $x \in \mathbf{U}(\mathbb{A})$ and $g \in W$,*

$$\mathrm{H}_{\mathcal{L}}(xg) < (1+\epsilon)\, \mathrm{H}_{\mathcal{L}}(x).$$

Proof. Since $\|\cdot\|_v$ is locally constant for all $v \in R_f$, $W_{\mathrm{H}_{\mathcal{L}}} \cap \mathbf{G}(K_v)$ is an open subgroup of $\mathbf{G}(K_v)$ for each $v \in R_f$. Since $\mathbf{G}$ acts on $\mathbf{U}$ via the linear action of SL_{d+1} on $\mathbb{P}^d$ and s is invariant, $W_{\mathrm{H}} \cap \mathbf{G}(K_v) = \mathcal{G}(\mathcal{O}_v)$ for almost all $v \in R_f$ by (3) of Def. 2.3. It follows that $W_{\mathrm{H}_{\mathcal{L}}}$ is open. By the properness of $\mathrm{H}_{\mathcal{L}}$, $W_{\mathrm{H}_{\mathcal{L}}}$ is compact. Any compact subset B of $\mathbf{G}(\mathbb{A})$ is contained in $\prod_{v\in S} B_v \times \prod_{v\notin S} \mathcal{G}(\mathcal{O}_v)$ for some finite $S \subset R$ where B_v is a compact subset in $\mathbf{G}(K_v)$. By enlarging S, we may assume $\prod_{v\notin S} \mathcal{G}(\mathcal{O}_v) \subset W_{\mathrm{H}}$. On the other hand, for each $v \in R$, there exists $c_v > 1$ such that

$$\mathrm{H}_{\mathcal{L},v}(xg) \le c_v \cdot \max_{i,j} |g_{ij}|_v \cdot \mathrm{H}_{\mathcal{L},v}(x)$$

for all $g = (g_{ij}) \in \mathbf{G}(K_v)$ and $x \in \mathbf{U}(K_v)$. Hence it suffices to take $c = \prod_{v\in S}(c_v \cdot \max_{g\in B_v} |g_{ij}|_v)$ for the claim (2).

The claim (3) follows from the claim (1) and (3) of Def. 2.3. □

We will call the height function $\mathrm{H}_{\mathcal{L}}$ *regular* if the function $\prod_{v\in R_\infty} \mathrm{H}^2_{\mathcal{L},v}$ is regular on $\mathbf{U}_\infty$, considered as the real algebraic variety via the restriction of scalars. For instance, the following height function is given by a regular adelic metrization :

$$\mathrm{H}_{\mathcal{L},v}(x) = \begin{cases} \frac{(\sum_i |\mathrm{s}_i(x)|_v^2)^{1/2}}{|\mathrm{s}(x)|_v} & \text{for archimedean } v, \\ \frac{\max_i |\mathrm{s}_i(x)|_v}{|\mathrm{s}(x)|_v} & \text{for non-archimedean } v. \end{cases} \tag{2.8}$$

The following example shows that our settings apply to any affine homogeneous varieties:

Example 2.9. Denote by $\mathbb{A}^d$ the d-dimensional affine space. Let $\mathbf{U} = v_0\mathbf{G} \subset \mathbb{A}^d$ be an affine homogeneous K- variety for a connected K-group $\mathbf{G} \subset \mathrm{GL}_d$ and a non-zero $v_0 \in \mathbb{A}^d(K)$. Via the embedding $\iota : \mathbb{A}^d \hookrightarrow \mathbb{P}^d$ given by

$$\iota(x_0, \cdots, x_{d-1}) \mapsto (x_0 : \cdots : x_{d-1} : 1)$$

and the embedding $\mathrm{GL}_d \to \mathrm{PGL}_{d+1}$ by $A \mapsto \mathrm{diag}(A, 1)$, the Zariski closure $\mathbf{X} \subset \mathbb{P}^d$ of $\iota(\mathbf{U})$ is a $\mathbf{G}$-equivariant compactification.

Consider the line bundle $L = \iota^*(\mathcal{O}_{\mathbb{P}^d}(1))$ and sections $\mathrm{s}_i = \iota^*(x_i)$ for $0 \le i \le d$. Since $\iota(\mathbf{U}) = \{\mathrm{s}_d \neq 0\}$ for the $\mathbf{G}$-invariant section s_d, we can choose an adelic metrization $\mathcal{L}$ of L so that the local height functions $\mathrm{H}_{\mathcal{L},v}$ on $\mathbf{U}(K_v)$ are given by

$$\begin{cases} \left(|x_0|_v^2 + \cdots + |x_{d-1}|_v^2 + 1\right)^{1/2} & \text{for archimedean } v, \\ \max\{|x_0|_v, \ldots, |x_{d-1}|_v, 1\} & \text{for non-archimedean } v. \end{cases} \tag{2.10}$$

2.2. **Tamagawa volumes of height balls.** We assume that $\mathbf{L} := \mathrm{stab}_{\mathbf{G}}(u_0)$ is semisimple, and s is an invariant global section of L such that $\mathbf{U} = \{\mathrm{s} \neq 0\}$. Fix an adelic metrization $\mathcal{L}$ of L and consider the height function $\mathrm{H} = \mathrm{H}_{\mathcal{L}}$ on $\mathbf{U}(\mathbb{A})$ defined in (2.6). For simplicity, we set $\mathrm{H}_v = \mathrm{H}_{\mathcal{L},v}$. We observe that $\mathbf{U}$ is a geometrically irreducible nonsingular algebraic variety and that $\mathbf{U}$ supports a nowhere zero differential form ω of top degree. We refer to [72] for the following discussion on the Tamagawa measure on $\mathbf{U}(\mathbb{A})$. To the form ω, we can associate measures μ_v on each $\mathbf{U}(K_v)$. Then $\mu_v(\mathcal{U}(\mathcal{O}_v)) = \frac{\#\mathbf{U}(k_v)}{q_v^{\dim \mathbf{U}}}$ for almost all $v \in R$. Since $\mathbf{U}$ is a homogeneous space of a connected semisimple algebraic group with the stabilizer subgroup being semisimple, $\prod_v \mu_v(\mathcal{U}(\mathcal{O}_v))$ converges absolutely, and hence ω defines the Tamagawa measure

$$\mu = |\Delta_K|^{-\frac{1}{2}\dim \mathbf{U}} \prod_v \mu_v$$

on the space $\mathbf{U}(\mathbb{A})$ where Δ_K is the discriminant of K.

For $t > 0$, set

$$B_t := \{y \in \mathbf{U}(\mathbb{A}) : \mathrm{H}(y) < t\}.$$

In this section, we review a theorem of Chambert-Loir and Tschinkel on asymptotic properties (as $t \to \infty$) of the Tamagawa volume

$$V(t) := \mu(B_t).$$

First, we assume that $\mathbf{X}$ is smooth and $\mathbf{X}\backslash\mathbf{U}$ is a divisor with normal crossings of irreducible components D_α, $\alpha \in \mathcal{A}$, defined over finite field extensions K_α of K. By extending ω to $\mathbf{X}$, which we denote by ω by abuse of notation, we obtain a non-zero differential form on $\mathbf{X}$ of top degree. Since $\{\mathrm{s} = 0\} = \mathbf{X}\backslash\mathbf{U}$ and ω is nowhere zero on $\mathbf{U}$,

$$\mathrm{div}(\mathrm{s}) = \sum_{\alpha\in\mathcal{A}} m_\alpha D_\alpha \quad \text{and} \quad -\mathrm{div}(\omega) = \sum_{\alpha\in\mathcal{A}} n_\alpha D_\alpha$$

for $m_\alpha \in \mathbb{N}$ and $n_\alpha \in \mathbb{Z}$. The Galois group $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$ acts on $\mathcal{A}$. We denote by $\mathcal{A}/\Gamma_K$ the set of Γ_K-orbits. Define

$$a(L) = \max_{\alpha\in\mathcal{A}} \left\{\frac{n_\alpha}{m_\alpha}\right\} \quad \text{and} \quad b(L) = \#\left\{\alpha \in \mathcal{A}/\Gamma_K : \frac{n_\alpha}{m_\alpha} = a(L)\right\}. \tag{2.11}$$

Lemma 2.12. (1) *D_α's are not rationally equivalent;*

(2) *$a(L)$ and $b(L)$ are independent of choices of* s *and ω*

Proof. If $D_\alpha = \mathrm{div}(f) + D_\beta$ for some $f \in K(X)^*$, then the poles as well as the zeros of f must lie outside $\mathbf{U}$, and hence f is constant by Lemma 2.1, proving (1). Since s is unique up to constant, again by Lemma 2.1, the independence of m_α's on s is clear. Similarly, any non-zero differential form on $\mathbf{X}$ of top degree, which is nowhere zero on $\mathbf{U}$, is a multiple of ω by a constant. Hence n_α's are determined independently on the choice of ω. $\square$

In general, we take an equivariant resolution of singularities $\pi : \tilde{\mathbf{X}} \to \mathbf{X}$ such that $\tilde{\mathbf{X}}$ is smooth and the boundary $\pi^{-1}(\mathbf{X}\backslash\mathbf{U})$ is a divisor with normal crossings. Then the constants $a(L)$ and $b(L)$ are defined as above with respect to the pull-backs $\pi^*(\mathrm{s})$ and $\pi^*(\omega)$.

We consider the Mellin transform of $V(t)$:

$$\eta(s) := \int_0^\infty t^{-s}\, dV(t) = \int_{\mathbf{U}(\mathbb{A})} \mathrm{H}(x)^{-s}\, d\mu(x).$$

Hence

$$\eta(s) = |\Delta_K|^{-\frac{1}{2}\dim X} \prod_v \eta_v(s)$$

where

$$\eta_v(s) := \int_{\mathbf{U}(K_v)} \mathrm{H}_v(x)^{-s}\, d\mu_v(x).$$

Let

$$\Omega_t = \{s \in \mathbb{C} :\ \mathrm{Re}(m_\alpha s - n_\alpha) > t, \alpha \in \mathcal{A}\}.$$

Theorem 2.13 (Chambert-Loir, Tschinkel)**.** (1) *For each $v \in R$, the integral $\eta_v(s)$ is absolutely convergent for $s \in \Omega_{-1}$.*

(2) *The integral $\eta(s)$ converges absolutely for $s \in \Omega_0$, and*

$$\eta(s) = \phi(s) \prod_{\alpha \in \mathcal{A}/\Gamma_K} \zeta_{K_\alpha}(m_\alpha s - n_\alpha + 1)$$

where ζ_{K_α} is the Dedekind zeta function of K_α, and $\phi(s)$ is a bounded holomorphic function for $s \in \Omega_{-1/2+\epsilon}$, $\epsilon > 0$.

The first claim follows from [20, Lemma 8.2] for non-archimedean place. For archimedean v, if H_v^2 is regular on $\mathbf{U}$, the same proof applies. Since any two norms on a finite dimensional vector spaces are equivalent to each other, this implies the first claim for any local height H_v. The second claim is [20, Corollary 11.4].

Corollary 2.14. *If $a(L) > 0$, then there exist a polynomial P of degree $b(L) - 1$ and $\delta > 0$ such that*

$$V(t) = t^{a(L)} P(\log t) + O(t^{a(L)-\delta}) \quad \text{as } t \to \infty.$$

Proof. It follows from Theorem 2.13 and the properties of the Dedekind zeta functions that for some $\epsilon > 0$, $\eta(s)$ has a meromorphic continuation to the region $\Omega_{-1/2+\epsilon}$ with a single pole at $s = a(L)$ of order $b(L)$. Moreover, in this region, $\eta(s)$ satisfies the bound

$$\left| \frac{(s-a(L))^{b(L)}}{s^{b(L)}} \eta(s) \right| \leq c \cdot |1 + \mathrm{Im}(s)|^N$$

for some $c, N > 0$. Hence, the claim follows from the Tauberian theorem (see [36, Thm. 4.4] and [19, Appendix]). □

2.3. **Volumes of homogeneous varieties.** We additionally assume that the subgroup $\mathbf{L}$ is connected. We recall the properties of orbits of algebraic groups over local fields and over adeles.

Lemma 2.15.
(1) *For each $v \in R$, the space $\mathbf{U}(K_v)$ consists of finitely many $\mathbf{G}(K_v)$-orbits, and each orbit is open and closed.*
(2) *For almost all v, $\mathcal{G}(\mathcal{O}_v)$ acts transitively on $\mathcal{U}(\mathcal{O}_v)$.*
(3) *The orbits of $\mathbf{G}(\mathbb{A})$ in $\mathbf{U}(\mathbb{A})$ are open and closed.*

Proof. The orbits in (1) are open by [59, Ch.3,§3.1]. This also implies that every orbit is closed. The finiteness of $\mathbf{G}(K_v)$-orbits follows from finiteness of Galois cohomology over local fields (see [59, Ch.3, §6.4]). (2) follows from Lang's theorem [43] and Hensel's lemma (see [11, Lemma 1.6.4]). (3) follows from (1) and (2). □

Theorem 2.16. *Assume that there are finitely many $\mathbf{G}(\mathbb{A})$-orbits in $\mathbf{U}(\mathbb{A})$. Let $x \in \mathbf{U}(\mathbb{A})$ and*

$$V(x,t) := \mu(x\mathbf{G}(\mathbb{A}) \cap B_t).$$

Then $a(L) > 0$ and there exist a nonzero polynomial P_x of degree $b(L) - 1$ and $\delta > 0$ such that

$$V(x,t) = t^{a(L)} P_x(\log t) + O(t^{a(L)-\delta}) \quad \text{as } t \to \infty.$$

Proof. As in the proof of Corollary 2.14, we consider the Mellin transform

$$\begin{aligned}
\eta(x,s) &:= \int_0^\infty t^{-s}\, dV(x,t) \\
&= \int_{x\mathbf{G}(\mathbb{A})} \mathrm{H}(y)^{-s}\, d\mu(y) = |\Delta_K|^{-\frac{1}{2}\dim X} \cdot \prod_v \eta_v(x_v, s)
\end{aligned}$$

where

$$\eta_v(x_v, s) := \int_{x_v\mathbf{G}(K_v)} \mathrm{H}_v(y)^{-s}\, d\mu_v(y).$$

By Theorem A.1.2, our assumption implies that for almost all v, $x_v\mathbf{G}(K_v) = \mathbf{U}(K_v)$ and hence $\eta_v(x_v, s) = \eta_v(s)$. Also, by Theorem 2.13(1), $\eta_v(x_v, s)$ is absolutely convergent for $s \in \Omega_{-1+\epsilon}$, $\epsilon > 0$. Hence, it follows from Theorem 2.13(2), that

$$\eta(x,s) = \phi(x,s) \prod_{\alpha\in\mathcal{A}/\Gamma_K} \zeta_{K_\alpha}(m_\alpha s - n_\alpha + 1) \tag{2.17}$$

where $\phi(x,s)$ is a bounded holomorphic function for $s \in \Omega_{-1/2+\epsilon}$, $\epsilon > 0$.

Note that for almost all v, $\mathbf{G}$ is quasi-split over K_v, and hence there is a unipotent one-parameter subgroup of $\mathbf{G}(K_v)$ acting nontrivially on $\mathbf{U}(K_v)$. It was shown in [10] that this property implies that for some $a' > 0$,

$$\mu_v(\{y \in x_v\mathbf{G}(K_v) :\, \mathrm{H}_v(y) < t\}) \ge c\, t^{a'}$$

for all large $t > 0$. This implies that $\eta_v(x_v, s)$ has a pole in the region $\mathrm{Re}(s) \ge a'$. Hence, $a(L) > 0$.

Now the claim follows from (2.17) using Tauberian theorem. □

Denoting by $\mathbf{G}_\infty^\circ$ the identity component of $\mathbf{G}_\infty$, we set for $x \in \mathbf{U}_\infty$,

$$\tilde{V}_\infty(x,t) := \mu_\infty(\{y \in x\mathbf{G}_\infty^\circ : \mathrm{H}_\infty(y) < t\})$$

where $\mu_\infty := \prod_{v\in R_\infty} \mu_v$ and $\mathrm{H}_\infty := \prod_{v\in R_\infty} \mathrm{H}_v$. The following is proved in [10, Lemma 7.8].

Theorem 2.18. *If H_∞ is regular and is not constant on $x\mathbf{G}_\infty^\circ$ for $x \in \mathbf{U}_\infty$, then there exist $c_0, \kappa > 0$ such that for all $t > 0$ and $\epsilon \in (0,1)$,*

$$\tilde{V}_\infty(x, t(1+\epsilon)) - \tilde{V}_\infty(x,t) \le c_0\epsilon^\kappa\, (\tilde{V}_\infty(x,t) + 1).$$

Proposition 2.19. *Assume that there are only finitely many $\mathbf{G}(\mathbb{A})$-orbits in $\mathbf{U}(\mathbb{A})$, and that H is regular. Let $M = \mathbf{G}_\infty^\circ M_f$ for a finite index closed subgroup M_f of $\mathbf{G}(\mathbb{A}_f)$, $x \in \mathbf{U}(\mathbb{A})$, and*

$$V^M(x,t) := \mu(xM \cap B_t).$$

Then

(1)
$$V^M(x,t) \asymp t^{a(L)}(\log t)^{b(L)-1}.$$

(2) *If H_∞ is not constant on $x_\infty\mathbf{G}_\infty^\circ$ where $x = x_\infty x_f \in \mathbf{U}_\infty\mathbf{U}_{\mathbb{A}_f}$, there exist $c, \kappa, t_0 > 0$ such that for every $t > t_0$ and $\epsilon \in (0,1)$,*

$$V^M(x,(1+\epsilon)t) - V^M(x,t) \le c\,\epsilon^\kappa V^M(x,t).$$

Proof. Since $V^M(x,t) \le V(x,t)$, the upper estimate follows from Theorem 2.16. To prove the lower estimate, we write $\mathbf{G}(\mathbb{A}) = \cup_{i=1}^n Mg_i$ for some $g_i \in \mathbf{G}(\mathbb{A})$. Then by 2.7(2) and invariance of μ, we obtain that for some $c > 1$,

$$V(x,t) \le \sum_{i=1}^n V^{Mg_i}(x,t) \le n\, V^M(x, c\cdot t) \quad \text{for all } t > 0.$$

Hence the lower estimate in (1) follows from Theorem 2.16. To prove (2), consider the decompositions

$$\mathbf{U}(\mathbb{A}) = \mathbf{U}_\infty \mathbf{U}_{\mathbb{A}_f} \quad \text{and} \quad \mu = \mu_\infty \otimes \mu_f$$

where $\mu_f = \prod_{v\in R_f} \mu_v$.

Set

$$\begin{aligned}\tilde{V}_\infty(t) &:= \mu_\infty(\{y \in x_\infty \mathbf{G}_\infty^\circ : \mathrm{H}_\infty(y) < t\}),\\ \tilde{V}_f(t) &:= \mu_f(\{y \in x_f M_f : \mathrm{H}_f(y) < t\})\end{aligned}$$

where $\mathrm{H}_f = \prod_{v\in R_f} \mathrm{H}_v$.

We claim that there exist $\rho_1, \rho_2 > 0$ such that for every $t > 0$,

$$\tilde{V}_f(t) \le \rho_1 V^M(x, \rho_2 t). \tag{2.20}$$

Let Ω be a compact subset of $x_\infty \mathbf{G}_\infty^\circ$ such that $\mu_\infty(\Omega) > 0$ and $\rho_2 = \max_{x\in\Omega} \mathrm{H}_\infty(x)$. Then

$$\Omega \cdot \{y \in x_f M_f : \mathrm{H}_f(y) < t\} \subset \{y \in xM : \mathrm{H}(y) < \rho_2 t\},$$

and hence

$$\tilde{V}_f(t) \le \frac{V^M(\rho_2 t)}{\mu_\infty(\Omega)}.$$

By Theorem 2.18, there exist $c_0, \kappa > 0$ such that for all $t > 0$ and $\epsilon \in (0, 1)$,

$$\tilde{V}_\infty(t(1+\epsilon)) - \tilde{V}_\infty(t) \le c_0 \epsilon^\kappa (\tilde{V}_\infty(t) + 1).$$

Let $\alpha = \inf_{y\in x_\infty \mathbf{G}_\infty^\circ} \mathrm{H}_\infty(y) > 0$. Then using (2.18) and (2.20),

$$\begin{aligned}
&V^M((1+\epsilon)t) - V^M(t)\\
&= \int_{y\in x_f M_f} \left(\tilde{V}_\infty(\tfrac{(1+\epsilon)t}{\mathrm{H}_f(y)}) - \tilde{V}_\infty(\tfrac{t}{\mathrm{H}_f(y)})\right) d\mu_f(y)\\
&= \int_{y\in x_f M_f : \mathrm{H}_f(y) < \alpha^{-1}2t} \left(\tilde{V}_\infty(\tfrac{(1+\epsilon)t}{\mathrm{H}_f(y)}) - \tilde{V}_\infty(\tfrac{t}{\mathrm{H}_f(y)})\right) d\mu_f(y)\\
&\le c\epsilon^\kappa \int_{y\in x_f M_f : \mathrm{H}_f(y) < \alpha^{-1}2t} (\tilde{V}_\infty(\tfrac{t}{\mathrm{H}_f(y)}) + 1)\, d\mu_f(y)\\
&\le c\epsilon^\kappa \left(\int_{y\in x_f M_f} \tilde{V}_\infty(\tfrac{t}{\mathrm{H}_f(y)}) d\mu_f(y) + \mu_f(\{y \in x_f M_f : \mathrm{H}_f(y) < \alpha^{-1}2t\})\right)\\
&= c\epsilon^\kappa \left(V^M(t) + \tilde{V}_f(\alpha^{-1}2t)\right)\\
&\le c\epsilon^\kappa \left(V^M(t) + \rho_1 V^M(\rho_2\alpha^{-1}2t)\right)\\
&\le c'\epsilon^\kappa V^M(t)
\end{aligned}$$

for some $c' > 1$, where the last inequality holds by the claim (1).

This completes the proof.

$\square$

Theorem 2.21. *Assume that there are only finitely many* $\mathbf{G}(\mathbb{A})$*-orbits in* $\mathbf{U}(\mathbb{A})$*. Let* M *be a finite index closed subgroup of* $\mathbf{G}(\mathbb{A})$ *and* W *a compact open subgroup of* $\mathbf{G}(\mathbb{A}_f)$ *contained in* $M \cap W_{\mathrm{H}}$*. Fixing* $x \in \mathbf{U}(\mathbb{A})$ *and* $t > 0$*, set* $\tilde{B}_t := B_t \cap xM$*.*

(1) *We have*

$$\mu(\tilde{B}_t) \asymp t^{a(L)}(\log t)^{b(L)-1}.$$

(2) *Suppose that* H *is regular. Then for any* $\epsilon > 0$*, there exists a neighborhood* U_ϵ *of* e *in* M *such that for all sufficiently large* t*,*

$$(1-\epsilon)\mu(\tilde{B}_t U_\epsilon W) \le \mu(\tilde{B}_t) \le (1+\epsilon)\mu(\cap_{u \in U_\epsilon W} \tilde{B}_t u). \tag{2.22}$$

Proof. Consider the subgroups $M_\infty = M \cap \mathbf{G}_\infty$ and $M_f = M \cap \mathbf{G}_{\mathbb{A}_f}$, which are closed subgroups of $\mathbf{G}_\infty$ and $\mathbf{G}_{\mathbb{A}_f}$ respectively. Then $\mathbf{G}_\infty^\circ$ is a finite index subgroup of $\mathbf{G}_\infty$ contained in M_∞, and $M_0 := \mathbf{G}_\infty^\circ M_f$ is a finite index subgroup of M. Hence, $xM = \sqcup_{i=1}^n x m_i M_0$ for some $m_i \in M$. Therefore in proving the above claims, we may assume without loss of generality that $M = \mathbf{G}_\infty^\circ M_f$ for some finite index subgroup M_f of $\mathbf{G}(\mathbb{A}_f)$. Note that any height function $\mathrm{H} = \mathrm{H}_{\mathcal{L}}$ on $\mathbf{U}(\mathbb{A})$ is equivalent to a regular height function, i.e., there is an adelic metrization $\mathcal{L}'$ such that for some $c \ge 1$,

$$c^{-1} \cdot \mathrm{H}_{\mathcal{L}}(x) \le \mathrm{H}_{\mathcal{L}'}(x) \le c \cdot \mathrm{H}_{\mathcal{L}}(x)$$

for all $x \in \mathbf{U}(\mathbb{A})$. Hence the claim (1) follows from Proposition 2.19.

Let x_∞ denote the $\mathbf{U}_\infty$-component of x. If H_∞ is constant on $x_\infty \mathbf{G}_\infty^\circ$, then H is invariant under $\mathbf{G}_\infty^\circ$. Hence H is invariant under $\mathbf{G}_\infty^\circ \times W$. Therefore by taking U_ϵ to be $\mathbf{G}_\infty^\circ \times W$, $(B_t \cap xM)u = B_t \cap xM$ for all $u \in U_\epsilon$, and hence $\tilde{B}_t$ satisfies (2.22).

Now suppose that H_∞ is non-constant on $x_\infty \mathbf{G}_\infty^\circ$. For $\epsilon > 0$ small, take a neighborhood of V_ϵ of e in G_∞° such that

$$B_t V_\epsilon \subset B_{(1+\epsilon)t} \quad \text{and} \quad B_{(1-\epsilon)t} \subset \cap_{v \in V_\epsilon} B_t v$$

for all large T. Set $U_\epsilon = V_\epsilon \times W$. Then for all large t,

$$\tilde{B}_t U_\epsilon W \subset \tilde{B}_{(1+\epsilon)t} \quad \text{and} \quad \tilde{B}_{(1-\epsilon)t} \subset \cap_{u \in U_\epsilon W} \tilde{B}_t.$$

Hence by Proposition 2.19 (2), (2.22) is proved. □

3. Wonderful varieties

3.1. **Symmetric varieties.** We review some basic properties of symmetric varieties and their wonderful compactifications due to De Concini and Procesi (see [21] for details).

Let $\mathbf{G}$ be a connected semisimple algebraic subgroup and $\sigma : \mathbf{G} \to \mathbf{G}$ an involution of $\mathbf{G}$. We denote by $\mathbf{L}$ the normalizer of the subgroup $\mathbf{G}^\sigma$ of invariants of σ. Then $\mathbf{L}\backslash\mathbf{G}$ is a *symmetric variety*.

A torus $\mathbf{T} \subset \mathbf{G}$ is called σ*-split* if $\sigma(t) = t^{-1}$ for every $t \in \mathbf{T}$. Let $\mathbf{T}_1$ be a σ-split torus of maximal dimension and $\mathbf{T}$ a maximal torus containing $\mathbf{T}_1$.

Then $\mathbf{T}$ is invariant under σ and it is an almost direct product $\mathbf{T} = \mathbf{T}_1\mathbf{T}_0$ where $\mathbf{T}_0$ is the subtorus of $\mathbf{T}$ on which σ acts trivially.

Let Φ be the set of roots of $\mathbf{T}$. We set

$$\Phi_0 = \{\alpha \in \Phi : \alpha^\sigma = \alpha\} \quad \text{and} \quad \Phi_1 = \Phi \backslash \Phi_0.$$

One can choose a Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$ such that the corresponding set Φ^+ of positive roots has the property that

$$(\Phi_1 \cap \Phi^+)^\sigma = -(\Phi_1 \cap \Phi^+). \tag{3.1}$$

For a root $\alpha \in \Phi$, we set $\tilde{\alpha} = \alpha - \alpha^\sigma$. The set $\tilde{\Phi} = \{\tilde{\alpha}\}$ is a (possibly nonreduced) root system of rank $\dim(\mathbf{T}_1)$ with the set of simple roots $\Delta_\sigma := \tilde{\Delta}\backslash\{0\}$.

Let Λ be the weight lattice and $\Lambda^+ \subset \Lambda$ the set of dominant integral weights. For $\lambda \in \Lambda^+$, we denote by $\iota_\lambda : \mathbf{G} \to \mathrm{GL}(V_\lambda)$ the corresponding irreducible representation with the highest weight λ. The weight λ is called *spherical* if there exists a nonzero vector $v_0 \in V_\alpha$ such that $\mathrm{Lie}(\mathbf{L}) \cdot v_0 = 0$. Let $\Omega^+ \subset \Lambda^+$ be the subset of spherical weights. Every spherical weight λ satisfies $\lambda^\sigma = -\lambda$. Since every dominant weight lies in the interior of the cone generated by positive roots, this implies that every dominant spherical λ can written as

$$\lambda = \sum_{\alpha \in \Delta_\sigma} n_\alpha \alpha$$

for some $n_\alpha \in \mathbb{Q}^+$. A weight λ is called *σ-regular* if $(\lambda, \alpha) \neq 0$ for all $\alpha \in \Delta_\sigma$.

3.2. **Wonderful compactification of a symmetric variety.** Given a σ-regular spherical representation $\iota : \mathbf{G} \to \mathrm{GL}(V)$ and a nonzero vector $v_0 \in V$ fixed by $\mathbf{L}$, one defines the *wonderful compactification* $\mathbf{X}$ of $\mathbf{L}\backslash\mathbf{G}$ as the closure of $\mathbf{U} := [v_0]\mathbf{G}$ in the projective space $\mathbb{P}(V)$. It was proved in [21] that $\mathbf{X}$ satisfies the following properties:

(1) $\mathbf{X}$ is a Fano variety.
(2) $\mathbf{X}\backslash\mathbf{U}$ is a divisor with normal crossings and has smooth irreducible components $\mathbf{X}_1,\ldots,\mathbf{X}_l$ where $l = \dim(\mathbf{T}_1)$.
(3) The closures of $\mathbf{G}$-orbits in $\mathbf{X}$ are precisely the partial intersections of $\mathbf{X}_i$'s.
(4) $\mathbf{X}$ contains the unique closed $\mathbf{G}$-orbit $\mathbf{Y} := \cap_{i=1}^l \mathbf{X}_i$ isomorphic to $\mathbf{P}\backslash\mathbf{G}$ where $\mathbf{P}$ is the parabolic subgroup with the unipotent radical $\exp(\oplus_{\alpha \in \Phi_1 \cap \Phi^+} \mathfrak{g}_\alpha)$.

We also recall a description of the Picard group of $\mathbf{X}$. The map $\mathrm{Pic}(\mathbf{X}) \to \mathrm{Pic}(\mathbf{Y})$ induced by the inclusion $\mathbf{Y} \to \mathbf{X}$ is injective. The Picard group of $\mathbf{Y} \simeq \mathbf{P}\backslash\mathbf{G}$ can be identified with a sublattice of the weight lattice Λ and under this identification

$$\begin{aligned}
\mathrm{Pic}(\mathbf{X}) &\longrightarrow \text{sublattice generated by } \Omega^+, \\
[\mathbf{X}_i] &\longrightarrow \alpha, \qquad \alpha \in \Delta_\sigma, \\
-K_{\mathbf{X}} &\longrightarrow \sum_{\beta \in \Phi_1 \cap \Phi^+} \beta + \sum_{\alpha \in \Delta_\sigma} \alpha.
\end{aligned}$$

Now we assume that $\mathbf{G}$ is defined over a number field K, the representation ι is K-rational and $v_0 \in V(K)$. Then the action of the Galois group Γ_K preserves the unique open $\mathbf{G}$-orbit $\mathbf{U}$ and permutes the boundary components $\mathbf{X}_1, \ldots, \mathbf{X}_l$. The identification of $\mathrm{Pic}(\mathbf{X})$ with a sublattice of the weight lattice Λ is Γ_K-equivariant with respect to the twisted Galois action on Λ.

3.3. **Wonderful varieties.** A generalization of the wonderful compactification was introduced in [45]. A smooth connected projective $\mathbf{G}$-variety $\mathbf{X}$ is called *wonderful* of rank l if

(1) $\mathbf{X}$ contains l irreducible $\mathbf{G}$-invariant divisors $\mathbf{X}_1, \cdots, \mathbf{X}_l$ with strict normal crossings.
(2) $\mathbf{G}$ has exactly 2^l orbits in $\mathbf{X}$.

It follows that $\mathbf{X}$ contains unique open $\mathbf{G}$-orbit, which we denote by $\mathbf{U}$, and that the irreducible components of the divisor $\mathbf{X} \setminus \mathbf{U}$ are $\mathbf{X}_1, \cdots, \mathbf{X}_l$. Fix $u_0 \in \mathbf{U}$ and set $\mathbf{L} = \mathrm{Stab}_{\mathbf{G}}(u_0)$.

In the following, we assume that the subgroup $\mathbf{L}$ is semisimple. A description of the $\mathrm{Pic}(\mathbf{X})$ was given by Brion [14, Proposition 2.2.1]. Since $\mathbf{L}$ is semisimple, $\mathrm{Pic}(\mathbf{L}\backslash\mathbf{G})$ is finite, and it follows that $\mathrm{Pic}(\mathbf{X})$ is a finite extension of the free abelian group generated by $[\mathbf{X}_i]$, $i = 1, \ldots, l$. Then by [14, Lemma 2.3.1], the cone $\Lambda_{\mathrm{eff}}(\mathbf{X}) \subset \mathrm{Pic}(\mathbf{X}) \otimes \mathbb{R}$ of effective divisors is generated by $[\mathbf{X}_i]$, $i = 1, \ldots, l$. The cone of ample divisors was computed in [12, Section 2.6]. Combining this description with [14, Lemma 2.1.2], it follows that the ample cone is contained in the interior of effective cone. The canonical class $K_{\mathbf{X}}$ was computed in [16]. The formula from [16] implies, in particular, that $-K_{\mathbf{X}}$ lies in the interior of the effective cone $\Lambda_{\mathrm{eff}}(\mathbf{X})$.

For an ample line bundle L on $\mathbf{X}$, we define

$$\begin{aligned}
a_L &:= \inf\{a : aL + K_{\mathbf{X}} \in \Lambda_{\mathrm{eff}}(\mathbf{X})\}, \\
b_L &:= \text{the maximal codimension of the face of } \Lambda_{\mathrm{eff}}(\mathbf{X}) \text{ containing } a_L L + K_{\mathbf{X}}.
\end{aligned}$$

Since L and $-K_{\mathbf{X}}$ belong to the interior of $\Lambda_{\mathrm{eff}}(\mathbf{X})$, the parameter a_L is well-defined and $a_L > 0$.

Remark 3.2. In the case when $\mathbf{X}$ is the wonderful compactification of a symmetric variety, and L is the restriction of $\mathcal{O}_{\mathbb{P}(V)}(1)$ to $\mathbf{X}$, the parameters a_L and b_L can be computed in terms of the highest weight λ_ι of the representation ι as follows. Writting

$$\sum_{\beta\in\Phi_1\cap\Phi^+}\beta=\sum_{\alpha\in\Delta_\sigma}m_\alpha\alpha \quad\text{and}\quad \lambda_\iota=\sum_{\alpha\in\Delta_\sigma}n_\alpha\alpha,$$

we have

$$a_L=\max\left\{\frac{m_\alpha+1}{n_\alpha}:\alpha\in\Delta_\sigma\right\}\quad\text{and}\quad b_L=\#\left\{\alpha\in\Delta_\sigma/\Gamma_K:\,a_L=\frac{m_\alpha+1}{n_\alpha}\right\}.$$

Luna showed that any wonderful variety is spherical. It follows that $\mathbf{L}$ has finite index in its normalizer. Hence using Theorem 2.2, we have an invariant global section s of any ample line bundle L such that $\mathbf{U}\subset\{\mathrm{s}\neq 0\}$. Since any ample line bundle is contained in the interior of the cone of effective divisors, it follows that $\mathbf{U}=\{\mathrm{s}\neq 0\}$.

Corollary 3.3. *For any adelic metrization $\mathcal{L}$ of a very ample line bundle L of a wonderful variety $\mathbf{X}$, there exist a polynomial $P_{\mathrm{H}_\mathcal{L}}$ of degree b_L-1 and $\delta>0$ such that*

$$\mu(\{x\in\mathbf{U}(\mathbb{A}):\,\mathrm{H}_\mathcal{L}(x)<t\})=t^{a_L}P_{\mathrm{H}_\mathcal{L}}(\log t)+O(t^{a_L-\delta})\quad \textit{as } t\to\infty.$$

Proof. Since $\mathbf{X}\setminus\mathbf{U}$ is a divisor whose irreducible components are given by $\mathbf{X}_i$, $1\le i\le l$, and $\Lambda_{\mathrm{eff}}(X)$ is generated by $\mathbf{X}_i$'s, we have $a(L)=a_L$ and $b(L)=b_L$ for $a(L)$ and $b(L)$ defined in (2.11). Hence the claim is a special case of Corollary 2.14. □

In the same way, the following is a special case of Theorem 2.16:

Corollary 3.4. *Assume that there are only finitely many $\mathbf{G}(\mathbb{A})$-orbits in $\mathbf{U}(\mathbb{A})$. Then for $\mathcal{L}$ as in the above corollary and for every $x\in\mathbf{U}(\mathbb{A})$, there exist a polynomial $P_{\mathrm{H}_\mathcal{L},x}$ of degree b_L-1 and $\delta>0$ such that*

$$\mu(\{y\in x\mathbf{G}(\mathbb{A}):\,\mathrm{H}_\mathcal{L}(y)<t\})=t^{a_L}P_{\mathrm{H}_\mathcal{L},x}(\log t)+O(t^{a_L-\delta})\quad \textit{as } t\to\infty.$$

3.4. **Examples.**

(1) (group varieties) Let $\iota:\mathbf{L}\to\mathrm{GL}(W)$ be an adjoint semisimple algebraic group defined over a number field K. Then $\iota(\mathbf{L})$ is a homogeneous variety of $\mathbf{G}=\mathbf{L}\times\mathbf{L}$ with the action

$$(l_1,l_2)\cdot x=\iota(l_1)^{-1}\cdot x\cdot\iota(l_2).$$

The stabilizer of identity is the symmetric subgroup corresponding to the involution $\sigma(l_1,l_2)=(l_2,l_1)$. Let $\mathbf{S}$ be a maximal torus of $\mathbf{L}$ with a root system Φ_L and set of simple roots $\Delta_\mathbf{L}$. Then $\mathbf{T}=\mathbf{S}\times\mathbf{S}$ is a maximal torus of $\mathbf{G}$ and

$$\Phi_1^+=\{(\alpha,-\beta):\,\alpha,\beta\in\Phi_\mathbf{L}^+\},$$
$$\Delta_\sigma=\{(\alpha,-\alpha):\,\alpha\in\Delta_\mathbf{L}\}.$$

Let λ_ι be the highest weight of the representation ι and ρ the sum of roots in $\Phi_\mathbf{L}^+$. Then the highest weight for the corresponding representation of $\mathbf{G}$ is $(\lambda_\iota,-\lambda_\iota)$, and the sum of positive roots of $\mathbf{G}$ is

$(2\rho,-2\rho)$. Writing

$$2\rho=\sum_{\alpha\in\Delta_{\mathbf{L}}} m_\alpha\alpha \quad\text{and}\quad \lambda_\iota=\sum_{\alpha\in\Delta_{\mathbf{L}}} n_\alpha\alpha,$$

we have

$$a=\max\left\{\frac{m_\alpha+1}{n_\alpha}:\alpha\in\Delta_\sigma\right\}\quad\text{and}\quad b=\#\left\{\alpha\in\Delta_{\mathbf{L}}/\Gamma_K:\, a=\frac{m_\alpha+1}{n_\alpha}\right\}.$$

This formulas agree with the ones obtained in [36].

(2) (space of symplectic forms) Consider the (projectivized) space $\mathbf{U}$ of symplectic forms of dimension $2n$. It can be identified with the symmetric variety $\mathbf{U}=\mathbf{L}\backslash\mathbf{G}$ where $\mathbf{G}=\mathrm{PGL}_{2n}$ and $\mathbf{L}=\mathrm{PSp}_{2n}$. Note that $\mathbf{L}$ is the set of fixed points of the involution

$$\sigma(g)=-J^tg^{-1}J$$

where $J=\sum_{i=1}^n E_{i,2n-i+1}-\sum_{i=1}^n E_{n+i,n-i+1}$. Consider the maximal torus in $\mathrm{Lie}(\mathbf{G})$ given by

$$\mathfrak{t}=\{\mathrm{diag}(u_1,\dots,u_n,v_n,\dots,v_1):\sum_{i=1}^n(u_i+v_i)=0\}.$$

Then

$$\sigma(s_1,\dots,s_d,t_d,\dots,t_1)=(-t_1,\dots,-t_d,-s_d,\dots,-s_1),$$

and we have the decomposition $\mathfrak{t}=\mathfrak{t}_0+\mathfrak{t}_1$ where

$$\mathfrak{t}_0=\{u_i=-v_i,\,1\le i\le n\}\quad\text{and}\quad \mathfrak{t}_1=\{u_i=v_i,\,1\le i\le n\}.$$

The root system Φ is given by

$$\Phi=\{\alpha_{ij}:=s_i-s_j,\,\beta_{ij}:=t_i-t_j,\,\gamma_{kl}:=s_k-t_l:\ \ 1\le i\ne j,k,l\le n\},$$

and $\Phi_0=\{\gamma_{kk}:\ \ 1\le k\le n\}$. If we choose the set of positive roots as

$$\Phi^+=\{\alpha_{ij},\beta_{ij},\gamma_{kl}:\ \ 1\le i<j,k,l\le n\},$$

then (3.1) holds, the set of simple roots is

$$\Delta=\{\alpha_{i,i+1},\beta_{i,i+1},\gamma_{n,n}:\ \ 1\le i\le n-1\},$$

and

$$\tilde\Delta=\{\alpha_i:=\alpha_{i,i+1}+\beta_{i,i+1}:\ \ 1\le i\le n-1\}\cup\{0\}.$$

The sum of positive roots is given by

$$2\rho=2\left(\sum_{i=1}^{n-1} i(2n-i)\alpha_i+n^2\gamma_{n,n}\right).$$

Since $\{\alpha_i\}$ forms a basis of $\mathfrak{t}_1^*$ and

$$\left(\sum_{\beta\in\Phi_1\cap\Phi^+}\beta\right)|_{\mathfrak{t}_1}=2\rho|_{\mathfrak{t}_1},$$

it follows that

$$\sum_{\beta\in\Phi_1\cap\Phi^+}\beta = 2\sum_{i=1}^{n-1} i(2n-i)\alpha_i.$$

Now take an irreducible spherical representation $\iota : \mathbf{G} \to \mathrm{GL}(V)$ with its regular highest weight given by

$$\lambda_\iota = \sum_{i=1}^{n-1} n_i\alpha_i$$

and $v_0 \in V(K)$ such that $\mathbf{L} = \mathrm{Stab}_{\mathbf{G}}([v_0])$. We then have an embedding

$$\iota : \mathbf{U} \to \mathbf{X} := \overline{[v_0]\mathbf{G}} :\ u \mapsto [v_0]u$$

of the space of symplectic forms in its wonderful compactification $\mathbf{X}$. The parameters a and b are computed as follows

$$a = \max_{1\le i\le n-1}\left\{\frac{2i(2n-i)+1}{n_i}\right\}; \quad \text{and}$$
$$b = \#\left\{i = 1,\dots,n-1 :\ a = \frac{2i(2n-i)+1}{n_i}\right\}.$$

4. Equidistributions of Adelic periods

Let $\mathbf{G} \subset \mathrm{GL}_N$ be a connected semisimple K-group. Let S be a finite subset of R which contains all archimedean valuations $v \in R$ such that $\mathbf{G}(K_v)$ is non-compact. This assumption is needed so that the diagonal embedding of $\mathbf{G}(\mathcal{O}_S)$ into $\mathbf{G}_S$ is a discrete subgroup of $\mathbf{G}_S$. Let $\Gamma \subset \mathbf{G}(\mathcal{O}_S)$ be a finite index subgroup; hence Γ is a lattice in $\mathbf{G}_S$.

Definition 4.1.
- *S is called isotropic for $\mathbf{G}$ if for any connected normal K-subgroup $\mathbf{N}$ of $\mathbf{G}$, $\mathbf{N}_S = \prod_{v\in S}\mathbf{N}(K_v)$ is non-compact.*
- *S is called strongly isotropic for $\mathbf{G}$ if S contains v such that every K_v-normal subgroup $\mathbf{N}$ of $\mathbf{G}$ is isotropic over K_v, i.e., $\mathbf{N}(K_v)$ is non-compact.*

Clearly a strongly isotropic subset for $\mathbf{G}$ is isotropic for $\mathbf{G}$.

For any connected semisimple K-subgroup $\mathbf{L}$ of $\mathbf{G}$, $\pi : \tilde{\mathbf{L}} \to \mathbf{L}$ denotes the simply connected covering, that is, $\tilde{\mathbf{L}}$ is a connected simply connected semisimple K-group and π is a K-isogeny. Note that π induces a map $\tilde{\mathbf{L}}(K_v) \to \mathbf{L}(K_v)$ for each $v \in R$, which is no more surjective in general.

Definition 4.2. $\mathbf{G}$ *satisfies strong approximation property with respect to S if the diagonal embedding of $\mathbf{G}(K)$ into $\mathbf{G}(\mathbb{A}_S)$ is dense.*

For the property of strong approximation theorems for algebraic groups, we refer to [59], for instance, see Proposition 7.2 and Theorem 7.12 in [59] for the following:

Theorem 4.3.
- *If S is isotropic for* $\mathbf{G}$*, then* $\tilde{\mathbf{G}}$ *satisfies the strong approximation property with respect to* S*.*
- *If $v \in S$ is isotropic for* $\mathbf{G}$*, then* $\tilde{\mathbf{G}}(\mathcal{O}_S)$ *is dense in* $\tilde{\mathbf{G}}_{S\setminus\{v\}}$*.*

Following Tomanov [67], we define the following:

Definition 4.4. *A connected K-subgroup* $\mathbf{P}$ *of* $\mathbf{G}$ *is in* class $\mathcal{F}$ *relative to S if the radical of* $\mathbf{P}$ *is unipotent and every K-simple factor of* $\mathbf{P}$ *is K_v-isotropic for some $v \in S$.*

The following is well-known, see [30, Lemma 5.1], for example.

Lemma 4.5. *Let* $\mathbf{L} \subset \mathbf{G}$ *be connected reductive algebraic K-subgroups with no non-trivial K-character. The following are equivalent:*

(1) *the centralizer of* $\mathbf{L}$ *is anisotropic over K.*
(2) $\mathbf{L}$ *is not contained in any proper K-parabolic subgroup of* $\mathbf{G}$*.*
(3) *any K-subgroup of* $\mathbf{G}$ *containing* $\mathbf{L}$ *is reductive.*

Set $X_S := \Gamma\backslash\mathbf{G}_S$, and let $\mathcal{P}(X_S)$ denote the space of all Borel probability measures of X_S. Let $\{\mathbf{L}_i\}$ be a sequence of connected semisimple K-subgroups of $\mathbf{G}$. We denote by $\nu_i \in \mathcal{P}(X_S)$ the unique invariant probability measure supported on $Y_{i,S} := \Gamma\backslash\Gamma\pi(\tilde{\mathbf{L}}_{i,S})$. For a given $g_i \in \mathbf{G}_S$, $g_i\nu_i$ denotes the translated measure: $(g_i\nu_i)(E) = \nu_i(Eg_i^{-1})$ for Borel subsets $E \subset X_S$

Theorem 4.6. *Let S be strongly isotropic for all* $\mathbf{L}_i$*.*

(1) *Suppose that the centralizer of each* $\mathbf{L}_i$ *is anisotropic over K. Then* $\{g_i\nu_i\}$ *is relatively compact in* $\mathcal{P}(X_S)$*.*
(2) *If $g_i\nu_i$ weakly converges to $\nu \in \mathcal{P}(X_S)$ as $i \to \infty$, then the followings hold:*
    (a) *There exists a connected K-subgroup* $\mathbf{M}$ *in class $\mathcal{F}$ (with respect to S) such that ν is the invariant measure supported on* $\Gamma\backslash\Gamma M g$ *for some closed subgroup M of* $\mathbf{M}_S$ *with finite index and for some* $g \in \mathbf{G}_S$*.*
    (b) *There exists a sequence $\{\gamma_i \in \Gamma\}$ such that for all sufficiently large i,*
    $$\gamma_i\mathbf{L}_i\gamma_i^{-1} \subset \mathbf{M}.$$
    (c) *There exists $\{h_i \in \pi(\tilde{\mathbf{L}}_{i,S})\}$ such that $\gamma_i h_i g_i$ converges to g as $i \to \infty$.*
    (d) *If the centralizers of* $\mathbf{L}_i$*'s are K-anisotropic,* $\mathbf{M}$ *is semisimple.*

Definition 4.7. *For a closed subgroup L of* $\mathbf{G}_S$*, the Mumford-Tate subgroup of L, denoted by* $\mathrm{MT}(L)$*, is defined to be the smallest connected K-subgroup of* $\mathbf{G}$ *such that*

$$\overline{L}^0 \subset \prod_{v\in S} \mathrm{MT}(L)$$

where $\overline{L}^\circ$ *denotes the identity component of the Zariski closure of L in* $\mathbf{G}_S$*.*

In this terminology, $\mathbf{M}$ in the above theorem 4.6 is the Mumford-Tate subgroup of M.

Theorem 4.6 will be proved in sections 6 and 7 (see (6.2) and (7.2)). We will deduce Theorem 1.7 from Theorem 4.6 in the rest of this section.

Lemma 4.8. *If G_0 is a subgroup of $\mathbf{G}(\mathbb{A})$ and $\mathbf{G}(K)G_0$ contains $[\mathbf{G}(\mathbb{A}),\mathbf{G}(\mathbb{A})]$, then $\mathbf{G}(K)G_0$ is a normal subgroup of $\mathbf{G}(\mathbb{A})$.*

Proof. Let $\gamma_i \in \mathbf{G}(K)$ and $g_i \in G_0$. Using the notation $[g,h] = ghg^{-1}h^{-1}$,

$$\gamma_1 g_1 \gamma_2 g_2 = \gamma_1\gamma_2[\gamma_2^{-1}, g_1]g_1 g_2 \in \mathbf{G}(K)G_0;$$

$$(\gamma_1 g_1)^{-1} = \gamma_1^{-1}[\gamma_1, g_1^{-1}]g_1^{-1} \in \mathbf{G}(K)G_0;$$

and for any $g \in \mathbf{G}(\mathbb{A})$,

$$g(\gamma_1 g_1^{-1})g^{-1} = \gamma_1[\gamma_1^{-1}, g^{-1}][g^{-1}, g_1]g_1 \in \mathbf{G}(K)[\mathbf{G}(\mathbb{A}),\mathbf{G}(\mathbb{A})]G_0 = \mathbf{G}(K)G_0.$$

This proves the claim. □

Proposition 4.9. *Let S be isotropic for $\mathbf{G}$. For any compact open subgroup W_S of $\mathbf{G}(\mathbb{A}_S)$, the product $G_{W_S} := \mathbf{G}(K)\pi(\tilde{\mathbf{G}}_S)W_S$ is a co-abelian (normal) subgroup of finite index of $\mathbf{G}(\mathbb{A})$, which contains $\pi(\tilde{\mathbf{G}}(\mathbb{A}))$.*

Proof. Consider the exact sequence $1 \to F \to \tilde{\mathbf{G}} \to \mathbf{G} \to 1$. This induces the exact sequence

$$\tilde{\mathbf{G}}(\mathbb{A}) \to_\pi \mathbf{G}(\mathbb{A}) \to \prod_v H^1(K_v, F)$$

(see the proof of Proposition 8.2 in [59]). Since $\prod_v H^1(K_v,F)$ is abelian, it follows that $[\mathbf{G}(\mathbb{A}),\mathbf{G}(\mathbb{A})] \subset \pi(\tilde{\mathbf{G}}(\mathbb{A}))$.

Since $\tilde{\mathbf{G}}$ has the strong approximation property with respect to S,

$$\tilde{\mathbf{G}}(\mathbb{A}_S) = \tilde{\mathbf{G}}(K)\pi^{-1}(W_S).$$

Therefore we have

$$\begin{aligned}[\mathbf{G}(\mathbb{A}),\mathbf{G}(\mathbb{A})] &\subset \pi(\tilde{\mathbf{G}}(\mathbb{A})) = \pi(\tilde{\mathbf{G}}(\mathbb{A}_S))\pi(\tilde{\mathbf{G}}_S) \\ &\subset \mathbf{G}(K)\pi(\tilde{\mathbf{G}}_S)W_S.\end{aligned}$$

Hence the claim follows from the above lemma. □

Corollary 4.10. (1) *If S is strongly isotropic for $\mathbf{G}$ and G_0 is a subgroup of finite index in $\mathbf{G}_S$ and W_S is an open compact subgroup of $\mathbf{G}(\mathbb{A}_S)$, then $\mathbf{G}(K)G_0W_S$ is a normal subgroup of finite index of $\mathbf{G}(\mathbb{A})$.*

(2) *For any compact open subgroup W of $\mathbf{G}(\mathbb{A}_f)$, the product*

$$G_W := \{\gamma x w \in \mathbf{G}(\mathbb{A}) : \gamma \in \mathbf{G}(K),\ x \in \pi(\tilde{\mathbf{G}}(\mathbb{A})),\ w \in W\}$$

is a normal subgroup of finite index in $\mathbf{G}(\mathbb{A})$ which contains $\pi(\tilde{\mathbf{G}}(\mathbb{A}))$.

Proof. Let $v \in S$ be strongly isotropic. Then by [8, Coro. 6.7], $\pi(\tilde{\mathbf{G}}(K_v))$ coincides with the subgroup $\mathbf{G}(K_v)^+$ generated by all unipotent one parameter subgroups of $\mathbf{G}(K_v)$, and hence G_0 contains $\pi(\tilde{\mathbf{G}}(K_v))$. Choose any compact open subgroup W_0 of $\mathbf{G}_{S-\{v\}}$. Then

$$\mathbf{G}(K)G_0W_S \supset \mathbf{G}(K)\pi(\tilde{\mathbf{G}}(K_v))(W_0W_S)$$

which is a normal subgroup of $\mathbf{G}(\mathbb{A})$ with finite index and contains $[\mathbf{G}(\mathbb{A}), \mathbf{G}(\mathbb{A})]$ by the previous corollary. Therefore by Lemma 4.8, the first claim follows. For the second claim, let S be a strongly isotropic subset of $\mathbf{G}$. Let $W_S < W$ be a compact open subgroup of $\mathbf{G}(\mathbb{A}_S)$. Then G_{W_S} is a co-abelian normal subgroup of $\mathbf{G}(\mathbb{A})$ of finite index by Proposition 4.9. Since $G_W = G_{W_S}W$, the claim follows. $\square$

For an isotropic set S for $\mathbf{G}$, and a compact open subgroup W_S of $\mathbf{G}(\mathbb{A}_S)$, every element g of G_{W_S} can be written as

$$g = (\gamma_g, \gamma_g)(g_S, w)$$

where $\gamma_g \in \mathbf{G}(K)$ and $g_S \in \pi(\tilde{\mathbf{G}}_S)$ and $w \in W_S$ (here we are using the identification $\mathbf{G}(\mathbb{A}) = \mathbf{G}_S \times \mathbf{G}(\mathbb{A}_S)$). The choice of $g_S \in \mathbf{G}(K)$ is unique up to the left multiplication by the elements of the group

$$\Gamma := \{\gamma \in \mathbf{G}(K) : \gamma \in W_S,\ \ \gamma \in \pi(\tilde{\mathbf{G}}_S)\} = (\mathbf{G}(K) \cap W_S) \cap \pi(\tilde{\mathbf{G}}_S).$$

Lemma 4.11. *Let $\mathbf{L}$ be a connected semisimple K-subgroup of $\mathbf{G}$ and assume that S is isotropic both for $\mathbf{L}$ and $\mathbf{G}$. Let $g \in G_{W_S}$.*

(1) *The map $g \mapsto g_S$ induces a $\pi(\tilde{\mathbf{G}}_S)$-equivariant homeomorphism, say Φ, between $\mathbf{G}(K)\backslash G_{W_S}/W_S$ and $\Gamma\backslash\pi(\tilde{\mathbf{G}}_S)$ where $\Gamma = \mathbf{G}(K) \cap \pi(\tilde{\mathbf{G}}_S) \cap W_S$.*
(2) *The map Φ maps $\mathbf{G}(K)\backslash\mathbf{G}(K)\pi(\tilde{\mathbf{L}}(\mathbb{A}))gW_S/W_S$ onto $\Gamma\backslash\Gamma(\gamma_g^{-1}\pi(\tilde{\mathbf{L}}_S)\gamma_g)g_S$, inducing a measurable isomorphism between them.*
(3) *If μ is the invariant probability measure supported on $\mathbf{G}(K)\backslash\mathbf{G}(K)\pi(\tilde{\mathbf{L}}(\mathbb{A}))$ (considered as a measure on $\mathbf{G}(K)\backslash G_{W_S}$), then the measure $g.\mu$, considered as a functional on $C_c(\mathbf{G}(K)\backslash G_{W_S})^{W_S}$, is mapped by Φ to the invariant probability measure supported on $\Gamma\backslash\Gamma(\gamma_g^{-1}\pi(\tilde{\mathbf{L}}_S)\gamma_g)g_S$, which will be denoted by $\Phi_*(g.\mu)$.*

Proof. It is easy to check that the map $g \mapsto (g_S, w)$ induces a $\pi(\tilde{\mathbf{G}}_S)$-equivariant homeomorphism between $\mathbf{G}(K)\backslash G_{W_S}$ and $\Gamma\backslash(\pi(\tilde{\mathbf{G}}_S)\times W_S)$. The first claim then follows. For (2), let $h \in \pi(\tilde{\mathbf{L}}(\mathbb{A}))$. Since S is isotropic for $\mathbf{L}$, we have

$$\pi(\tilde{\mathbf{L}}(\mathbb{A})) \subset \mathbf{L}(K)\pi(\tilde{\mathbf{L}}_S)(gW_Sg^{-1} \cap \pi(\tilde{\mathbf{L}}(\mathbb{A}_S))).$$

Without loss of generality, we may assume that the W_S-component of g is e, i.e., $w = e$. We can write $h = (\delta, \delta)(h_S, gw'g^{-1})$ where $\delta \in \mathbf{L}(K)$, $h_S \in \pi(\tilde{\mathbf{L}}_S)$ and $w' \in W_S \cap g^{-1}\pi(\tilde{\mathbf{L}}(\mathbb{A}_S))g$. Note that $gw'g^{-1} = \gamma_g w'\gamma_g^{-1}$. So $hg = (\delta\gamma_g, \delta\gamma_g)(\gamma_g^{-1}h_S\gamma_g g_S, w')$ and hence

$$\Phi[hg] = \Gamma\backslash\Gamma\gamma_g^{-1}h_S\gamma_g g_S.$$

This also explains the measurable isomorphism between

$$\mathbf{G}(K)\backslash\mathbf{G}(K)\pi(\tilde{\mathbf{L}}(\mathbb{A}))gW_S/W_S \quad \text{and}$$

$$\Gamma\backslash\Gamma(\gamma_g^{-1}\pi(\tilde{\mathbf{L}}_S)\gamma_g)g_S,$$

and proves the third claim. $\square$

Proof of Theorem 1.7, assuming Theorem 4.6 Let $\{\mathbf{L}_i\}$, g_i and μ_i be as in the introduction. Let S be any isotropic subset for $\mathbf{G}$ which intersects with $\cap_i \mathcal{I}_{\mathbf{L}_i}$ non-trivially and contains all archimedean valuations v such that $\mathbf{G}(K_v)$ is non-compact. Fixing some compact open subgroup W_S of $\mathbf{G}(\mathbb{A}_S)$, let Φ be the map defined in Lemma 4.11 for this choice of S and W_S. If the first claim in Theorem 1.7 does not hold, then $\Phi_*(g_i\mu_i)$ is not relatively compact in $\Gamma\backslash\Gamma\pi(\tilde{\mathbf{G}}_S)$, which contradicts Theorem 4.6.

To prove the second claim, let W_S be a compact open subgroup of $\mathbf{G}(\mathbb{A}_S)$, and set $X_{W_S} = \mathbf{G}(K)\backslash G_{W_S}$. Letting Φ be the map defined in Lemma 4.11 for this choice of S and W_S, we have $\Phi_*(g_i\mu_i)$ weakly converges to $\Phi_*(\mu)$ in the space of Borel measures of $X_S := \Gamma\backslash\mathbf{G}_S$ and $\Gamma := \mathbf{G}(K) \cap W_S \cap \pi(\tilde{G}_S)$.

Write

$$g_i = (\gamma_{g_i}, \gamma_{g_i})(g_{i,S}, w_i)$$

where $\gamma_{g_i} \in \Gamma$, $g_{i,S} \in \pi(\tilde{\mathbf{G}}_S)$ and $w_i \in W_S$. Then the measure $\Phi_*(g_i\mu_i)$ is precisely same as $g_{i,S}\nu_i$ where ν_i is the invariant probability measure supported on $\Gamma\backslash\Gamma(\gamma_{g_i}^{-1}\pi(\tilde{\mathbf{L}}_{i,S})\gamma_{g_i})$.

Applying Theorem 4.6 to $\mathbf{G}_S$ and the groups $\gamma_{g_i}^{-1}\mathbf{L}_i\gamma_{g_i}$ and $g_{i,S}$, we obtain a connected K-group $\mathbf{M} \in \mathcal{F}$ (with respect to S), $g \in \pi(\tilde{\mathbf{G}}_S)$, $\gamma_i \in \Gamma$ and $h_i \in \gamma_{g_i}^{-1}\pi(\tilde{\mathbf{L}}_i)\gamma_{g_i}$, which depend on a priori S and W_S, such that

$$\gamma_i\gamma_{g_i}^{-1}\mathbf{L}_i\gamma_{g_i}\gamma_i^{-1} \subset \mathbf{M}$$

and $\gamma_i h_i g_{i,S} \to g$, and that for some finite index subgroup M of $\mathbf{M}_S$, $\Phi_*(\mu)$ is the invariant probability measure supported on $\Gamma\backslash\Gamma M g$.

We now claim that $\mathbf{M}$ can be taken simultaneously for any S as above and all W_S. We denote (M, g) by $i(S, W_S)$. Let W_S and $W_{S'}$ be open compact subgroups of $\mathbf{G}(\mathbb{A}_S)$ and $\mathbf{G}(\mathbb{A}_{S'})$ respectively. Without loss of generality, we may assume that $S \subset S'$ and $W_{S'} \subset W_S$. Let $W_0 < W_S$ be a compact open subgroup of $\pi(\tilde{\mathbf{G}}_{S'-S})$ and set $W' := W_{S'}W_0$. Then $G_{W_{S'}} = G_{W'}$. We set $\Gamma := \mathbf{G}(K) \cap W_S \cap \pi(\tilde{\mathbf{G}}_S)$ and $\Gamma' := \mathbf{G}(K) \cap W' \cap \pi(\tilde{\mathbf{G}}_S)$. If Φ denotes the bijection of $\mathbf{G}(K)\backslash G_{W_S}/W_S$ and $\Gamma\backslash\pi(\tilde{\mathbf{G}}_S)$ and Φ' similarly for W'_S, then $\Phi_*(\mu)$ and $\Phi'_*(\mu)$ are invariant measures supported on $\Gamma\backslash\Gamma M g$ and $\Gamma\backslash\Gamma M' g'$ for some $g, g' \in \pi(\tilde{\mathbf{G}}_S)$. Here M and M' are finite index subgroups of $\mathbf{M}_S$ and $\mathbf{M}'_S$ respectively where $\mathbf{M}$ and $\mathbf{M}'$ denote the Mumford-Tate subgroups of M and M' respectively. Since both $\Phi'_*(\mu)$ and $\Phi_*(\mu)$ are the limits of images of $g_i\mu_i$, it is clear that the canonical projection from $\Gamma'\backslash\pi(\tilde{\mathbf{G}}_S) \to \Gamma\backslash\pi(\tilde{\mathbf{G}}_S)$ maps $\Phi'_*(\mu)$ to $\Phi_*(\mu)$. Therefore $g' = \gamma m g$ for some $\gamma \in \Gamma$ and $m \in M$, and $\Phi_*(\mu)$ is invariant under $g^{-1}Mg$ which implies that $g^{-1}\mathbf{M}g = g'^{-1}\mathbf{M}'g'$ (see Lemma 6.7 below). Hence $m^{-1}\gamma^{-1}\mathbf{M}'\gamma m = \mathbf{M}$ or equivalently $\gamma^{-1}\mathbf{M}'\gamma =$

$\mathbf{M}$. Therefore by replacing M' by $\gamma^{-1}M'\gamma$, $\Phi'_*(\mu)$ is the invariant probability measure supported on $\Gamma'\backslash\Gamma'M'g'$ where the Mumford-Tate subgroup of M' is $\mathbf{M}$. Hence $\mathbf{M} = \mathbf{M}'$.

Therefore for a fixed $\mathbf{M}$, we have associated to every S and W_S a finite index subgroup M of $\mathbf{M}_S$ and $g \in \mathbf{G}_S$ such that $i(S, W_S) = (M, g)$, proving claim.

Now fix one S which is also strongly isotropic for $\mathbf{M}$, and set $i(S, W_S) = (M, g)$. Then $M_0 := \mathbf{M}(K)M(\mathbf{M}(\mathbb{A}_S) \cap W_S)$ is a finite index normal subgroup of $\mathbf{M}(\mathbb{A})$ by Corollary 4.10. Let dm and dw denote the Haar measures on M and $\mathbf{M}(\mathbb{A}_S) \cap W_S$ respectively such that dm and $d(m \otimes w)$ induce probability measures on $\Gamma\backslash\Gamma M$ and $\Gamma\backslash\Gamma(M \times (\mathbf{M}(\mathbb{A}_S) \cap W_S))$ respectively.

For $f \in C_c(X_{W_S})^{W_S}$,

$$\begin{aligned}\mu(f) = \Phi_*(\mu)(\Phi(f)) &= \int_{\Gamma\backslash\Gamma M} \Phi(f)(mg)\ dm \\ &= \int_{\Gamma\backslash\Gamma(M\times(\mathbf{M}(\mathbb{A}_S)\cap W_S))} \Phi(f)(mg)\ dm\, dw \\ &= \int_{\mathbf{G}(K)\backslash\mathbf{G}(K)M_0} f(m_0 g)\ dm_0\end{aligned}$$

where dm_0 is the invariant probability measure on $\mathbf{G}(K)\backslash\mathbf{G}(K)M_0$. Since $C_c(X, W_f)$ and $C_c(X_{W_S})^{W_S}$ can be canonically identified, this finishes the proof.

If the sequence $g_i\mu_i$ weakly converges to $\mu \in \mathcal{P}(X)$, we say that the orbits $Y_i g_i$ become equidistributed in X with respect to the measure μ.

Corollary 4.12. *Let $\mathbf{G}$ be simply connected and $\{\mathbf{L}_i\}$ be a sequence of semisimple simply connected maximal connected K-subgroups of $\mathbf{G}$ and $\cap_i \mathfrak{I}_{\mathbf{L}_i} \neq \emptyset$. Then for any sequence $g_i \in \mathbf{G}(\mathbb{A})$, either of the following holds:*

1. *the sequence $x_0\mathbf{L}_i(\mathbb{A})g_i$ is equidistributed in $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$ with respect to the invariant measure as $i \to \infty$,*
2. *there exist $i_0 \in \mathbb{N}$, $\{\delta_i \in \mathbf{G}(K)\}$ and $g \in \mathbf{G}(\mathbb{A})$ such that for infinitely many i,*

$$\delta_i^{-1}\mathbf{L}_i\delta_i = \mathbf{L}_{i_0}; \qquad \text{and hence } x_0\mathbf{L}_i(\mathbb{A})g_i = x_0\mathbf{L}_{i_0}(\mathbb{A})\delta_i g_i$$

and $l_i\delta_i g_i$ converges to g for some $l_i \in \mathbf{L}_{i_0}(\mathbb{A})$.

Proof. Since $[\mathrm{N}_{\mathbf{G}}(\mathbf{L}_i) : \mathbf{L}_i] < \infty$ and $\mathbf{L}_i$ are semisimple, their centralizers are K-anisotropic. Hence by Theorem 1.7, $\{g_i\mu_i\}$ are weakly compact in the space of probability measures on $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$. Let μ be a weak-limit and let $\mathbf{M}$ be as in Theorem 1.7. If $\mathbf{M} \neq \mathbf{G}$, by passing to a subsequence, we have $\mathbf{L}_i$'s are conjugate with each other by elements of $\mathbf{G}(K)$. Hence we may assume $\mathbf{L}_i = \delta_i^{-1}\mathbf{L}_{i_0}\delta_i$ for some $\delta_i \in \mathbf{G}(K)$ and $\delta_i h_i g_i \to g$ for some $h_i = \delta_i^{-1}l_i\delta_i$ with $l_i \in \mathbf{L}_{i_0}(\mathbb{A})$. Hence (2) happens.

Now suppose for every weak-limit μ, we have $\mathbf{M} = \mathbf{G}$. Fix a finite subset $S \subset R$ such that $R_\infty \subset S$ and $S \cap (\cap_i \mathfrak{I}_{\mathbf{L}_i}) \cap \mathfrak{I}_{\mathbf{G}} \neq \emptyset$. Since $\mathbf{G}$ is simply connected, $G_{W_S} = \mathbf{G}(\mathbb{A})$ for any compact open subgroup W_S of $\mathbf{G}(\mathbb{A}_S)$, and the restriction of μ to $C_c(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))^{W_S}$ is the Tamagawa measure, since $M_0 = \mathbf{G}(\mathbb{A})$ for any W_S.

Since $\cup_{W_S} C_c(X)^{W_S}$ is dense in $C_c(X)$, this implies μ is an invariant measure. Therefore $g_i\mu_i$ converges to the invariant measure and yields the equidistribution (1).

Proof of Corollary 1.8: If we set $\mathbf{G}_0 := \mathbf{G} \times \mathbf{G}$ and $\Delta(\mathbf{G})$ denotes the diagonal embedding of $\mathbf{G}$ into $\mathbf{G}_0$, it can be easily seen that the above Adelic mixing is equivalent to the equidistribution of the translates $x_0\Delta(\mathbf{G})(\mathbb{A})(e, g_i)$ in the space $\mathbf{G}_0(K)\backslash\mathbf{G}_0(\mathbb{A})$ for any $g_i \to \infty$; for the function $f := f_1 \otimes f_2$, $f_i \in C_c(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$,

$$\int_{x_0\Delta(\mathbf{G})(\mathbb{A})} f(x, xg_i)dy = \int_X f_1(x)f_2(xg_i)dx.$$

Since $\mathbf{G}$ is almost K-simple, $\Delta(\mathbf{G})$ is a maximal connected K-subgroup of $\mathbf{G}_0$, we may apply Corollary 1.8. If the second case happens, we have δ_i belongs to the normalizer of $\Delta(\mathbf{G})$. Since $\Delta(\mathbf{G})$ has finite index in its normalizer, by passing to a subsequence, we may assume $\delta_i = e$. Now since $g_i \to \infty$, we cannot have $l_i \in \Delta(\mathbf{G})(\mathbb{A})$ such that l_ig_i is convergent. Therefore (2) of Corollary 1.8 cannot happen, and consequently the claim is proved. □

Remark 4.13. In the above and the next corollary, the assumption on the maximality of $\mathbf{L}$ appears more than what we need, which is that $\mathbf{L}$ is maximal as a semisimple K-group and $[\mathrm{N}_{\mathbf{G}}(\mathbf{L}_i) : \mathbf{L}_i] < \infty$. However for $\mathbf{L}$ semisimple, $[\mathrm{N}_{\mathbf{G}}(\mathbf{L}) : \mathbf{L}] < \infty$ is same as the centralizer of $\mathbf{L}$ being finite, and any connected K-group containing a semisimple group with a finite centralizer is automatically semisimple (cf. [28]).

We now prove an analogue of Corollary 4.12 when $\mathbf{G}$ and $\mathbf{L}$ are not necessarily simply connected.

Corollary 4.14. *Let $\mathbf{L}$ be a semisimple maximal connected K-subgroup of $\mathbf{G}$. Let W be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$, and let $g_i \in G_W$ be a sequence going to infinity modulo $\mathbf{L}(\mathbb{A})$. Let ν be the invariant probability measure supported on $\mathbf{L}(K)\backslash(\mathbf{L}(\mathbb{A})\cap G_W)$ considered as a measure on $X_W := \mathbf{G}(K)\backslash G_W$. Then for any $f \in C_c(X_W)^W$,*

$$\lim_{i\to\infty} \int_{x\in X_W} f(xg_i)\, d\nu(x) = \int_{X_W} f d\mu$$

where μ is the probability Haar measure on X_W.

Proof. Let S be a strongly isotropic subset for $\mathbf{L}$. Since G_W contains $\mathbf{L}(K)(\pi(\tilde{\mathbf{G}}_S)\cap \mathbf{L}_S)(W_S \cap \mathbf{L}(\mathbb{A}_S))$, by Corollary 4.10, G_W contains $\pi(\tilde{\mathbf{L}}(\mathbb{A}))$. Also, by the same corollary, for each $g_i \in G_W$, $\mathbf{L}(K)\pi(\tilde{\mathbf{L}}(\mathbb{A}))(g_iWg_i^{-1} \cap \mathbf{L}(\mathbb{A}_f))$ is a normal subgroup of $\mathbf{L}(\mathbb{A}) \cap G_W$ with finite index. Hence there exists a finite subset $\Delta_{g_i} \subset \mathbf{L}(\mathbb{A}) \cap G_W$ such that

$$\mathbf{L}(\mathbb{A}) \cap G_W = \cup_{x \in \Delta_{g_i}} \mathbf{L}(K)\pi(\tilde{\mathbf{L}}(\mathbb{A}))x(g_i W g_i^{-1} \cap \mathbf{L}(\mathbb{A}_f))$$

where the union is a disjoint union. Therefore for $f \in C_c(X_W)^W$, the integral $(g_i\nu)(f)$ is equal to a finite linear combination of integrals of f against invariant measures on $x_0\pi(\tilde{\mathbf{L}}(\mathbb{A}))xg_i$, $x \in \Delta_{g_i}$.

Hence it suffices to show the following: for any $x_i \in \Delta_{g_i}$, and $f \in C_c(X_W)^W$,

$$\int_{x_0\pi(\tilde{\mathbf{L}}(\mathbb{A}))x_i g_i} f\, d\mu_i \to \int f\, d\mu$$

where μ_i is the invariant probability measure supported on $x_0\pi(\tilde{\mathbf{L}}(\mathbb{A}))x_i g_i$.

We apply Theorem 1.7 for any weak-limit ν of μ_i. By (1), we have $\nu \in \mathcal{P}(X_W)$. We claim $\mathbf{M} = \mathbf{G}$. Suppose not. Since $\mathbf{L}$ is maximal, we have $\delta_i \in \mathbf{G}(K)$ and $h_i \in \pi(\tilde{\mathbf{L}}(\mathbb{A}))$, $g \in G_W$ such that $\delta_i \mathbf{L} \delta_i^{-1} = \delta_j \mathbf{L} \delta_j^{-1}$ for all large i and $\delta_i h_i x_i g_i \to g$. Since $\mathbf{L}$ has a finite index in the normalizer of $\mathbf{L}$, by passing to a subsequence, there exist $\delta_0 \in \mathbf{G}(K)$, and $\delta_i \in \delta_0\mathbf{L}(\mathbb{A})$ such that $(\delta_0^{-1}\delta_i)h_i x_i g_i \to \delta_0^{-1} g$. Since $\delta_0^{-1}\delta_i h_i \in \mathbf{L}(\mathbb{A})$ and $x_i g_i \to \infty$ modulo $\mathbf{L}(\mathbb{A})$, this is a contradiction. Hence by Theorem 1.7, ν is an invariant measure supported on $x_0 M_0 g$ where M_0 contains $\mathbf{G}(K)\pi(\tilde{\mathbf{G}}(\mathbb{A}))W$. Since $G_W = \mathbf{G}(K)\pi(\tilde{\mathbf{G}}(\mathbb{A}))W$, we conclude that $\nu = \mu$, proving the claim. $\square$

5. Counting rational points of bounded height

The basic strategy is due to Duke, Rudnick, Sarnak [25], and to Eskin-McMullen [29], which can be summarized as follows. Let $L \subset G$ be unimodular locally compact groups and $Z := L\backslash G$. Let μ_G, μ_L and μ be invariant measures on G, L and Z respectively which are compatible with each other, that is, if for any $f \in C_c(G)$,

$$\int f d\mu_G = \int_{L\backslash G}\int_L f(hg) d\mu_L(h) d\mu(Lg).$$

Definition 5.1. *For a fixed compact subgroup W of G, a family $\{B_T \subset Z\}$ of compact subsets is called W-well-rounded if $B_T W = B_T$ for all large T and for every small $\epsilon > 0$, there exists a neighborhood U_ϵ of e in G such that for all sufficiently large T,*

$$(1-\epsilon)\mu(B_T U_\epsilon W) \le \mu(B_T) \le (1+\epsilon)\mu(\cap_{u \in U_\epsilon W} B_T u). \tag{5.2}$$

Note that this is a slight variant of the notion of well-roundedness introduced in [29].

Proposition 5.3. *Let $\Gamma \subset G$ be a lattice such that $\Gamma \cap L$ is a lattice in L. Let $W \subset G$ be a compact subgroup. Suppose that for $Y := [e]L \subset \Gamma\backslash G$, the* *translates Yg become equidistributed in $\Gamma\backslash G$ as $g \to \infty$ in Z with respect to $C_c(\Gamma\backslash G)^W$, that is, for any $f \in C_c(\Gamma\backslash G)^W$,*

$$\int_Y f(yg)d\mu_L(y) \to \int_{\Gamma\backslash G} f\, d\mu.$$

Then for any W-well-rounded sequence $\{B_T \subset Z\}$ of compact subsets whose volume going to infinity, we have

$$\#z_0\Gamma \cap B_T \sim \frac{\mu_L(L \cap \Gamma\backslash L)}{\mu_G(\Gamma\backslash G)}\mu(B_T).$$

Proof. Without loss of generality, we may assume that $\mu_L(L \cap \Gamma\backslash L) = 1 = \mu_G(\Gamma\backslash G)$. Let U_ϵ be as in the definition 5.2. We may assume that U_ϵ is symmetric and $U_\epsilon \cap \Gamma = \{e\}$. If we define a function on $\Gamma\backslash G$ by

$$F_{B_T}(g) := \sum_{\gamma \in \Gamma\cap L\backslash\Gamma} \chi_{B_T}(z_0\gamma g)$$

where χ_{B_T} is the indicator function of B_T, then $F_{B_T}(e) = \#(z_0\Gamma \cap B_T)$. Let ψ_ϵ be a non-negative W-invariant continuous function on $\Gamma\backslash G$ with support in $\Gamma\backslash\Gamma U_\epsilon W$ and with integral one. Set $F_T^+ = F_{B_T U_\epsilon W}$ and $F_T^- = F_{\cap_{u\in U_\epsilon W} B_T u}$. Observe that for any $g \in U_\epsilon W$,

$$F_T^-(g) \le F_{B_T}(e) \le F_T^+(g),$$

and hence

$$\langle F_T^-, \psi_\epsilon\rangle \le F_{B_T}(e) \le \langle F_T^+, \psi_\epsilon\rangle$$

where the inner product is taken place in $L^2(\Gamma\backslash G)$. One can easily see that

$$\langle F_T^+, \psi_\epsilon\rangle = \int_{g\in B_T U_\epsilon W} \int_{y\in Y} \psi_\epsilon(yg)d\mu_L(y)d\mu(g)$$

By the assumption,

$$\int_{y\in Y} \psi_\epsilon(yg)dy \to 1$$

as $g \to \infty$ on Z and hence if the volume of B_T goes to infinity as $T \to \infty$, we have

$$\langle F_T^+, \psi_\epsilon\rangle \sim \mathrm{vol}(B_T U_\epsilon W).$$

Similarly, we have

$$\langle F_T^-, \psi_\epsilon\rangle \sim \mathrm{vol}(\cap_{u\in U_\epsilon W} B_T u).$$

Using the W-well-roundedness assumption on B_T, it is easy deduce that $F_{B_T}(e) \sim \mu(B_T)$ (see [10] for details).

□

Let $\mathbf{G}$ be a connected semisimple algebraic group defined over K, with a given K-representation $\iota : \mathbf{G} \to \mathrm{GL}_{d+1}$. Let $\mathbf{U} := u_0\mathbf{G} \subset \mathbb{P}^d$ for $u_0 \in \mathbb{P}^d(K)$, and fix a height function $\mathrm{H}_{\mathcal{O}(1)}$ on $\mathbb{P}^d(K)$ as in the introduction. That is, $\mathrm{H}_{\mathcal{O}(1)} = \prod_{v\in R} \mathrm{H}_v$ where H_v is a norm on K_v^{d+1} and is a max norm for almost all v.

We set

$$N_T(\mathbf{U}) := \#\{x \in \mathbf{U}(K) : \mathrm{H}_{\mathcal{O}(1)}(x) < T\}.$$

We assume that

(i) $\mathbf{L} = \mathrm{Stab}_{\mathbf{G}}(u_0)$ is a semisimple maximal proper connected K-subgroup of $\mathbf{G}$.

(ii) There are only finitely many $\mathbf{G}(\mathbb{A})$-orbits on $\mathbf{U}(\mathbb{A})$.

We note that (ii) is equivalent to saying that for almost all $v \in R$, $\mathbf{G}(K_v)$ acts transitively on $\mathbf{U}(K_v)$ (see Thm. A.1.2). Denote by $\mathbf{X} \subset \mathbb{P}^d$ the Zariski closure of $\mathbf{U}$ and by L the line bundle which is the pull back of $\mathcal{O}_{\mathbb{P}^d}(1)$. We assume that there is a global section s of L such that $\mathbf{U} = \{\mathrm{s} \neq 0\}$. By Theorem 2.2, s is $\mathbf{G}$-invariant. Let $\mathrm{s}_0, \cdots, \mathrm{s}_d$ be the global sections of L which are the pull-backs of the coordinates x_i's. Using the height function $\mathrm{H}_{\mathcal{O}(1)} = \prod_{v \in R} \mathrm{H}_v$, we define the adelic height function $\mathrm{H}_{\mathcal{L}} = \prod_v \mathrm{H}_{\mathcal{L},v} : \mathbf{U}(\mathbb{A}) \to \mathbb{R}_{>0}$ where

$$\mathrm{H}_{\mathcal{L},v}(x) = \mathrm{H}_v\left(\frac{\mathrm{s}_0(x)}{\mathrm{s}(x)}, \cdots, \frac{\mathrm{s}_d(x)}{\mathrm{s}(x)}\right).$$

Set

$$B_T := \{x \in \mathbf{U}(\mathbb{A}) : \mathrm{H}_{\mathcal{L}}(x) < T\}.$$

The assumption (ii) implies that the set $\mathbf{U}(K)$ consists of finitely many $\mathbf{G}(K)$-orbits (Theorem A.1.2). Choose a set $u_1, \ldots, u_l \in \mathbf{U}(K)$ of representatives of these orbits, and denote by $\mathbf{L}_1, \ldots, \mathbf{L}_l$ their stabilizers in $\mathbf{G}$. Then

$$N_T(\mathbf{U}) = \sum_{i=1}^{l} \#(B_T \cap u_i\mathbf{G}(K)).$$

A naive heuristic

$$\#B_T \cap u_i\mathbf{G}(K) \sim_T \mathrm{vol}(B_T \cap u_i\mathbf{G}(\mathbb{A}))$$

does not hold in general unless $\mathbf{G}$ is simply connected. To correct this problem, we consider the following finite index subgroup of $\mathbf{G}(\mathbb{A})$:

Recall from Lemma 2.7 that the following is a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$:

$$W_{\mathrm{H}_{\mathcal{L}}} := \{g \in \mathbf{G}(\mathbb{A}_f) : \mathrm{H}_{\mathcal{L}}(ug) = \mathrm{H}_{\mathcal{L}}(u) \text{ for all } u \in \mathbf{U}(\mathbb{A})\}.$$

Recall from Corollary 4.10 that for any compact open subgroup W of $\mathbf{G}(\mathbb{A}_f)$,

$$G_W := \{\gamma x w \in \mathbf{G}(\mathbb{A}) : \gamma \in \mathbf{G}(K),\ x \in \pi(\tilde{\mathbf{G}}(\mathbb{A})),\ w \in W\}$$

is a normal subgroup of $\mathbf{G}(\mathbb{A})$ with finite index.

Let μ be the Tamagawa measure on $\mathbf{U}(\mathbb{A})$, and choose invariant measures μ_G and μ_{L_i} on the adelic spaces $\mathbf{G}(\mathbb{A})$ and $\mathbf{L}_i(\mathbb{A})$ respectively so that $\mu_G = \mu \times \mu_{L_i}$ locally.

The main theorem 1.1 in the introduction follows from the following:

Theorem 5.4. (1) *If the height function* $\mathrm{H}_{\mathfrak{L}}$ *is regular, then for any co-finite subgroup* W *of* $W_{\mathrm{H}_{\mathfrak{L}}}$

$$N_T(\mathbf{U}) \sim_T \sum_{i=1}^{l} \frac{\mu_{L_i}(\mathbf{L}_i(K))\backslash G_W \cap \mathbf{L}_i(\mathbb{A}))}{\mu_G(\mathbf{G}(K)\backslash G_W)} \mu(u_i G_W \cap B_T).$$

(2) *For* $a = a(L)$ *and* $b = b(L)$ *defined as in* (2.11),

$$N_T(\mathbf{U}) \asymp T^a (\log T)^{b-1}.$$

(3) *Suppose that* $\mathbf{G}$ *is simply connected, or that* $\mathbf{G}(\mathbb{A}) = G_{W_{\mathrm{H}_{\mathfrak{L}}}}$. *Then for some* $c > 0$,

$$N_T(\mathbf{U}) \sim c \cdot T^a (\log T)^{b-1}.$$

Proof. Fixing $1 \leq i \leq l$, we apply the above proposition to $G = G_W$, $L = \mathbf{L}_i(\mathbb{A}) \cap G_W$ and $Y = \mathbf{G}(K)\backslash \mathbf{G}(K) L \subset \mathbf{G}(K)\backslash G$.

By Corollary 4.14, the translates Yg become equidistributed in $\mathbf{G}(K)\backslash G_W$ with respect to $C_c(\mathbf{G}(K)\backslash G_W)^W$.

And by Theorem 2.21, the family $\{B_T \cap u_i G_W\}$ is W-well-rounded. Hence (1) follow from Proposition 5.3. (2) follows from (1) using Corollary 2.21. For (3), first note that $G_W = \mathbf{G}(\mathbb{A})$ for $\mathbf{G}$ simply connected. Theorem 2.16 implies $B_T \cap u_i \mathbf{G}(\mathbb{A})$ is W-well-rounded, and hence (1) holds under the hypothesis of (3), without assuming that $\mathrm{H}_{\mathfrak{L}}$ is regular. It remains to apply the asymptotic given Theorem 2.16 once more. □

Proof of Corollary 1.5 Since $\mathbf{X}$ is smooth, L^k is $\mathbf{G}$-linearized for some k (cf. [41]). Therefore, by replacing L by L^k if necessary, we are in the setup of Theorem 5.4. Since $a_L = a(L)$ and $b_L = b(L)$ (see the proof of Corollary 3.3), the claim follows from Theorem 5.4.

Proof of Corollary 1.9: Let $\|\cdot\|_p$ be denote the max norm on $\mathbb{Q}_p^N$ for each p. Fix any compact subset $\Omega \subset v_0\mathbf{G}(\mathbb{R})$ with boundary of measure zero and $\mathrm{vol}(\Omega) > 0$. If $m = \prod_{p:prime} p^{m_p}$ (of course, $m_p = 0$ for almost all p), set

$$B_m := \{(x_p) \in v_0\mathbf{G}(\mathbb{A}) : x_\infty \in \Omega, \|x_p\|_p = p^{m_p} \text{ for each } p\}.$$

That is, for $B'_m := v_0\mathbf{G}(\mathbb{A}_f) \cap \prod_p \mathbf{U}(m^{-1}\mathbb{Z}_p)$ $B_m := \Omega \times B'_m$. Since B'_m is invariant under the subgroup $\prod_p \mathbf{G}(\mathbb{Z}_p)$, the family $\{B_m\}$ is clearly well-rounded. Moreover since $\mathbf{G}$ is simply connected, $G_{W_S} = \mathbf{G}(\mathbb{A})$ for any strongly isotropic S for $\mathbf{G}$.

By the computation in [10],

$$\mu(B_m) := \mu_\infty(\Omega) \prod_p \mu_p(\mathbf{U}(m^{-1}\mathbb{Z}_p) \cap v_0\mathbf{G}(\mathbb{Q}_p)) \to \infty$$

if $m \to \infty$, subject to $B_m \neq \emptyset$.

Therefore by Proposition 5.3, we have, as $m \to \infty$, subject to $B_m \neq \emptyset$,

$$\# v_0\mathbf{G}(\mathbb{Q}) \cap B_m \sim \mu(B_m)$$

Observe that if $x \in \mathbf{U}(\mathbb{Q}) \cap B_m$, then $x \in \mathbf{U}(m^{-1}\mathbb{Z}) \cap \Omega$, and hence

$$\# v_0 \mathbf{G}(\mathbb{Q}) \cap B_m \leq \# \mathbf{U}(m^{-1}\mathbb{Z}) \cap \Omega.$$

Consequently, for all sufficiently large m, $B_m \neq \emptyset$ implies $\mathbf{U}(m^{-1}\mathbb{Z}) \neq \emptyset$.

In the case when $\mathbf{L}$ is simply connected, there is exactly one $\mathbf{G}(\mathbb{Q})$-orbit in each $\mathbf{G}(\mathbb{R})$-orbit and hence for $\Omega \subset v_0\mathbf{G}(\mathbb{R})$

$$\# v_0 \mathbf{G}(\mathbb{Q}) \cap B_m = \# \mathbf{U}(\mathbb{Q}) \cap B_m = \# \mathbf{U}(m^{-1}\mathbb{Z}) \cap \Omega.$$

Hence the above argument shows (2).

6. Limits of invariant measures for unipotent flows

6.1. **Statements of Main Theorem.** Let K be a number field, and $\mathbf{G}$ be a connected K-group with no non-trivial K-character. Let S be a finite set of (normalized) valuations of K including all the archimedean $v \in R$ such that $\mathbf{G}(K_v)$ is non-compact. For each valuation $v \in S$, we denote by $|\cdot|_v$ the normalized absolute value on the completion field K_v, and by θ_v the normalized Haar measure on K_v.

Let G be a finite index subgroup of

$$\mathbf{G}_S := \prod_{v \in S} \mathbf{G}(K_v),$$

and Γ be an S-arithmetic subgroup of G, that is, $\Gamma \subset \mathbf{G}(K)$ is commensurable with $\mathbf{G}(\mathcal{O}_S)$, where $\mathcal{O}_S$ denotes the ring of S-integers in K. Then Γ is a lattice in G by a theorem of Borel [6].

Recall the definition of *class* $\mathcal{F}$-subgroups of $\mathbf{G}$ from 4.4. Equivalently, a connected K-subgroup $\mathbf{P}$ of $\mathbf{G}$ is in class $\mathcal{F}$ relative to S if for each proper normal K-subgroup $\mathbf{Q}$ of $\mathbf{P}$ there exists $v \in S$ such that $(\mathbf{P}/\mathbf{Q})(K_v)$ contains a non-trivial unipotent element.

Note that for every subgroup L of finite index in $\mathbf{P}_S$ with $\mathbf{P} \in \mathcal{F}$, the orbit $\Gamma\backslash\Gamma L$ is closed and supports a finite L-invariant measure.

For a closed subgroup L of $\mathbf{G}_S$, we denote by L_u the closed subgroup of L generated by all unipotent one-parameter subgroups of L. We note that since G has a finite index in $\mathbf{G}_S$, every one-parameter unipotent subgroup of $\mathbf{G}_S$ is contained in G.

Definition 6.1. *We say that a closed subgroup L of G is in class $\mathcal{H}$ if there exists a connected K-subgroup $\mathbf{P}$ in* class $\mathcal{F}$ *relative to S such that L has a finite index in $\mathbf{P}_S$ and L_u acts ergodically on $\Gamma\backslash\Gamma L$ with respect to the L-invariant probability measure.*

Set $X = \Gamma\backslash G$. We denote by $\mathcal{P}(X)$ the space of probability measures on X equipped with the weak* topology. For $\mu \in \mathcal{P}(X)$ and $d \in G$, the translate $d\mu$ is defined by $d\mu(E) = \mu(Ed^{-1})$ for any Borel subset E of X. We also define the invariance subgroup for $\mu \in \mathcal{P}(X)$ by

$$\Lambda(\mu) = \{d \in G : d\mu = \mu\}.$$

For a unipotent one parameter subgroup $u : K_v \to \mathbf{G}(K_v)$, $x \in X$ and $\mu \in \mathcal{P}(X)$, the trajectory xU is said to be uniformly distributed relative to μ if for every $f \in C_c(X)$,

$$\lim_{T\to\infty} \frac{1}{\theta_v(I_T)} \int_{t\in I_T} f(xu(t))\, d\theta(t) = \int_X f(x)\, d\mu(x)$$

where $I_T = \{t \in K_v : |t|_v < T\}$.

We present a generalization of the theorem of Mozes and Shah in [52] in the S-arithmetic setting, which is the main result of this section:

Theorem 6.2. *Let $v \in S$ and $\{U_i\}$ be a sequence of one-parameter unipotent subgroups of $\mathbf{G}(K_v)$. Let $\{\mu_i : i \in \mathbb{N}\}$ be a sequence of U_i-invariant ergodic measures in $\mathcal{P}(X)$. Suppose that $\mu_i \to \mu$ in $\mathcal{P}(X)$ and let $x = \Gamma\backslash\Gamma g \in \mathrm{supp}(\mu)$. Then the following holds:*

(1) *There exists a closed subgroup $L \in \mathcal{H}$ such that μ is an invariant measure supported on $\Gamma\backslash\Gamma Lg$. In particular*
$$\mathrm{supp}(\mu) = x\Lambda(\mu).$$
(2) *Let $z_i \to e$ be a sequence in G such that $xz_i \in \mathrm{supp}(\mu_i)$ and the trajectory $\{xz_iU_i\}$ is uniformly distributed with respect to μ_i. Then there exists i_0 such that for all $i \geq i_0$,*
$$\mathrm{supp}(\mu_i) \subset \mathrm{supp}(\mu)z_i \quad \textit{and} \quad \Lambda(\mu_i) \subset z_i^{-1}\Lambda(\mu)z_i.$$
(3) *Denote by H the closed subgroup generated by the set $\{z_iU_i{z_i}^{-1} : i \geq i_0\}$. Then $H \subset g^{-1}Lg$ and μ is H-ergodic.*

We state some corollaries of the above theorem 6.2, as in [52]. Let $Q(X)$ denote the set $\mathcal{P}(X)$ of probability measures μ on X such that the group generated by all unipotent one-parameter subgroups of G contained in $\Lambda(\mu)$ acts ergodically on X with respect to μ. The following is an immediate consequence of the above theorem:

Corollary 6.3. (1) *$Q(X)$ is a closed subset of $\mathcal{P}(X)$.*
(2) *For $x \in X$, $Q(x) := \{\mu \in Q(X) : x \in supp(\mu)\}$ is a closed subset of $\mathcal{P}(X)$.*

Let $X \cup \{\infty\}$ denote the one-point compactification of X. As well known, $\mathcal{P}(X \cup \{\infty\})$ is compact with respect to the weak*-topology.

Combined with a theorem proved by Kleinbock and Tomanov (see Theorem 7.4), we can also deduce:

Corollary 6.4. (1) *Let $\{\mu_i\} \in Q(X)$ be a sequence of measures converging weakly to a measure $\mu \in \mathcal{P}(X \cup \{\infty\})$. Then either $\mu \in Q(X)$ or $\mu(\{\infty\}) = 1$.*
(2) *For $x \in X$, $Q(x)$ is compact with respect to the weak*-topology.*

6.2. **Deduction of Theorem 4.6 (2) from Theorem 6.2.** We will now deduce Theorem 4.6 (2) from Theorem 6.2.

Lemma 6.5. *Let* $\mathbf{L}$ *be a connected semisimple* K*-subgroup of* $\mathbf{G}$*. If* S *is strongly isotropic for* $\mathbf{L}$*, then there exists a one-parameter unipotent subgroup* $U = \{u(t)\}$ *of* $\tilde{\mathbf{L}}_S$ *which acts ergodically on* $\Gamma\backslash\Gamma\pi(\tilde{\mathbf{L}}_S)$*.*

Proof. Let $v \in S$ be strongly isotropic for $\mathbf{L}$. Denote by $\mathbf{L}(K_v)^+$ the subgroup generated by all unipotent one-parameter subgroups in $\mathbf{L}(K_v)$. Then by [8],

$$\pi(\tilde{\mathbf{L}}(K_v)) = \mathbf{L}(K_v)^+.$$

First, we show that $\mathbf{L}(K_v)^+$ acts ergodically on $\Gamma\backslash\Gamma\tilde{\mathbf{L}}_S$. Since $\tilde{\mathbf{L}}$ satisfies the strong approximation property with respect to $\{v\}$ and $\pi^{-1}(\Gamma) \cap \tilde{\mathbf{L}}_S$ is an S-arithmetic subgroup of $\tilde{\mathbf{L}}_S$, it follows that the diagonal embedding of $\pi^{-1}(\Gamma) \cap \tilde{\mathbf{L}}_S$ is dense in $\prod_{v\in S\setminus\{v\}} \tilde{\mathbf{L}}(K_v)$ by Theorem 4.3. This implies that $(\Gamma \cap \pi(\tilde{\mathbf{L}}_S))\mathbf{L}(K_v)^+$ is dense in $\pi(\tilde{\mathbf{L}}_S)$.

Since $\tilde{\mathbf{L}}(K_v)$ is a normal subgroup of $\tilde{\mathbf{L}}_S$, this implies that $\pi^{-1}(\Gamma) \cap \tilde{\mathbf{L}}_S$ acts ergodically on $\tilde{\mathbf{L}}(K_v)\backslash\tilde{\mathbf{L}}_S$. By the duality, this implies that $\tilde{\mathbf{L}}(K_v)$ acts ergodically on $\pi^{-1}(\Gamma) \cap \tilde{\mathbf{L}}_S\backslash\tilde{\mathbf{L}}_S$, and hence on $\Gamma\backslash\Gamma\tilde{\mathbf{L}}_S$.

Since every K_v-simple factor of $\mathbf{L}$ is K_v-isotropic, there exists a unipotent one-parameter subgroup U of $\tilde{\mathbf{L}}(K_v)$ such that $\tilde{\mathbf{L}}(K_v)$ is the smallest normal subgroup containing U. Now by the S-algebraic version of Mautner phenomenon (Proposition 6.21), any U-invariant function in $L^2(\Gamma\backslash\Gamma\tilde{\mathbf{L}}_S)$ is $\tilde{\mathbf{L}}(K_v)$-invariant, and consequently a constant function. Hence the ergodicity of the U-action $\Gamma\backslash\Gamma\tilde{\mathbf{L}}_S$ follows. $\square$

Proof of Theorem 4.6 (2): Fix $v \in S$ which is strongly isotropic for all $\mathbf{L}_i$. It follows from Lemma 6.5 that measures ν_i is ergodic with respect to one-parameter unipotent subgroups $U_i' := g_i^{-1}U_i g_i$, where $U_i = \{u_i(t)\} \subset \mathbf{L}_i(K_v)^+$ is as in Lemma 6.5. Hence, we may apply Theorem 6.2(1) to conclude that ν is an invariant measure supported on $\Gamma\backslash\Gamma Mg$ for some closed subgroup $M \in \mathcal{H}$ and $g \in \mathbf{G}_S$. In particular, M is a finite index subgroup in $\mathbf{M}_S$ where $\mathbf{M}$ is the Mumford-Tate subgroup of M which is in class $\mathcal{F}$ (see Def. 4.7). By a pointwise ergodicity theorem, there exists a sequence $z_i = g^{-1}\gamma_i h_i g_i \to e$ for some $\gamma_i \in \Gamma$ and $h_i \in \tilde{\mathbf{L}}_i(K_v)$ and the trajectory $\Gamma\backslash\Gamma g z_i U_i'$ is uniformly distributed with respect to $g_i\nu_i$. By Theorem 6.2(2), we have

$$g_i^{-1}\tilde{\mathbf{L}}_i g_i \subset z_i^{-1}(g^{-1}\mathbf{M}g)z_i$$

for all large i. This implies that

$$\gamma_i\tilde{\mathbf{L}}_i\gamma_i^{-1} \subset \mathbf{M}$$

as well as that $\gamma_i h_i g_i$ converges to g as i tends to infinity. Now for (d), suppose that the centralizers of $\mathbf{L}_i$ are K-anisotropic. Since $\mathbf{M}$ is reductive by Lemma 4.5 and it belongs to class $\mathcal{F}$ with respect to S, $\mathbf{M}$ is semisimple.

6.3. **Measures invariant under unipotent flows.** The crucial ingredient in our proof of Theorem 6.2 is a fundamental theorem of Ratner [60] on the classification of the measures in $\mathcal{P}(X)$ which are ergodic with respect to unipotent subgroups of G. In the S-arithmetic case, also see [49].

We will use the following more precise description due to Tomanov:

Theorem 6.6. [67, Theorem 2] *Let W be a subgroup of G generated by unipotent one-parameter subgroups.*

(1) *For any W-invariant ergodic probability measure μ on X, there exist a subgroup $L \in \mathcal{H}$ and $g \in G$ such that $W \subset g^{-1}Lg$ and μ is the invariant measure supported on $\Gamma\backslash\Gamma Lg$.*
(2) *For every $g \in G$, there exists a closed subgroup $L \in \mathcal{H}$ such that $W \subset g^{-1}Lg$ and*
$$\overline{\Gamma\backslash\Gamma gW} = \Gamma\backslash\Gamma Lg.$$

Although it is assumed in [67] that S contains all archimedean valuations of K and $G = \mathbf{G}_S$, these assumptions are not used in the proof.

Lemma 6.7. *For $P, Q \in \mathcal{H}$, we have $\mathrm{MT}(P) \subset \mathrm{MT}(Q)$ if and only if $\Gamma\backslash\Gamma P \subset \Gamma\backslash\Gamma Q$.*

Proof. Suppose that $\mathrm{MT}(P) \subset \mathrm{MT}(Q)$. Then $P \cap Q$ has finite index in P and hence $P_u \subset Q_u$. Since P_u is normal in P and it acts ergodically on $\Gamma\backslash\Gamma P$, it follows that $\overline{\Gamma\backslash\Gamma P_u} = \Gamma\backslash\Gamma P$. Hence,
$$\Gamma\backslash\Gamma P \subset \Gamma\backslash\Gamma Q.$$

Conversely, suppose that $\Gamma\backslash\Gamma P \subset \Gamma\backslash\Gamma Q$. Since P and Q have finite indices in $\mathrm{MT}(P)_S$ and $\mathrm{MT}(Q)_S$ respectively, it follows that the Lie algebra of $\mathrm{MT}(P)_S$ is contained in the Lie algebra of $\mathrm{MT}(Q)_S$. Hence, $\mathrm{MT}(Q)_S$ contains an open subgroup of $\mathrm{MT}(P)_S$. Since such groups are Zariski dense in $\mathrm{MT}(P)$, we deduce that $\mathrm{MT}(P) < \mathrm{MT}(Q)$. $\square$

Let W be a closed subgroup of G generated by one parameter unipotent subgroups in it. For each $L \in \mathcal{H}$, define
$$\begin{aligned} \mathcal{N}(L, W) &= \{g \in G : W \subset g^{-1}Lg\}, \\ \mathcal{S}(L, W) &= \cup_{M \in \mathcal{H}, \mathrm{MT}(M) \subsetneq \mathrm{MT}(L)} \mathcal{N}(M, W), \\ \mathcal{T}_L(W) &= \pi(\mathcal{N}(L, W) - \mathcal{S}(L, W)) \end{aligned}$$
where $\pi : G \to \Gamma\backslash G$ denotes the canonical projection.

Note that for $L \in \mathcal{H}$, L has finite index in $\mathrm{MT}(L)_S$ and hence L contains the closed subgroup of $\mathrm{MT}(L)_S$ generated by all unipotent elements of $\mathrm{MT}(L)_S$. Hence
$$\mathcal{N}(L, W) = \{g \in G : W \subset g^{-1} \cdot \mathrm{MT}(L)_S \cdot g\}.$$

Note also that for any $P, Q \in \mathcal{H}$ with $\mathrm{MT}(P) = \mathrm{MT}(Q)$,
$$\mathcal{N}(P, W) = \mathcal{N}(Q, W); \quad \mathcal{S}(P, W) = \mathcal{S}(Q, W) \quad \text{and hence} \quad \mathcal{T}_P(W) = \mathcal{T}_Q(W).$$

Lemma 6.8. *For any $g \in \mathcal{N}(L, W) \backslash \mathcal{S}(L, W)$, the closure of $\Gamma\backslash\Gamma gW$ is equal to $\Gamma\backslash\Gamma Lg$.*

Proof. By Theorem 6.6, there exists $M \in \mathcal{H}$ such that $W \subset g^{-1}Mg$ and $\overline{\Gamma\backslash\Gamma gW} = \Gamma\backslash\Gamma Mg$. Since $\Gamma\backslash\Gamma gW \subset \Gamma\backslash\Gamma Lg$ and $\Gamma\backslash\Gamma Lg$ is closed, we have

$$\Gamma\backslash\Gamma M \subset \Gamma\backslash\Gamma L.$$

Hence, by Lemma 6.7, $\mathrm{MT}(M) \subset \mathrm{MT}(L)$. Since $g \notin \mathcal{S}(L, W)$, $\mathrm{MT}(L) = \mathrm{MT}(M)$ and hence by Lemma 6.7,

$$\Gamma\backslash\Gamma L = \Gamma\backslash\Gamma M.$$

This proves the lemma. □

Note that Lemma 6.8 implies that

$$\mathcal{T}_L(W) = \pi(\mathcal{N}(L, W)) - \pi(\mathcal{S}(L, W)). \tag{6.9}$$

Lemma 6.10. *For $P, Q \in \mathcal{H}$, the following are equivalent:*

(i) $\mathcal{T}_P(W) \cap \mathcal{T}_Q(W) \neq \emptyset$;
(ii) $\mathrm{MT}(P) = \gamma\,\mathrm{MT}(Q)\gamma^{-1}$ *for some* $\gamma \in \Gamma$;
(iii) $\mathcal{T}_P(W) = \mathcal{T}_Q(W)$.

Proof. Suppose $g \in \mathcal{N}(P, W) - \mathcal{S}(P, W)$ and $\gamma g \in \mathcal{N}(Q, W) - \mathcal{S}(Q, W)$ for some $\gamma \in \Gamma$. Then by Lemma 6.8, the closure of $\Gamma\backslash\Gamma gW$ is equal to

$$\Gamma Pg = \Gamma Q\gamma g = \Gamma\gamma^{-1}Q\gamma g.$$

Hence by Lemma 6.7,

$$\mathrm{MT}(P) = \mathrm{MT}(\gamma Q\gamma^{-1}) = \gamma\,\mathrm{MT}(Q)\gamma^{-1}.$$

This shows (i) implies (ii). If (ii) holds, then $\mathcal{N}(P, W) = \gamma\mathcal{N}(Q, W)$ and $\mathcal{S}(P, W) = \gamma\mathcal{S}(Q, W)$. Hence, (iii) follows. The claim that (iii) implies (i) is obvious. □

Let $\mathcal{F}^*$ be the Γ-conjugacy class of Mumford-Tate subgroups of $L \in \mathcal{H}$. For each $[\mathbf{L}] \in \mathcal{F}^*$, choose one subgroup $L \in \mathcal{H}$ with $\mathrm{MT}(L) = \mathbf{L}$. We collect them to a set $\mathcal{H}^*$. Note that $\mathcal{H}^*$ is countable and the sets $\mathcal{T}_L(W)$, $H \in \mathcal{H}^*$, are disjoint from each other.

Theorem 6.11. *Let $\mu \in \mathcal{P}(X)$ be a W-invariant measure. For every $L \in \mathcal{H}$, let μ_L denote the restriction of μ to $T_L(W)$. Then*

(1) $\mu = \sum_{L\in\mathcal{H}^*} \mu_L$.
(2) *Each μ_L is W-invariant. For any W-ergodic component $\nu \in \mathcal{P}(X)$ of μ_L, there exists $g \in \mathcal{N}(L, W)$ such that ν is the unique $g^{-1}Lg$-invariant measure on $\Gamma\backslash\Gamma Lg$.*

Proof. We first disintegrate μ into W-ergodic components. By Theorem 6.6, each of them is of the form νg where $L \in \mathcal{H}$, $g \in \mathcal{N}(L, W) - \mathcal{S}(L, W)$, and ν is the normalized L-invariant measure on $\Gamma\backslash\Gamma L$. Now the claim follows from Lemma 6.8, (6.9), and Lemma 6.10. □

6.4. **Linearization.** Let $L \in \mathcal{H}$. Let $\mathfrak{g}$ denote the Lie algebra of $\mathbf{G}$ and $\mathfrak{l}$ the Lie subalgebra of $\mathrm{MT}(L)$. For $d = \dim(\mathfrak{l})$, we consider the K-rational representation

$$\wedge^d \mathrm{Ad} : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_L) \quad \text{where} \quad \mathbf{V}_L := \wedge^d \mathfrak{g}.$$

We set $V_L = \prod_{v \in S} \mathbf{V}_L(K_v)$ and fix $p_L \in (\wedge^d \mathfrak{l})(K)$, $p_L \neq 0$.

Consider the orbit map $\eta_L : G \to V_L$ given by

$$\eta_L((g_v)_{v \in S}) := (p_L g_v)_{v \in S}.$$

Let

$$\Gamma_L := \{\gamma \in \Gamma : \gamma^{-1}\,\mathrm{MT}(L)\gamma = \mathrm{MT}(L)\},$$
$$\Gamma_L^0 := \{\gamma \in \Gamma : \eta_L(\gamma) = p_L\} = \{\gamma \in \Gamma_L : \det(\mathrm{Ad}(\gamma)|_{\mathfrak{l}}) = 1\}.$$

By Lemma 6.7, we have $\Gamma\backslash\Gamma L = \Gamma\backslash\Gamma L\gamma$ for $\gamma \in \Gamma_L$. This implies that $\gamma \in \Gamma_L$ preserves the volume and

$$\prod_{v \in S} |\det(\mathrm{Ad}(\gamma)|_{\mathfrak{l}})|_v = 1. \tag{6.12}$$

Hence, $\eta_L(\Gamma_L) \subset \mathcal{O}_S^\times \cdot p_L$ where $\mathcal{O}_S^\times$ denotes the group of units in $\mathcal{O}_S$.

Following Tomanov [67, 4.6], we define the notion of $S(v_0)$-small subsets of V_L. We fix $\delta > 0$ such that for any $w \in S$, any $\alpha \in \mathcal{O}_S^\times$ satisfying $\max_{v \in S\backslash\{w\}} |1 - \alpha|_v < \delta$ is a root of unity in K.

Definition 6.13. *Let $v_0 \in S$. A subset $C = \prod_{v \in S} C_v \subset V_L$ is $S(v_0)$-small if for any $v \in S\backslash\{v_0\}$ and $\alpha \in K_v^\times$, $\alpha C_v \cap C_v \neq \emptyset$ implies that $|1 - \alpha|_v < \delta$.*

Then for $\alpha \in \mathcal{O}_S^\times$ and $S(v_0)$-small subset C of V_L,

$$\alpha C \cap C \neq \emptyset \quad \Rightarrow \quad \alpha \in \mu_K$$

where μ_K is the set of roots of unity in K.

We set

$$\bar{V}_L = V_L / \{\alpha \in \mu_K : \alpha p_L \in \eta_L(\Gamma_L)\}.$$

Now $\bar{\eta}_L$ denotes the composition map of η_L with the quotient map $V_L \to \bar{V}_L$.

Since Γ is an S-arithmetic subgroup of G and p_L is rational, it is clear that $\bar{\eta}_L(\Gamma)$ is a discrete subset in $\bar{V}_L$, and the map

$$\Gamma_L^0\backslash G \to \Gamma\backslash G \times \bar{V}_L : \ \Gamma_L^0 g \mapsto (\Gamma g, \bar{\eta}_L(g)) \tag{6.14}$$

is proper (see [67, 4.7]).

Denote by A_L the Zariski closure of $\bar{\eta}_L(\mathcal{N}(L, W))$ in $\bar{V}_L$. Then (see [67, 4.5])

$$\bar{\eta}_L^{-1}(A_L) = \mathcal{N}(L, W). \tag{6.15}$$

Proposition 6.16. *Let D be a compact $S(v_0)$-small subset of A_L for some $v_0 \in S$. Define*

$$\mathcal{S}(D) = \{g \in \bar{\eta}_L^{-1}(D) : \gamma g \in \bar{\eta}_L^{-1}(D) \text{ for some } \gamma \in \Gamma - \Gamma_L\}.$$

Then

(1) $\mathcal{S}(D) \subset \mathcal{S}(L, W)$;
(2) $\pi(\mathcal{S}(D))$ *is closed in* X;
(3) *for any compact subset* $B \subset X \setminus \pi(\mathcal{S}(D))$, *there exists a neighborhood* Φ *of* D *in* $\bar{V}_L$ *such that for each* $y \in \pi(\bar{\eta}_L^{-1}(\Phi)) \cap B$, *the set* $\bar{\eta}_L(\pi^{-1}(y)) \cap \Phi$ *consists of a single element.*

Proof. Suppose that $g \in \mathcal{S}(D)$. Then $\gamma g \in \bar{\eta}_L^{-1}(D)$ for some $\gamma \in \Gamma - \Gamma_L$. By (6.15), both g and γg belong to $\mathcal{N}(L, W)$. Then

$$\overline{\Gamma \backslash \Gamma g W} \subset \Gamma \backslash \Gamma L \gamma g.$$

Suppose $g \notin \mathcal{S}(L, W)$. Then by Lemma 6.8,

$$\overline{\Gamma \backslash \Gamma g W} = \Gamma \backslash \Gamma L g.$$

Hence by Lemma 6.7,

$$\mathrm{MT}(L) \subset \gamma^{-1}\,\mathrm{MT}(L)\gamma.$$

Therefore, $\gamma \in \Gamma_L$, which gives a contradiction. This shows (1).

If (3) fails, then there exist $g_i \in \pi^{-1}(B)$ and $\gamma_i \in \Gamma$ with $\bar{\eta}_L(g_i) \neq \bar{\eta}_L(\gamma_i g_i)$ and $\bar{\eta}_L(g_i), \bar{\eta}_L(\gamma_i g_i)$ converge to elements of D. Since the map (6.14) is proper, by passing to a subsequence, there exist $\delta_i, \delta_i' \in \Gamma_L^0$ such that $\delta_i g_i \to g$ and $\delta_i' \gamma_i g_i \to g'$ for some $g, g' \in G$. Hence by passing to a subsequence, $\delta_i' \gamma_i \delta_i^{-1} = g' g^{-1}$ for all large i. Hence $\delta_0 := g' g^{-1} \in \Gamma$. Then $\bar{\eta}_L(\Gamma_L^0 g), \bar{\eta}_L(\Gamma_L^0 \delta_0 g) \in D$. Since $\Gamma_L^0 g \notin \mathcal{S}(D)$, $\delta_0 \in \Gamma_L$. Hence

$$\bar{\eta}_L(g) \in D \cap \alpha D$$

for some $\alpha \in \mathcal{O}_S^\times$. Since D is $S(v_0)$-small, it follows from (6.12) that $\alpha \in \mu_K$ and hence $\bar{\eta}_L(\gamma_i) = \bar{\eta}_L(\delta_i)$. This gives a contradiction.

Claim (2) can be proved similarly.

□

By an interval I in K_v, we mean a subset of the form $\{x \in K_v : |x - x_0|_v < T\}$ for some $x_0 \in K_v$ and $T > 0$. We call x_0 the center of I.

We will need the following property of polynomial maps in the proof of our main proposition 6.19.

Proposition 6.17. [67, 4.2] *Let A_v be a Zariski closed subset of K_v^m, $C_v \subset A_v$ a compact subset, and $\epsilon > 0$. Then there exists a compact neighborhood $D_v \subset A_v$ of C_v such that for any neighborhood Φ_v of D_v in K_v^m there exists a neighborhood $\Psi_v \subset \Phi_v$ such that for any one parameter unipotent subgroup $u(t)$ of $\mathbf{G}(K_v)$, any bounded interval I in K_v and any $w \in K_v^m$ such that $wu(t_0) \notin \Phi_v$ for some $t_0 \in I$,*

$$\theta_v(\{t \in I : wu(t) \in \Psi_v\}) \le \epsilon \cdot \theta_v(\{t \in I : wu(t) \in \Phi_v\}).$$

We will also use the following simple lemma from [67] to relate the behavior of unipotent one parameter subgroups over $\mathbb{C}$ with those over $\mathbb{R}$.

Lemma 6.18. *Let $K_v = \mathbb{C}$, $I = \{t \in \mathbb{C} : |t| \leq 1\}$, $\epsilon > 0$ and A measurable subset of I such that for any $x \in I$.*

$$\epsilon\theta_0\{a \in \mathbb{R} : ax \in I\} \geq \theta_0\{a \in \mathbb{R} : ax \in I \cap A\}.$$

Then $\theta_v(A) \leq \epsilon\pi$ where θ_0 is the Lebesgue measure on $\mathbb{R}$.

The following proposition is a main tool in the proof of Theorem 6.2.

Proposition 6.19. *Fix $v_0 \in S$. Let $C \subset A_L$ be a compact subset and $\epsilon > 0$ be given. Then there exists a closed subset R of X contained in $\pi(\mathcal{S}(L, W))$ such that for any compact subset $B \subset X \setminus R$, there exists a neighborhood Ψ of C in V_L such that for any one parameter unipotent subgroup $u(t)$ of $\mathbf{G}(K_{v_0})$ and any $x \in B$, at least one of the following holds:*

(1) *There exists $w \in \bar{\eta}_L(\pi^{-1}(x)) \cap \overline{\Psi}$ such that*
$$\{u(t)\} \subset \{g \in G : wg = w\}$$
(2) *For any sufficiently large bounded interval $I \subset K_{v_0}$ centered at zero,*
$$\theta_{v_0}(\{t \in I : xu(t) \in B \cap \pi(\bar{\eta}_L^{-1}(\Psi))\}) \leq \epsilon \cdot \theta_{v_0}(I).$$

Proof. Since C can be covered by finitely many compact $S(v_0)$-small sets, it suffices to prove the proposition for a $S(v_0)$-small subset $C = \prod_{v \in S} C_v$ with C_v compact. For C_{v_0} and $\epsilon > 0$, let D_{v_0} be as in Proposition 6.19 and $D_v = C_v$ for $v \in S \backslash \{v_0\}$. Then the set $D := \prod_{v \in S} D_v$ is also $S(v_0)$-small. For $S(D)$ defined in Proposition 6.16, set $R = \pi(S(D))$. For a given B, let Φ be a neighborhood of D as in Proposition 6.16. Passing to a smaller neighborhood, we may assume that Φ is of the form $\prod_{v \in S} \Phi_v$. Let $\Psi_{v_0} \subset \Phi_{v_0}$ be a neighborhood of C_{v_0} so that the statement of Proposition 6.17 holds. We set $\Psi := \prod_v \Psi_v$ where $\Psi_v = \Phi_v$ for $v \neq v_0$.

Let $\Omega := B \cap \pi(\bar{\eta}_L^{-1}(\Psi))$ and $J = \{t \in K_{v_0} : xu(t) \in \Omega\}$.

Assume that v_0 is non-archimedean. For each $t \in J$, there exists a unique $w_t \in \eta_L(\pi^{-1}(x))$ such that $w_t u(t) \in \Phi$. By uniqueness, $w_t u(t) \in \Psi$. Note that the map $t \mapsto w_t$ is a locally constant. For each $t \in J$, let $I(t)$ be the maximal interval containing t such that $w_t u(I(t)) \subset \Phi$. By the nonarchimedean property of K_{v_0}, the intervals $I(t)$ are either disjoint or equal. Since $s \mapsto w_t u(s)$, $s \in I(t)$, is a polynomial map, it is either constant or unbounded. Hence if some $I(t)$ is unbounded for $t \in J$, $w_t u(K_{v_0}) = w_t$ and hence the first case happens. Now suppose that $I(t)$ is bounded for any $t \in J$. Let $J(t)$ be the minimal interval containing $I(t)$ such that $w_t u(J(t)) \cap \Phi^c \neq \emptyset$. Note that $\theta_{v_0}(J(t)) \leq q_0 \cdot \theta_{v_0}(I(t))$ where q_0 is the cardinality of the residue field of K_{v_0}. By Proposition 6.17,

$$\begin{aligned}\theta_{v_0}(\{s \in I(t) : w_t u(s) \in \Psi\}) &\leq \theta_{v_0}(\{s \in J(t) : w_t u(s) \in \Psi\}) \\ &\leq \epsilon\theta_{v_0}(J(t)) \leq \epsilon \cdot q_0 \cdot \theta_{v_0}(I(t)).\end{aligned}$$

Now for any interval I centered at zero, we have

$$\{s \in I : xu(s) \in \Omega\} = \bigcup_{t \in J} I(t) \cap I.$$

If I is sufficiently large, it follows from the nonarchimedean property of v_0 that either $I \cap I(t) = \emptyset$ or $I(t) \subset I$. Hence

$$\begin{aligned}\theta_{v_0}(\{s \in I : xu(s) \in \Omega\}) &= \sum_{I(t) \subset I} \theta_{v_0}(\{s \in I(t) : xu(s) \in \Omega\}) \\ &\leq \epsilon \cdot q_0 \cdot \sum_{I(t) \subset I} \theta_{v_0}(I(t)) \leq \epsilon \cdot q_0 \cdot \theta_{v_0}(I).\end{aligned}$$

This proves the claim for v_0 non-archimedean. The case when $K_{v_0} = \mathbb{R}$ is proved in [52]. Consider the case of $K_{v_0} = \mathbb{C}$. By the restriction of scalars, we may consider $\mathbf{G}(K_{v_0})$ as a real Lie group and hence the statement holds for any restriction u_r of $u : \mathbb{C} \to \mathbf{G}(K_{v_0})$ to a one dimensional real subspace r of $\mathbb{C}$. Suppose (1) holds for some real subspace r, i.e., $u_r(\mathbb{R})$ stabilizes a vector $w = p_L(\gamma g)$ with $\pi(g) = x$. Then

$$gu(r)g^{-1} \subset \gamma^{-1}\{y \in N(L) : \mathrm{Ad}(y)|_{\mathfrak{l}} = 1\}\gamma.$$

Since the right hand side of the above is a K-subgroup, it follows that $gu(K_{v_0})g^{-1}$ is also contained in the same group and hence u satisfies (1). Therefore if (1) fails for u, then (2) holds for u_r for any one dimensional real subspace $r \subset \mathbb{C}$ and for any interval of r. By (6.18), this implies (2).

□

6.5. **Proof of Theorem 6.2.** Set $W := \Lambda(\mu)_u$ and $U_i = \{u_i(t) : t \in K_v\}$. Then $\dim(W) \geq 1$ by [MS, Lemma 2.2] whose proof works in the same way for K_v. By Theorem 6.11, there exists $L \in \mathcal{H}$ such that

$$\mu(\pi(\mathcal{S}(L, W)) = 0 \quad \text{and} \quad \mu(\pi(\mathcal{N}(L, W)) > 0.$$

Therefore we can find a compact set $C_1 \subset \mathcal{N}(L, W) \backslash \mathcal{S}(L, W)$ such that $\mu(\pi(C_1)) > 0$. Note that by (6.9), $\pi(C_1) \cap \pi(\mathcal{S}(L, W)) = \emptyset$. Let $z_i \to e \in G$ be a sequence such that $xz_i \in \mathrm{supp}(\mu_i)$ and the trajectory $\{xz_iu_i(t) : t \in K_v\}$ is uniformly distributed with respect to μ_i as $T \to \infty$ when the averages are taken over the sets $I_T := \{s \in K_v : |s|_v \leq T\}$.

By the pointwise ergodic theorem, such a sequence $\{z_i\}$ always exists.

Pick $y \in \mathrm{supp}(\mu) \cap \pi(C_1)$. Then there exists a sequence $y_i \in xz_iU_i$ which converges to y. Let $h_i \to e$ be a sequence satisfying $y_i = yh_i$ for each i. Set

$$\mu_i' = \mu_i h_i \quad \text{and} \quad u_i'(t) = h_i u_i(t) h_i^{-1}.$$

Then $\mu_i' \to \mu$ as $i \to \infty$, $y \in \mathrm{supp}(\mu_i')$ and $\{yu_i'(t)\}$ is uniformly distributed with respect to μ_i'.

Let R and Ψ be as Proposition 6.19 with respect to $C := \bar{\eta}_L(C_1)$ and $\epsilon := \mu(\pi(C_1))/2$. We can choose a compact neighborhood B of $\pi(C_1)$ such that $B \cap R = \emptyset$. Put

$$\Omega := \pi(\bar{\eta}_L^{-1}(\Psi)) \cap B.$$

Since $\pi(C_1) \subset \Omega$, we have $\mu_i'(\Omega) > \epsilon$ for all sufficiently large i. Hence for sufficiently large T and i,

$$\theta_{v_0}(\{s \in I_T : yu_i'(t) \in \Omega\}) > \epsilon \cdot \theta_{v_0}(I_T)$$

By Proposition 6.19, there exists $g_0 \in \pi^{-1}(y)$ so that $w = p_L(g_0) \in \bar{\eta}_L(\pi^{-1}(y)) \cap \overline{\Psi}$ and $wu_i'(t) = w$ for all $t \in K_v$. Consider the K-subgroup $\mathbf{M} := \mathrm{Stab}_{\mathbf{G}}(wg_0^{-1})$. We observe that

$$g_0\{u_i'(t)\}g_0^{-1} \subset \mathbf{M}_S \quad \text{and} \quad \mathrm{supp}(\mu_i')g_0^{-1} \subset \Gamma\backslash\Gamma\mathbf{M}_S. \tag{6.20}$$

We use induction on $\dim(\mathbf{G})$ to show that

$$\text{(a)} \quad \mathrm{supp}(\mu) = y\Lambda(\mu) \qquad \text{and} \qquad \text{(b)} \quad \Lambda(\mu_i') \subset \Lambda(\mu)$$

for all sufficiently large i.

If $\dim(\mathbf{M}) < \dim(\mathbf{G})$, since (6.20) and $g_0^{-1}\mu_i' \to g_0^{-1}\mu$, we can apply inductive hypothesis to the space $\Gamma\backslash\Gamma\mathbf{M}_S$ and the measure $g_0^{-1}\mu$. This yields (a) and (b).

If $\dim(\mathbf{M}) = \dim(\mathbf{G})$, then $\mathrm{MT}(L)$ is a normal subgroup of $\mathbf{G}$.

Since $\mathcal{N}(L, W) = G$ and $\mu(\pi(\mathcal{S}(L, W))) = 0$, we have $\mu = \mu_L$. By Theorem 6.11, every W-ergodic component of μ is $g^{-1}Lg$-invariant for some $g \in G$. Since $\mathrm{MT}(L)$ is a normal subgroup of $\mathbf{G}$, $g^{-1}Lg$ is a subgroup of $\mathrm{MT}(L)_S$ and $[\mathrm{MT}(L)_S : L] = [\mathrm{MT}(L)_S : g^{-1}Lg]$. Since $\mathrm{MT}(L)_S$ has only finitely many closed subgroups of bounded index, we obtain a finite index subgroup L_0 of $\mathrm{MT}(L)_S$ such that L_0 is normal in G and every W-ergodic component of μ, and hence μ itself, is L_0-invariant.

Denoting by $\rho : G \to L_0\backslash G$ the quotient homomorphism, we set $\bar{X} = \rho(\Gamma)\backslash(L_0\backslash G)$ and obtain the push-forward map $\bar{\rho}_* : \mathcal{P}(X) \to \mathcal{P}(\bar{X})$ of measures.

Since $\dim(\mathrm{MT}(L)) \geq \dim W > 1$, we may apply the induction to the measures $\bar{\rho}_*(\mu_i')$ and $\bar{\rho}_*(\mu)$ and obtain

$$\mathrm{supp}(\bar{\rho}_*(\mu)) = \bar{y}\Lambda(\bar{\rho}_*(\mu))$$

and for all large i,

$$\Lambda(\bar{\rho}_*(\mu_i)) \subset \Lambda(\bar{\rho}_*(\mu)).$$

Since μ is L_0-invariant, applying [22] in the same way as in [52], this implies

$$\rho^{-1}(\Lambda(\bar{\rho}_*(\mu))) = \Lambda(\mu).$$

It is easy to deduce (a) and (b) now.

We finally claim that (a) and (b) imply (1)–(3). Since μ_i' are $\{u_i'(t)\}$-ergodic measures and $y \in \mathrm{supp}(\mu')$, by Theorem 6.6, μ_i' is a $\Lambda(\mu_i')$-invariant measure supported on $y\Lambda(\mu_i')$. Hence, by (b),

$$\mathrm{supp}(\mu_i) = \mathrm{supp}(\mu_i')h_i = y\Lambda(\mu_i')h_i \subset y\Lambda(\mu)h_i = x\Lambda(\mu)h_i.$$

Since

$$xz_i \in \mathrm{supp}(\mu_i) \subset x\Lambda(\mu)h_i,$$

and $z_i, h_i \to e$, it follows that $z_i h_i^{-1} \in \Lambda(\mu)$. Therefore,

$$\mathrm{supp}(\mu_i) \subset x\Lambda(\mu)h_i = x\Lambda(\mu)z_i \quad \text{and} \quad \Lambda(\mu_i) = h_i^{-1}\Lambda(\mu_i')h_i \subset z_i^{-1}\Lambda(\mu)z_i.$$

This proves (2).

There exists i_0 such that for all $i \geq i_0$, $z_i U_i z_i^{-1} \subset \Lambda(\mu)$. Let H be the subgroup of G (in fact of $\mathbf{G}(K_v)$) generated by all $z_i U_i z_i^{-1}$, $i \geq i_0$. By (2), $H \subset \Lambda(\mu)$ and hence by (a)

$$\overline{xH} \subset x\Lambda(\mu) = \mathrm{supp}(\mu).$$

On the other hand, by Theorem 6.6,

$$\overline{xH} = \Gamma\backslash\Gamma Lg$$

for some $L \in \mathcal{H}$ such that $H \subset g^{-1}Lg$. Since $\mu_i \to \mu$, it follows that $\Gamma\backslash\Gamma Lg = \mathrm{supp}(\mu)$.

Since $\Gamma\backslash\Gamma Lg = x\Lambda(\mu)$, μ is the unique invariant probability measure supported on $\Gamma\backslash\Gamma Lg$, as required in (1). Since $L \in \mathcal{H}$ and $H \subset g^{-1}L_u g$, by the following proposition 6.21, μ is ergodic with respect to H. This finishes the proof.

Proposition 6.21. *Let L be a closed subgroup of finite index in $\mathbf{P}_S$ for some $\mathbf{P} \in \mathcal{F}$, and let H be a closed subgroup of L generated by unipotent one-parameter groups such that $\overline{\Gamma\backslash\Gamma H} = \Gamma\backslash\Gamma L$. Then the translation action of H on $\Gamma\backslash\Gamma L$ is ergodic.*

Proof. By an S-algebraic version of the Mautner lemma (see below Prop. 6.22) there exists a closed normal subgroup $M \subset \mathbf{P}_S$ containing H such that the triple $(H, M, \mathbf{P}_S)$ has the Mautner property, that is, for any continuous unitary representation of $\mathbf{P}_S$, any H-invariant vector is also M-invariant. Since $M \cap L$ is normal in L and $\overline{\Gamma\backslash\Gamma(M \cap L)} = \Gamma\backslash\Gamma L$, it follows that $M \cap L$ acts ergodically on $\Gamma\backslash\Gamma L$. Applying the Mautner property to the unitary representation $\mathrm{Ind}_L^{\mathbf{P}_S} L^2(\Gamma\backslash\Gamma L)$, we deduce that H acts ergodically on $\Gamma\backslash\Gamma L$. $\square$

We recall an S-arithmetic version of the Mautner lemma:

Proposition 6.22. [50, Corollary 2.8] *Let $L \subset \mathbf{G}_S$ be a closed subgroup generated by unipotent one-parameter subgroups in it. Then there exists a closed normal subgroup $M \subset \mathbf{G}_S$ containing L such that the triple $(L, M, \mathbf{G}_S)$ has the Mautner property, that is, for any continuous unitary representation of $\mathbf{G}_S$, any L-invariant vector is also M-invariant.*

7. Non-divergence of unipotent flows

7.1. **Statement of Main theorem.** Let $\mathbf{G}$ be a connected semisimple algebraic K-group, S a finite set of normalized absolute values of K including all the archimedean v such that $\mathbf{G}(K_v)$ is non-compact and $\Gamma \subset G$ an S-arithmetic lattice. Here we also assume that $\mathbf{G}$ is K-isotropic, equivalently, that Γ is a non-uniform lattice (otherwise, the main theorem of this section holds trivially). Note this also implies that $\mathbf{G}(K_v)$ is non-compact for every valuation v of R. We generalize the main theorem of Dani-Margulis in [23] to an S-algebraic setting. Some of our arguments follow closely those in [30].

Let $\mathbf{A}$ be a maximal K-split torus of $\mathbf{G}$ and choose a system $\{\alpha_1, \dots, \alpha_r\}$ of simple K-roots for $(\mathbf{G}, \mathbf{A})$. For each i, let $\mathbf{P}_i$ be the standard maximal parabolic subgroup corresponding to $\{\alpha_1, \dots, \alpha_r\} - \{\alpha_i\}$.

The subgroup $\mathbf{P} := \cap_{1 \le i \le r} \mathbf{P}_i$ is a minimal K-parabolic subgroup of $\mathbf{G}$, and there exists a finite subset $F \subset \mathbf{G}(K)$ such that

$$\mathbf{G}(K) = \Gamma F \mathbf{P}(K).$$

For $T > 1$, we set

$$J_T := \{x \in K_v : |x|_v < T\}.$$

Theorem 7.1. *Let $\epsilon > 0$. Then there exists a compact subset $C \subset \Gamma\backslash\mathbf{G}_S$ such that for any unipotent one parameter subgroup $U = \{u(t)\} \subset \mathbf{G}(K_v)$, and $g \in \mathbf{G}_S$, either one of the following holds:*

(1) *for all large $T > 0$,*

$$\theta_v\{t \in J_T : \Gamma\backslash\Gamma g u_t \in C\} \ge (1 - \epsilon)\, \theta_v(J_T);$$

(2) *there exist i and $\lambda \in \Gamma F$ such that*

$$gUg^{-1} \subset \lambda \mathbf{P}_i \lambda^{-1}.$$

7.2. **Deduction of Theorem 4.6 (1) from Theorem 7.1:** Suppose not. Let C be a compact subset as in Theorem 7.1. Then there exists $\epsilon > 0$ such that

$$g_i \nu_i(C) < 1 - \epsilon \quad \text{for all large } i,$$

by passing to a subsequence. Fix $v \in S$ that is strongly isotropic for all $\mathbf{L}_i$. Let $U_i = \{u_i(t)\} \subset \mathbf{L}_i(K_v)^+$ be as in Lemma 6.5 and let R_i denote a subset of full measure in $\pi(\tilde{\mathbf{L}}_{i,S})$ such that for every $h \in R_i$, the orbit $\Gamma\backslash\Gamma h U_i$ is uniformly distributed on $\Gamma\backslash\Gamma\tilde{\mathbf{L}}_{i,S}$. Hence for each i, there exists T_i such that

$$\theta_v\{t \in J_T : \Gamma\backslash\Gamma h u_i(t) g_i \in C\} \le (1 - \epsilon/2)\theta_v(J_T)$$

for all $T > T_i$.

Applying $U = g_i^{-1} U_i g_i$ and $g = hg_i$ to Theorem 7.1, there exist j_i and $\lambda_i \in \Gamma F$ such that

$$hU_i h^{-1} \subset \lambda_i \mathbf{P}_{j_i} \lambda_i^{-1}$$

for all $h \in R_i$, where $\mathbf{P}_{j_i}$ is a proper parabolic K-subgroup of $\mathbf{G}$.

Since the set $\{h \in \pi(\tilde{\mathbf{L}}_{i,S}) : hU_i h^{-1} \subset \lambda_i \mathbf{P}_{j_i} \lambda_i^{-1}\}$ is an analytic manifold which contains a subset of full measure in $\pi(\tilde{\mathbf{L}}_{i,S})$, it follows that this set is $\pi(\tilde{\mathbf{L}}_{i,S})$ itself. Since U_i is not contained in any proper normal subgroup of $\mathbf{L}_i(K_v)^+$, we have

$$\mathbf{L}_i \subset \lambda_i \mathbf{P}_{j_i} \lambda_i^{-1}.$$

This is a contradiction to the assumption by Lemma 4.5.

7.3. For each i, let $\mathbf{U}_i$ denote the unipotent radical of $\mathbf{P}_i$. Denote by $\mathfrak{u}_i$ the Lie algebra of $\mathbf{U}_i$ and by $\mathfrak{g}$ the Lie algebra of $\mathbf{G}$. For each $v \in S$, we fix a norm $\|\cdot\|_v$ on the K_v-vector space $\wedge^{\dim \mathfrak{u}_i}\mathfrak{g}(K_v)$ and choose a non-zero vector w_i of $\wedge^{\dim \mathfrak{u}_i}\mathfrak{u}_i(K)$ with $\|w_i\|_v = 1$ for all $v \in S$. Define $\Delta_i : \mathbf{G}_S \to \mathbb{R}^*$ by

$$\Delta_i((g_v)) := \prod_{v \in S} \|w_i g_v\|_v.$$

Fix $v \in S$. Let $\mathcal{P}_d$ denote the family of all polynomial maps $K_v \to \mathbf{G}(K_v)$ (resp. $\mathbb{R} \to \mathbf{G}(\mathbb{C})$) of degree at most d if $K_v \neq \mathbb{C}$ (resp. $K_v = \mathbb{C}$).

For $T > 1$ we set if $K_v \neq \mathbb{C}$

$$I_T := \{x \in K_v : |x|_v < T\},$$

and if $K_v = \mathbb{C}$,

$$I_T := \{x \in \mathbb{R} : |x| < T\}$$

where $|\cdot|$ is the usual absolute value of a real number. We keep this definition of I_T for the rest of this section. We will deduce Theorem 7.1 from the following:

Theorem 7.2. *Fix $\alpha, \epsilon > 0$ and $v \in S$. Then there exists a compact subset $C \subset \Gamma\backslash\mathbf{G}_S$ such that for any $u \in \mathcal{P}_d$ and any $T > 0$, either one of the following holds:*

(1) $\theta_v(\{s \in I_T : \Gamma\backslash\Gamma u(s) \in C\}) \geq (1-\epsilon)\theta_v(I_T)$
(2) *there exist $i \in \{1, \ldots, r\}$ and $\lambda \in \Gamma F$ such that*

$$\Delta_i(\lambda^{-1}u(s)) \leq \alpha \quad \textit{for all } s \in I_T,$$

Deduction of Theorem 7.1 from Theorem 7.2 First consider the case of $K_v \neq \mathbb{C}$. There is $d > 0$ such that for any $g \in \mathbf{G}_S$ and for any u one parameter unipotent subgroup of $\mathbf{G}(K_v)$, the maps $t \mapsto gu(t)$ belongs to $\mathcal{P}_d$. Hence if the first case of Theorem 7.1 fails, then there exists $1 \leq i \leq r$, $T_m \to \infty$, $0 < \alpha_m < 1$, $\alpha_m \to 0$, $\lambda_m \in \Gamma F$ such that

$$\Delta_i(\lambda_m^{-1} g u(s)) < \alpha_m$$

for all $s \in I_{T_m}$.

Since this implies $\Delta_i(\lambda_m^{-1}g) < 1$ and for a given $\theta > 0$, the number of the elements $\lambda \in \Gamma F$, modulo the stabilizer of w_i, such that $\Delta_i(\lambda^{-1}g) < \theta$ is finite, we can assume, by passing to a subsequence, that there exists $\lambda \in \Gamma F$ such that for each m,

$$\Delta_i(\lambda^{-1} g u(s)) < \alpha_m$$

for all $s \in I_{T_m}$.

Since any orbit of a unipotent one parameter subgroup is unbounded except for a fixed point, it follows that $\lambda^{-1}gu(s)g^{-1}\lambda$ fixes w_i for all $s \in K_v$. Therefore

$$\lambda^{-1} g U g^{-1} \lambda \subset \mathbf{P}_i.$$

Now consider the case when $K_v = \mathbb{C}$, Suppose (1) fails for some $g \in \mathbf{G}_S$ and for some unipotent one parameter subgroup $u : \mathbb{C} \to \mathbf{G}(\mathbb{C})$. By (6.18), we have a one dimensional real subspace $r = \mathbb{R}x \subset \mathbb{C}$, $x \in \mathbb{C}$, such that

$$|\{s \in [-T_m, T_m] : \Gamma\backslash\Gamma g u(sx) \in C\}| < 2(1-\epsilon)T_m$$

for some $T_m \to \infty$. By Theorem 7.2, for any $\alpha > 0$, there are i and $\lambda \in \Gamma F$ such that for all $s \in I_{T_m}$,

$$\Delta_i(\lambda^{-1} u_r(s)) \le \alpha$$

where $u_r(s) = u(sx)$.

By the same argument as in the above case, we deduce that for some $1 \le i \le r$ and $\lambda \in \Gamma F$, we have

$$g u(r) g^{-1} \subset \lambda \mathbf{P}_i \lambda^{-1}.$$

Since $\mathbf{P}_i$ is an algebraic K-subgroup, it follows that

$$g U g^{-1} \subset \lambda \mathbf{P}_i \lambda^{-1}.$$

This finishes the proof.

In order to prove Theorem 7.2, we use the following:

Theorem 7.3. *Let $\alpha > 0$ be given. There exists a compact subset $C \subset \Gamma\backslash\mathbf{G}_S$ such that for any $u \in \mathcal{P}_d$ and $T > 0$, one of the following holds:*

(1) *there exist $i \in \{1, \dots, r\}$ and $\lambda \in \Gamma F$ such that*

$$\Delta_i(\lambda^{-1} u(s)) \le \alpha \quad \textit{for all } s \in I_T,$$

(2) $\Gamma\backslash\Gamma u(I_T) \cap C \neq \emptyset$.

Theorem 7.3 implies Theorem 7.2 in view of the following theorem, proved by Kleinbock and Tomanov [39, Theorem 9.1]:

Theorem 7.4. *For a given compact subset $C \subset \Gamma\backslash\mathbf{G}_S$ and $\epsilon > 0$ there exists a compact subset $C' \subset \Gamma\backslash\mathbf{G}_S$ such that for any $u \in \mathcal{P}_d$, any $y \in \Gamma\backslash\mathbf{G}_S$ and $T > 0$ such that $y u(I_T) \cap C \neq \emptyset$,*

$$\theta_v(\{s \in I_T : y u(s) \in C'\}) \ge (1-\epsilon)\theta_v(I_T).$$

The rest of this section is devoted to a proof of Theorem 7.3. We start by constructing certain compact subsets in X which will serve as C in the theorem 7.3.

We denote by S_∞ the set of all archimedean absolute values and $S_f := S - S_\infty$. For a K-subgroup $\mathbf{M}$ of $\mathbf{G}$, and $S_0 \subset S$, we use the notation $\mathbf{M}_{S_0} = \prod_{v \in S_0} \mathbf{M}(K_v)$, $\mathbf{M}_\infty = \mathbf{M}_{S_\infty}$, and $\mathbf{M}_v = \mathbf{M}(K_v)$. For simplicity, we write M for $\mathbf{M}_S$ in this section.

We often write an element of $g \in M$ as (g_∞, g_f) where $g_\infty \in \mathbf{M}_\infty$ and $g_f \in \mathbf{M}_{S_f}$.

7.4. Description of compact subsets in X. For each $i = 1, \dots, r$, we set

$$\mathbf{Q}_i = \{x \in \mathbf{P}_i : \alpha_i(x) = 1\}$$

and

$$\mathbf{A}_i := \{x \in \mathbf{A} : \alpha_j(x) = 1 \quad \forall j \neq i\}.$$

For a subset $I \subset \{1, \dots, r\}$, we define

$$\mathbf{P}_I := \cap_{i\in I}\mathbf{P}_i, \;\; \mathbf{Q}_I := \cap_{i\in I}\mathbf{Q}_i, \;\; \mathbf{A}_I := \prod_{i\in I}\mathbf{A}_i.$$

Let $\mathbf{U}_I$ be the unipotent radical of $\mathbf{P}_I$ and $\mathbf{H}_I$ the centralizer of $\mathbf{A}_I$ in $\mathbf{Q}_I$. We have Langlands decomposition:

$$\mathbf{P}_I = \mathbf{A}_I\mathbf{Q}_I = \mathbf{A}_I\mathbf{H}_I\mathbf{U}_I.$$

There is $m_i \in \mathbb{N}$ such that for $x \in \mathbf{P}_i$,

$$\det(\operatorname{Ad} x)|_{\mathfrak{u}_i} = \alpha_i^{m_i}(x).$$

For each non-archimedean $v \in S$, we set

$$A_v^0 = \{x \in \mathbf{A}_v : \alpha_i(x) \in q_v^{\mathbb{Z}} \quad \forall i = 1, \dots, r\}$$

where q_v is the cardinality of the residue field of K_v. Since $\mathbf{A}$ is K-split, $A_v^0 \subset \mathbf{A}(K)$.

For archimedean v, we set

$$A_v^0 = \{x \in \mathbf{A}_v : \alpha_i(x) > 0 \quad \forall i = 1, \dots, r\}.$$

For $v \in S$, let W_v be a maximal compact subgroup of $\mathbf{G}_v$ such that $\mathbf{G}_v = W_v\mathbf{Q}_v$ for any parabolic K-subgroup $\mathbf{Q}$ containing $\mathbf{P}$ and $\mathbf{A}_v \subset W_v A_v^0$.

We set $W = \prod_{v\in S} W_v$, $W_f = \prod_{v\in S_f} W_v$ and $W_\infty = \prod_{v\in S_\infty} W_v$. Without loss of generality, we may assume that each norm $\|\cdot\|_v$ is W_v-invariant.

For a subset $I \subset \{1, \dots, r\}$, we set

$$A_{I,v}^0 := \mathbf{A}_{I,v} \cap A_v^0 \quad \text{and} \quad A_{I,\infty}^0 = \prod_{v\in S_\infty} A_{I,v}^0.$$

Lemma 7.5. *There exists a finite subset $Y \subset \mathbf{A}_I(K)$ such that*

$$\prod_{v\in S_f} A_{I,v}^0 \subset (\mathbf{A}_I \cap \Gamma)Y.$$

Proof. Since Γ is commensurable with $\mathbf{G}(\mathcal{O}_S)$, Γ contains a finite index subgroup of $\mathbf{A}_I(\mathcal{O}_S)$. Now the claim follows easily from the fact that the map $f : \mathbf{A}_I \to \mathbf{G}_m^l$, $l = |I|$, given by $x \mapsto (\alpha_1(x), \dots, \alpha_l(x))$, is a K-rational isomorphism, where $\mathbf{G}_m$ denotes the one-dimensional multiplicative group. $\square$

Lemma 7.6. *Given $I \subset \{1, \dots, r\}$, $j \in \{1, \dots, r\} - I$, and $0 < a \leq b$, there exists a compact subset M_0 of Q_I such that*

$$\{g \in Q_I : \Delta_j(g) \in [a, b]\} \subset (\mathbf{A}_j \cap \Gamma)Q_{I\cup\{j\}}M_0.$$

Proof. Since $\mathbf{A}_{I,v} \subset W_v A^0_{I,v}$ for each $v \in S$, we can show in a similar way as in the proof of [23, Lemma 1.5] that for any $j \notin I$,

$$Q_I = (\prod_{v \in S} A^0_{j,v}) Q_{I \cup \{j\}} (W \cap H_I).$$

Hence any $(g_v) \in Q_I$ is of the form $g_v = a_v q_v w_v$ with $a_v \in A^0_{j,v}$, $q_v \in \mathbf{Q}_{I \cup \{j\},v}$, and $w_v \in W_v$, and

$$\|w_j g_v\|_v = |\alpha_j(a_v)|_v^{m_j}.$$

Suppose $g = (g_v) \in Q_I$ satisfies $a < \Delta_j(g) < b$, i.e.,

$$\Delta_j(g) = \prod_{v \in S} |\alpha_j(a_v)|_v^{m_j} \in [a, b].$$

It follows from Lemma 7.5 that $d_0 := \prod_{v \in S_f} a_v \in (\mathbf{A}_j \cap \Gamma) Y$ where Y is a finite subset of $\mathbf{A}_j(K)$. If we set $d_v = a_v d_0^{-1}$ for $v \in S_\infty$ and $d_v = \prod_{w \in S_f \setminus \{v\}} a_w^{-1}$ for $v \in S_f$, then $d_0 d_v = a_v$ and $d_v \in W_v$ for $v \in S_f$, and

$$\prod_{v \in S_\infty} |\alpha_j(d_v)|_v^{m_j} = \prod_{v \in S} |\alpha_j(a_v)|_v^{m_j} \in [a, b]$$

This implies that there exists a compact set $M_\infty \subset \mathbf{A}_{j,\infty}$, depending only on $[a, b]$, such that

$$(d_v)_{v \in S} \in M_\infty \times (\prod_{v \in S_f} M_v)$$

where $M_v := \{a \in \mathbf{A}_{j,v} : \|a\|_v = 1\}$.

Therefore for $M := M_\infty \times \prod_{v \in S_f} M_v$,

$$Q_I \subset (\mathbf{A}_j \cap \Gamma) Y M Q_{I \cup \{j\}} (W \cap H_I).$$

Since $\mathbf{A}_j$ normalizes $\mathbf{Q}_{I \cup \{j\}}$, it follows that

$$Q_I \subset (\mathbf{A}_j \cap \Gamma) Q_{I \cup \{j\}} M_0$$

for some compact subset M_0 of Q_I. □

For $I \subset \{1, \dots, r\}$, we define a finite subset $F_I \subset \mathbf{Q}_I(K)$ such that

$$\mathbf{Q}_I(K) = (\Gamma \cap \mathbf{Q}_I) F_I (\mathbf{P} \cap \mathbf{Q}_I)(K).$$

Since $\mathbf{A}_I$ normalizes $\mathbf{Q}_I$, there exists a finite subset $\tilde{F}_I \subset \mathbf{Q}_I(K)$ such that

$$F_I^{-1}(\mathbf{Q}_I \cap \Gamma)(\mathbf{A}_I \cap \Gamma) \subset F_I^{-1}(\mathbf{A}_I \cap \Gamma)(\mathbf{Q}_I \cap \Gamma) \subset (\mathbf{A}_I \cap \Gamma)\tilde{F}_I^{-1}(\mathbf{Q}_I \cap \Gamma). \tag{7.7}$$

We put

$$\Lambda(I) := \tilde{F}_I^{-1}(Q_I \cap \Gamma) \subset \mathbf{Q}_I(K).$$

Note that $\mathbf{P}_\emptyset = \mathbf{Q}_\emptyset = \mathbf{G}$, $\mathbf{A}_\emptyset = \mathbf{A}$, $F_\emptyset = F = \tilde{F}_\emptyset$, and $\Lambda(\emptyset) = F^{-1}\Gamma$.

Lemma 7.8. *For $j \in \{1, \dots, r\}$ and $I \subset \{1, \dots, r\} - \{j\}$, there is a finite subset $E \subset \mathbf{P}(K)$ such that $\Lambda(I \cup \{j\})\Lambda(I) \subset E\Lambda(I)$.*

Proof. Same as Lemma 3.6 in [30]. □

Denote by $\mathcal{T}$ the set of all l-ordered tuples of integers $1 \le i_1, \cdots, i_l \le r$ for $1 \le l \le r$. For $I = (i_1, \cdots, i_l) \in \mathcal{T}$, there exists a finite subset $L(I) \subset \mathbf{G}(K)$ such that

$$\Lambda(\{i_1, \dots, i_l\}) \cdots \Lambda(\{i_1\})\Lambda(\emptyset) = L(I)\Gamma.$$

We set $L(\emptyset) = \{e\}$.

An l-tuple $((i_1, \lambda_1), \dots, (i_l, \lambda_l))$ is called an *admissible* sequence of length l if $i_1, \dots, i_l \subset \{1, \cdots, r\}$ are distinct and $\lambda_1, \dots, \lambda_l \in \mathbf{G}(K)$ satisfy $\lambda_j \lambda_{j-1}^{-1} \in \Lambda(\{i_1, \dots, i_{j-1}\})$ for all $j = 1, \dots, l$ (here we set $\lambda_0 = e$). For an admissible sequence ξ of length l, we denote by $\mathcal{C}(\xi)$ the set of all pairs (i, λ) where $1 \le i \le r$ and $\lambda \in \mathbf{G}(K)$ for which there exists an admissible sequence η of length $l+1$ extending ξ and containing (i, λ) as the last term. The support of ξ, denoted by $\operatorname{supp}(\xi)$, is defined to be the emptyset if $l = 0$; and otherwise the set $\{(i_1, \lambda_1), \dots, (i_l, \lambda_l)\}$ if $\xi = ((i_1, \lambda_1), \dots, (i_l, \lambda_l))$.

For any $0 < a < b$, $\alpha > 0$ and any admissible sequence ξ, we define

$$\begin{aligned}(7.9)\quad \mathcal{W}_{\alpha,a,b}(\xi) = \{g \in G : \Delta_j(\lambda g) \ge \alpha, \forall (j, \lambda) \in \mathcal{C}(\xi) \\ \text{and } a \le \Delta_i(\lambda g) \le b, \forall (i, \lambda) \in \operatorname{supp}(\xi)\}.\end{aligned}$$

The same proof of [23, Prop. 1.8] shows:

Lemma 7.10. *For any admissible sequence $\xi = \{(i_1, \lambda_1), \dots, (i_l, \lambda_l)\}$ of length $l \ge 1$, we have*

$$\mathcal{W}_{\alpha,a,b}(\xi) = \mathcal{W}_{\alpha,a,b}(I, \lambda_l)$$

where $I = \{i_1, \dots, i_l\}$ and

$$\begin{aligned}(7.11)\quad W_{\alpha,a,b}(I, \lambda) := \{g \in G : \Delta_j(\theta \lambda g) \ge \alpha, \forall j \notin I, \forall \theta \in \Lambda(I) \\ \textit{and } a \le \Delta_i(\lambda g) \le b, \forall i \in I\}.\end{aligned}$$

Note that λ_l arises in the above way if and only if $\lambda_l \in L(I)\Gamma$.

For any subset $I \subset \{1, \dots, r\}$, note that $W_\infty \cap \mathbf{H}_{I,\infty}$ is a maximal compact subgroup of $\mathbf{H}_{I,\infty}$. Set $J := \{1, \dots, r\} \setminus I$. By reduction theory, there exist a compact subset $C_I \subset \mathbf{U}_{J,\infty} \cap \mathbf{H}_{I,\infty}$, a finite subset $E_I \subset \mathbf{H}_I(K)$, and $t_I > 0$ such that

$$H_I = (\Gamma \cap H_I) E_I \left(C_I \Omega_I (W_\infty \cap \mathbf{H}_{I,\infty}) \times (W_f \cap \mathbf{H}_{I,S_f}) \right)$$

where

$$\Omega_I = \{(s_v)_{v \in S_\infty} : s_v \in A^0_{J,v},\ 0 < \alpha_j(s_v) \le t_I, \quad \forall j \in J,\ \forall v \in S_\infty\}.$$

We enlarge the finite subset F_I, chosen above, so that

$$(\Gamma \cap \mathbf{H}_I) E_I (\Gamma \cap \mathbf{U}_I) \subset (\Gamma \cap \mathbf{Q}_I) F_I.$$

We have $U_I = (\Gamma \cap U_I)D'_I$ for some $D'_I = D_I \times (\mathbf{U}_{I,S_f} \cap W_f)$ with $D_I \subset \mathbf{U}_{I,\infty}$. Then for $C'_I = C_I \Omega_I (W_\infty \cap \mathbf{H}_{I,\infty}) \times (W_f \cap \mathbf{H}_{I,S_f})$,

$$\begin{aligned} Q_I &= U_I H_I = U_I(\Gamma \cap H_I)E_I \\ &= (\Gamma \cap H_I)E_I U_I C' = (\Gamma \cap H_I)E_I(\Gamma \cap U_I)D'_I C'_I \\ &= (\Gamma \cap \mathbf{Q}_I)F_I(\Psi_I \Omega_I (W_\infty \cap \mathbf{Q}_{I,\infty}) \times (W_f \cap \mathbf{Q}_{I,S_f})) \end{aligned} \tag{7.12}$$

where Ψ_I is a compact subset of $(\mathbf{Q}_I \cap \mathbf{Q}_J)_\infty$.

In the proof of the next proposition, we use the following lemma, which follows from continuity of the norms:

Lemma 7.13. *Let $1 \le i \le r$ and C be a compact subset of G. Then for some $c > 0$,*

$$\Delta_i(gx) \ge c \cdot \Delta_i(g) \quad \text{for all } x \in C \text{ and } g \in G.$$

For $g = (g_v)_{v \in S_\infty} \in \mathbf{G}_\infty$, set

$$d_i(g) := \prod_{v \in S_\infty} \|w_i g_v\|_v.$$

Proposition 7.14. *For any admissible sequence ξ of length $0 \le l \le r$ and positive $a < b$ and $\alpha > 0$, the set $\Gamma \backslash \Gamma \mathcal{W}_{\alpha,a,b}(\xi)$ is relatively compact.*

Proof. For simplicity, write $\mathcal{W} = \mathcal{W}_{\alpha,a,b}(\xi)$.

Let ξ be the empty sequence. Then

$$\mathcal{W} = \{g \in G : \Delta_j(\lambda g) \ge \alpha, \quad \forall j,\ \forall \lambda \in \Lambda(\emptyset)\}.$$

Every $g \in \mathcal{W}$ has a decomposition $g = (\lambda, \lambda)(\psi \omega k_\infty, k_f)$, $\psi \in \Psi_\emptyset$, $w \in \Omega_\emptyset$, $k_\infty \in W_\infty$ and $k_f \in W_f$ as in (7.12) where $\lambda \in \Gamma \tilde{F}_\emptyset = \Lambda(\emptyset)^{-1}$. Hence,

$$\Delta_j(\psi \omega k_\infty, k_f) = d_j(\psi \omega k_\infty) = d_j(\omega) \ge c\alpha$$

where $c > 0$ is a constant depending on $\Psi_\emptyset$. Since $d_j(\omega) = \prod_{v \in S_\infty} |\alpha_j(\omega)|_v^{m_j}$, we have

$$\mathcal{W} \subset \Gamma \tilde{F}_\emptyset (\Psi_\emptyset \tilde{\Omega}_\emptyset W_\infty \times W_f)$$

where

$$\tilde{\Omega}_\emptyset = \{(\omega_v) \in A^0_{\emptyset,\infty} :\ (c\alpha)^{1/m_j} \le \prod_{v \in S_\infty} |\alpha_j(\omega_v)|_v \le t_\emptyset, \quad \forall j\}.$$

This shows that $\Gamma \backslash \Gamma \mathcal{W}$ is relatively compact in this case.

Now let $\xi = ((i_1, \lambda_1), \ldots, (i_l, \lambda_l))$ be an admissible sequence of length $l \ge 1$ and $I(j) = \{i_1, \ldots, i_j\}$. We claim that there exist compact subsets $M_1, \ldots, M_l$ such that for any $j = 1, \ldots, l$ and $g \in \mathcal{W}$,

$$\lambda_j \mathcal{W} \subset (S_{i_j} \cap \Gamma) Q_{I(j)} M_j^{-1}.$$

We prove the claim by induction. For $j = 1$, we can take $M_1 = M_0^{-1}$ where M_0 is as in Lemma 7.6 with $I = \emptyset$ and $j = i_1$. Suppose that the sets $M_1, \ldots, M_j$ have been found. By Lemma 7.13, there is $c \in (0,1)$ such that

$$\Delta_{i_{j+1}}(hx) \ge c \cdot \Delta_{i_{j+1}}(h)$$

for all $x \in M_j \cup M_j^{-1}$ and $h \in G$. By Lemma 7.6, there exists a compact set M_0 such that

$$\{g \in Q_{I(j)} : \Delta_{i_{j+1}}(g) \in [ca, c^{-1}b]\} \subset (A_{i_{j+1}} \cap \Gamma) Q_{I(j+1)} M_0. \tag{7.15}$$

Let $M_{j+1} = M_j M_0^{-1}$. For $g \in \mathcal{W}$, there exists $m_j \in M_j$ such that $\lambda_j g m_j \in (A_{i_j} \cap \Gamma) Q_{I(j)}$. Since $\lambda_{j+1}\lambda_j^{-1} \in Q_{I(j)}$ and $A_{i_j} \cap \Gamma$ normalizes $Q_{I(j)}$, we have

$$\lambda_{j+1} g m_j \in (A_{i_j} \cap \Gamma) Q_{I(j)}.$$

Hence for some $\gamma_j \in A_{i_j} \cap \Gamma$, $\gamma_j \lambda_{j+1} g m_j \in Q_{I(j)}$. Since $\alpha_{i_{j+1}}(\gamma_j) = 1$,

$$\Delta_{i_{j+1}}(\gamma_j \lambda_{j+1} g m_j) = \Delta_{i_{j+1}}(\lambda_{j+1} g m_j),$$

and

$$ca \le c\Delta_{i_{j+1}}(\lambda_{j+1} g) \le \Delta_{i_{j+1}}(\lambda_{j+1} g m_j) \le c^{-1}\Delta_{i_{j+1}}(\lambda_{j+1} g) \le c^{-1} b.$$

By (7.15), there exists $m_0 \in M_0$ such that

$$\gamma_j \lambda_{j+1} g m_j \in (A_{i_{j+1}} \cap \Gamma) Q_{I(j+1)} m_0.$$

So for $m_{j+1} = m_j m_0^{-1}$, we have

$$\lambda_{j+1} g m_{j+1} \in (A_{i_{j+1}} \cap \Gamma) Q_{I_{j+1}}$$

proving the claim.

By the above claim,

$$\lambda_l \mathcal{W} \subset (A_{i_l} \cap \Gamma) Q_I M_l^{-1} \tag{7.16}$$

where $I := \{i_1, \dots, i_l\}$. If $I = \{1, \dots, r\}$, then $\Gamma \cap Q_I \backslash Q_I$ is compact. Hence $\Gamma\backslash\Gamma\lambda_r^{-1}(A_{i_r} \cap \Gamma) Q_I M_r$ is compact, which implies that $\Gamma\backslash\Gamma\mathcal{W}$ is relatively compact.

Now suppose I is a proper subset. Then by (7.12) and (7.16), for $g \in \mathcal{W}$,

$$\delta\gamma\lambda_l g m = (\psi\omega k_\infty, k_f) \in \Psi_I \Omega_I W_\infty \times W_f$$

for some $\delta \in F_I^{-1}(Q_I \cap \Gamma)$, $\gamma \in A_{i_l} \cap \Gamma$, and $m \in M_l$. Hence, for every $j \notin I$,

$$|\alpha_j(\omega)|_\infty^{m_j} = d_j(\psi\omega k_\infty) = \Delta_j(\delta\gamma\lambda_l g m).$$

By (7.7), $\delta\gamma = \gamma'\theta$ where $\gamma' \in A_I \cap \Gamma$ and $\theta \in \Lambda(I)$. Since A_I acts trivially on the vectors w_j, $j \notin I$, we have

$$\Delta_j(\delta\gamma\lambda_l g m) = \Delta_j(\theta\lambda_l g m).$$

By Lemma 7.10, we have $\Delta_j(\theta\lambda_l g) \ge \alpha$. Hence, by Lemma 7.13, there exists $\beta > 0$, depending only on α and M_l, such that $\Delta_j(\theta\lambda_l g m) \ge \beta$. This shows that $\prod_{v \in S_\infty} |\alpha_j(\omega)|_v^{m_j} \ge \beta$ for $j \notin I$. Therefore, if we set

$$\tilde{\Omega}_I = \{\omega \in A_J : \beta^{1/m_j} \le \prod_{v \in S_\infty} |\alpha_j(\omega)|_v \le t_I,\ j \in J\}$$

where J is the complement of I, then

$$\mathcal{W} \subset \lambda_l^{-1}\Gamma F_I(\Psi_I \tilde{\Omega}_I W_\infty \times W_f) M_l^{-1},$$

and the later set is compact modulo Γ. $\square$

7.5. Proof of Theorem 7.3. Fix $v \in S$ and a vector space K_v^N. We define $\mathcal{P}_d^*$ is the set of polynomial maps $K_v \to K_v^N$ (res. $\mathbb{R} \to \mathbb{C}^N$) of degree less than d if $K_v \neq \mathbb{C}$, (resp. $K_v = \mathbb{C}$). We write $f \in \mathcal{P}_d^*$ as $(f_1, \cdots, f_N)$. We set

$$\|f(x)\|_v = \max_i |f_i(x)|_v.$$

Recall that by an interval of a non-archimedean local field, we mean a subset of K_v of the form $I = \{t \in K_v : |t - t_0|_v < \delta\}$. There is the unique k such that $q_v^k < \delta \leq q_v^{k+1}$. Then $2q_v^k$ is called the diameter of I. In the case when $K_v = \mathbb{C}$, as $\mathcal{P}_d^*$ consists of polynomial maps defined in $\mathbb{R}$, the intervals are understood as subsets of $\mathbb{R}$ and the meaning of diameter is then clear.

Lemma 7.17. *Given $M > 1$, there exists $\eta \in (0,1)$ such that for any $f \in \mathcal{P}_d^*$ and any interval I, there exists a subinterval $I_0 \subset I$ with $\operatorname{diam}(I_0) \geq \eta \cdot \operatorname{diam}(I)$ satisfying*

$$\sup_I \|f\|_v \leq M \cdot \inf_{I_0} \|f\|_v$$

Proof. For the archimedean version of this lemma, see [30, Corollary 2.18]. Let v be non-archimedean. Since I can be expressed as a disjoint union $\cup J$ of intervals so that on each interval J, there is i such that $\|f(x)\|_v = |f_i(x)|_v$ for all $x \in J$. Therefore it suffices to prove the above claim for $N = 1$.

There exists $t_0 \in I$ such that $\sup_I |f|_v = |f(t_0)|_v$. It follows from the Lagrange interpolation formula that there exists $M_i > 0$, depending on I, such that

$$\sup_I |f^{(i)}|_v \leq M_i \cdot \sup_I |f|_v \quad \text{for all } f \in \mathcal{P}_d^*$$

where $f^{(i)}$ is the i-th derivative of f. Let δ denote the diameter of I. Let $n \in \mathbb{N}$ be big enough so that

$$M^{-1} < 1 - \sum_{i=1}^{d} q_v^{-ni} \delta^i \frac{M_i}{i!}$$

and $I_0 := \{t : |t - t_0|_v \leq q_v^{-n}\delta\}$ is contained in I. Then using the Taylor formula, we deduce that for every $t \in I_0$,

$$|f(t)|_v \geq |f(t_0)|_v - \left(\sum_{i=1}^{d} (q_v^{-n}\delta)^i \frac{M_i}{i!} \right) \sup_I |f|_v = M^{-1} \sup_I |f|_v.$$

Hence $\sup_I |f|_v \leq M \inf_{I_0} |f|_v$ and the diameter of I_0 is $2q_v^{-n}\delta$. Hence this proves the claim. $\square$

Lemma 7.18. *Given $\eta \in (0,1)$, there exists $M > 1$ such that for any interval I and any subinterval $I_0 \subset I$ with $\operatorname{diam}(I_0) \geq \eta \cdot \operatorname{diam}(I)$,*

$$\sup_I \|f\|_v \leq M \cdot \sup_{I_0} \|f\|_v \quad \textit{for all } f \in \mathcal{P}_d^*. \tag{7.19}$$

Proof. For the archimedean case, this is proved in [30, Coro. 2.17]. We give a proof in the non-archimedean case. By Lemmas 2.1 and 2.4 in [39], there exist $C, \alpha > 0$ such that for any $\epsilon > 0$ and any interval I

$$\theta_v\{x \in I : \|f(x)\|_v < \epsilon\} \le C \left(\frac{\epsilon}{\sup_I \|f\|_v} \right)^\alpha \theta_v(I). \tag{7.20}$$

Choose $n \in \mathbb{N}$ and $M > 1$ satisfying $\eta > q_v^{-n}$ and $M^\alpha > C q_v^n$. Then for any subinterval $I_0 \subset I$ with $\mathrm{diam}(I_0) \ge \eta \cdot \mathrm{diam}(I)$, $\theta_v(I_0) > q_v^{-n} \theta_v(I)$. Applying (7.20) with $\epsilon = M^{-1} \cdot \sup_I \|f\|_v$, we deduce that

$$\theta_v \left\{ x \in I : M \cdot \|f(x)\|_v < \sup_I \|f\|_v \right\} \le q_v^{-n} \theta_v(I).$$

Therefore there exists $x \in I_0$ such that

$$M \cdot \|f(x)\|_v \ge \sup_I \|f\|_v.$$

This proves the lemma. □

Proposition 7.21. *There exists $M > 1$ such that for any $\alpha > 0$, any interval I, and a subfamily $\mathcal{F} \subset \mathcal{P}_d^*$ satisfying:*

(i) *For any $t_0 \in I$, $\#\{\phi_f(t) := \|f(t)\|_v : f \in \mathcal{F}, \|f(t_0)\|_v < \alpha\} < \infty$,*

(ii) *For any $f \in \mathcal{F}$, $\sup_{t \in I} \phi_f(t) \ge \alpha$,*

one of the followings holds:

(a) *There is $t_0 \in I$ such that*

$$\phi_f(t_0) \ge \alpha \quad \textit{for all } f \in \mathcal{F}.$$

(b) *There exist an interval $I_0 \subset B$ and $f_0 \in \mathcal{F}$ such that*

$$\phi_{f_0}(I_0) \subset [\alpha/M, \alpha M] \quad \textit{and} \quad \sup_{I_0} \phi_f \ge \alpha/M \quad \textit{for all } f \in \mathcal{F}.$$

Proof. Pick $t_0 \in I$ and suppose that (a) fails, that is,

$$\mathcal{F}_1 = \{\phi_f : f \in \mathcal{F}, \ \|f(t_0)\|_v < \alpha\} \ne \emptyset.$$

By (i), the set $\mathcal{F}_1$ is finite. By Lemma 7.18, there exists $M_1 > 1$ such that for every $\phi_f \in \mathcal{F}_1$ and $k \in \mathbb{Z}$,

$$\sup_{|t - t_0|_v \le q_v^{k+1}} \|f(t)\|_v \le M_1 \cdot \sup_{|t - t_0|_v \le q_v^k} \|f(t)\|_v.$$

We set $E = \{t : |t - t_0|_v \le q_v^k\}$, where k is the smallest integer such that $\sup_E \|f\|_v \ge \alpha$ for all $t \mapsto \|f(t)\|_v \in \mathcal{F}_1$. Such k exists by (ii). Then there is $\phi_{f_0} \in \mathcal{F}_1$ such that $\sup_E \|f_0\|_v \le \alpha M_1$. By Lemma 7.17, there exists a subinterval $I_0 \subset E$ such that $\mathrm{diam}(I_0) \ge \eta \cdot \mathrm{diam}(I)$ and

$$\inf_{I_0} \|f_0\|_v \ge \alpha / M_1.$$

By Lemma 7.18, there exists $M_2 > 1$ such that

$$\sup_{I_0} \|f\|_v \ge \alpha / M_2 \quad \text{for all } f \in \mathcal{F}.$$

This proves the proposition. □

Proof of Theorem 7.3. Suppose that condition (1) in the theorem fails. We will show that for some $I \in \mathcal{T}$, $\lambda \in L(I)\Gamma$, $\alpha_I > 0$, and $0 < a_I < b_I$ depending only on α,

$$xu(I_T) \cap \Gamma\backslash\Gamma W_{\alpha_I,a_I,b_I}(I,\lambda) \neq \emptyset. \tag{7.22}$$

By Proposition 7.14, this implies the theorem.

We construct inductively increasing sequence of tuples $J \in \mathcal{T}$, elements $\lambda \in L(I)\Gamma$, constants $0 < a_I < b_I$, $\alpha_I > 0$, and intervals $B \subset I_T$ satisfying the following properties:

(A) $\Delta_i(\lambda u(B)) \subset [a_I, b_I]$ for all $i \in I$.

(B) $\sup_B \phi \geq \alpha_I$ for all $\phi \in \mathcal{F}(I,\lambda)$ where $\mathcal{F}(I,\lambda)$ is the family of functions $K_v \to \mathbb{R}^+$ of the form

$$\phi(t) = \Delta_j(\theta\lambda u(t))$$

where $\theta \in \Lambda(I)$, $j \notin I$ and $u \in \mathcal{P}_d$

Note that for some fixed constant $d_j > 0$,

$$\phi(t) = (\prod_{w\in S\backslash v} \|w_j\theta\lambda\|_w)\cdot\|f(t)\|_v$$

and $f(t) := w_j\theta\lambda u(t) \in \mathcal{P}^*_{d_j}$ where $\mathcal{P}^*_{d_j}$ are polynomial maps into the vector space $\wedge^{\dim\mathfrak{u}_i}\mathfrak{g}(K_v)$ of degree at most d_j.

We start with $I = \emptyset$, $\lambda = e$, $\alpha_\emptyset = \alpha$, $B = I_T$ which satisfy (A) and (B) because (1) fails.

Property (B) implies that $\mathcal{F}(I,\lambda)$ satisfies condition (ii) of Proposition 7.21. We claim that $\mathcal{F}(I,\lambda)$ satisfies condition (i) as well, that is, there are only finitely many $\phi \in \mathcal{F}(I,\lambda)$ such that $\phi(t) < \alpha$ for a fixed α and t. Fix $\beta > 0$ and any rational vector w with co-prime entries in $\mathcal{O}$. For any $\gamma \in \Gamma$, $\gamma w = \alpha w'$ where $\alpha \in \mathcal{O}^*_S$ (here $\mathcal{O}^*_S$ denotes the unit group) and the entries of w' are relatively prime to each other in $\mathcal{O}$. Since $\prod_{v\in S}|\alpha|_v = 1$ for $\alpha \in \mathcal{O}^*_S$, the claim follows from the fact that $\tilde{F}_I$ is finite and that there are only finitely many vectors w' with coefficients in $\mathcal{O}$ whose entries are relative prime to each other and $\prod_{v\in S}\|w'\|_v < \beta$.

By Proposition 7.21, one of the following holds:

(a) For some $t_0 \in B$,

$$\Delta_j(\theta\lambda u(t_0)) \geq \alpha_I$$

for all $\theta \in \Lambda(I)$, $j \notin I$, and $u \in \mathcal{P}_d$. In this case, using (A), we have $\Gamma u(t_0) \in W_{\alpha_I,a_I,b_I}(I)$ and hence we stop the process.

(b) There exist $j_0 \notin I$, $\theta_0 \in \Lambda(I)$ and an interval $B_0 \subset B$ such that

$$\Delta_{j_0}(\theta_0\lambda u(B_0)) \subset [\alpha_I/M, \alpha_I M]$$

and for all $\theta \in \Lambda(I)$ and $j \notin I$,

$$\sup_{t\in B_0}\Delta_j(\theta\lambda u(t)) \geq \alpha_I/M.$$

In case (b), we set $I_1 = I \cup \{j_0\}$ and $\lambda_1 = \theta_0\lambda$. Then since $\Delta_i(\theta_0 g) = \Delta_i(g)$ for all $i \in I$ and $g \in \mathbf{G}_S$, condition (A) is satisfied for suitable $0 < a_{I_1} < b_{I_1}$ and B_0. By Lemma 7.8, there is a finite subset $E \subset \mathbf{P}(K)$ such that for any $\theta \in \Lambda(I \cup \{j_0\})$, there exists $\theta' \in \Lambda(I)$ and $x \in E$ such that $\theta\theta_0 = x\theta'$, Hence this implies for any $j \notin I_1$,

$$\sup_{t \in B_0} \Delta_j(\theta\lambda_1 u(t)) = \sup_{t \in B_0} \Delta_j(x\theta'\lambda u(t)) = \sup_{t \in B_0} \Delta_j(x)\Delta_j(\theta'\lambda u(t)) \geq \beta\alpha_I/M$$

where $\beta = \min_E \Delta_j > 0$ depends only on I and j_0. Hence Condition (B) is satisfied for the family $\mathcal{F}(I_1, \lambda_1)$, $\alpha_{I_1} = \beta\alpha_I/M$ and B_0. This completes the description of the inductive step. Since the cardinality of I increases, this process must stop after finitely many steps, and we deduce that (7.22) holds. □

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A. Appendix: Symmetric homogeneous spaces over number fields with finitely many orbits (by Mikhail Borovoi)

Let G be a connected linear algebraic group over a field K of characteristic 0. Let $H \subset G$ be a connected K-subgroup. Let $X = H\backslash G$ be the corresponding homogeneous space. The group $G(K)$ acts on $X(K)$ on the right. We consider the set of orbits $X(K)/G(K)$.

We fix an algebraic closure $\overline{K}$ of K and write $\overline{G} = G \times_K \overline{K}$, $\overline{H} = H \times_K \overline{K}$. We say that (G, H) is a *symmetric pair* if G is semisimple and $\overline{H}$ is the subgroup of invariants $\overline{G}^\theta$ of some involutive automorphism θ of $\overline{G}$. In this case we say also that H is a *symmetric subgroup of* G and that X is a *symmetric space of* G.

Let K be a number field. In this Appendix we give a list of all symmetric pairs (G, H) over K with adjoint absolutely simple G and semisimple H, such that the set of orbits $X(K)/G(K)$ is finite (Theorem A.5.2). The assumption that $X(K)/G(K)$ is finite is equivalent to the assumption that $G(K_v)$ acts on $X(K_v)$ transitively for almost all places v of K.

The plan of the Appendix is as follows. In Section A.1 we consider a connected K-group G and a connected K-subgroup $H \subset G$ over a number field K. We prove that the set of K-orbits $X(K)/G(K)$ is finite if and only if the set of adelic orbits $X(\mathbb{A})/G(\mathbb{A})$ is finite (here $\mathbb{A}$ is the adèle ring of K). We notice that the set $X(\mathbb{A})/G(\mathbb{A})$ is finite if and only if $\#X(K_v)/G(K_v) = 1$ for almost all v. We give a criterion when $\#X(K_v)/G(K_v) = 1$ for almost all v in terms of the induced homomorphism $\pi_1(\overline{H}) \to \pi_1(\overline{G})$, where π_1 is the algebraic fundamental group introduced in [Bo98, Sect. 1]. These results constitute Theorem A.1.2.

In Section A.2 we give corollaries of Theorem A.1.2. We show that the finiteness of $X(K)/G(K)$ is related to the following condition: the homomorphism $\pi_1(\overline{H}) \to \pi_1(\overline{G})$ is injective.

In Section A.3 we assume that K is algebraically closed and that both G and H are semisimple. We write G^{sc} and H^{sc} for the universal coverings. We show that the homomorphism $\pi_1(H) \to \pi_1(G)$ is injective if and only if the subgroup $H' := \mathrm{im}[H^{\mathrm{sc}} \to G^{\mathrm{sc}}] \subset G^{\mathrm{sc}}$ is simply connected.

In Section A.4 we again assume that K is algebraically closed. We give a list of all symmetric pairs (G, H) over K with simply connected absolutely almost simple G and semisimple H, such that H is simply connected (Theorem A.4.1).

In Section A.5 K is a number field and (G, H) is a symmetric pair over K, such that G is an absolutely almost simple K-group and H is semisimple K-subgroup. We consider two cases: either G is simply connected or G is adjoint. We give a list of all such symmetric pairs (G, H) with finite $X(K)/G(K)$ (Theorems A.5.1 and A.5.2). We show that for such (G, H) with finite set of K-orbits $X(K)/G(K)$, this set of K-orbits is related to the set of "real" orbits (Theorem A.5.3). In particular, if $K = \mathbb{Q}$, then any $G(\mathbb{R})$-orbit in $X(\mathbb{R})$ contains exactly one orbit of $G(\mathbb{Q})$ in $X(\mathbb{Q})$.

In Section A.6 (Addendum) we give examples of homogeneous spaces $X = H\backslash G$ (symmetric or not, with G absolutely almost simple or not), satisfying assumptions (i–iii) of Theorem 1.1 but not covered by Theorems A.5.1 and A.5.2.

The author is very grateful to È.B. Vinberg and A.G. Elashvili for their invaluable help in proving Theorem A.4.1.

A.1. Orbits over a number field and over adeles.

A.1.1. Let K be a number field, and let $\overline{K}$ be a fixed algebraic closure of K. Let G be a connected linear K-group. Let $H \subset G$ be a connected K-subgroup. Set $X = H\backslash G$, it is a right homogeneous space of G. We would like to investigate, when the set of orbits $X(K)/G(K)$ of $G(K)$ in $X(K)$ is finite.

We write $i\colon H \hookrightarrow G$ for the inclusion map. We consider the induced morphism of $\mathrm{Gal}(\overline{K}/K)$-modules

$$i_*\colon \pi_1(\overline{H}) \to \pi_1(\overline{G}),$$

where π_1 is the algebraic fundamental group introduced in [Bo98, Sect. 1], see also [CT06, §6].

Let $\mathfrak{g}$ denote the image of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{Aut}\ \pi_1(\overline{H}) \times \mathrm{Aut}\ \pi_1(\overline{G})$; it is a finite group. Let $L \subset \overline{K}$ be the field corresponding to the subgroup $\ker[\mathrm{Gal}(\overline{K}/K) \to \mathfrak{g}]$ of $\mathrm{Gal}(\overline{K}/K)$, then L/K is a finite Galois extension with Galois group $\mathfrak{g}$. For any place v of K, let $\mathfrak{g}_v \subset \mathfrak{g}$ denote a decomposition group of v (defined up to conjugacy). For almost all v the group $\mathfrak{g}_v$ is cyclic.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subgroup. We shall consider the group of coinvariants $\pi_1(\overline{H})_{\mathfrak{h}}$ and the subgroup of torsion elements $(\pi_1(\overline{H})_{\mathfrak{h}})_{\mathrm{tors}}$. We shall also consider the induced map

$$i_*\colon (\pi_1(\overline{H})_{\mathfrak{h}})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{h}})_{\mathrm{tors}}\ .$$

We write R for the set of all places of K. We write R_f (resp. R_∞) for the set of all finite (resp. infinite) places of K. We write K_v for the completion of K at $v \in R$, and $\mathbb{A}$ for the adèle ring of K.

Theorem A.1.2. *Let G be a connected linear algebraic group over a number field K. Let $H \subset G$ be a connected K-subgroup. Set $X = H\backslash G$. Then the following four conditions are equivalent:*
(i) The set of K-orbits $X(K)/G(K)$ is finite.
(ii) The set of adelic orbits $X(\mathbb{A})/G(\mathbb{A})$ is finite.
(iii) We have $\#X(K_v)/G(K_v) = 1$ for almost all places v of K.
(iv) For any **cyclic** *subgroup $\mathfrak{h} \subset \mathfrak{g}$ the map*

$$(\pi_1(\overline{H})_{\mathfrak{h}})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{h}})_{\mathrm{tors}}$$

is injective.

Proof. Write

$$\ker(K, H \to G) = \ker[H^1(K, H) \to H^1(K, G)],$$
$$\ker(K_v, H \to G) = \ker[H^1(K_v, H) \to H^1(K_v, G)].$$

We have canonical bijections

$$X(K)/G(K) \xrightarrow{\sim} \ker(K, H \to G),$$
$$X(K_v)/G(K_v) \xrightarrow{\sim} \ker(K_v, H \to G),$$

see [Se65, Ch. I §5.4, Cor. 1 of Prop. 36].

In [Bo98, Sections 2,3] we defined, for any connected group H over a field K of characteristic 0, an abelian group $H^1_{\mathrm{ab}}(K, H)$ and the abelianization map

$$\mathrm{ab}^1 \colon H^1(K, H) \to H^1_{\mathrm{ab}}(K, H)$$

(see also [CT06, Prop. 8.3] in any characteristic). Both $H^1_{\mathrm{ab}}(K, H)$ and ab^1 are functorial in H.

Now let K be a number field. Set $\Gamma = \mathrm{Gal}(\overline{K}/K)$, $\Gamma_v = \mathrm{Gal}(\overline{K}_v/K_v)$. We regard Γ_v as a subgroup of Γ.

For $v \in R_f$ we defined in [Bo98, Prop. 4.1(i)] a canonical isomorphism $\lambda_v \colon H^1_{\mathrm{ab}}(K_v, H) \xrightarrow{\sim} (\pi_1(\overline{H})_{\Gamma_v})_{\mathrm{tors}}$. Here we set

$$\lambda'_v = \lambda_v \colon H^1_{\mathrm{ab}}(K_v, H) \xrightarrow{\sim} (\pi_1(\overline{H})_{\Gamma_v})_{\mathrm{tors}} .$$

For $v \in R_\infty$ we defined in [Bo98, Prop. 4.2] a canonical isomorphism

$$\lambda_v \colon H^1_{\mathrm{ab}}(K_v, H) \xrightarrow{\sim} H^{-1}(\Gamma_v, \pi_1(\overline{H})).$$

Here we define a homomorphism λ'_v as the composition

$$\lambda'_v \colon H^1_{\mathrm{ab}}(K_v, H) \xrightarrow{\lambda_v} H^{-1}(\Gamma_v, \pi_1(\overline{H})) \hookrightarrow (\pi_1(\overline{H})_{\Gamma_v})_{\mathrm{tors}} .$$

For any $v \in R$ we define the Kottwitz map β_v as the composition

$$\beta_v \colon H^1(K_v, H) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(K_v, H) \xrightarrow{\lambda'_v} (\pi_1(\overline{H})_{\Gamma_v})_{\mathrm{tors}} .$$

This map β_v is functorial in H. Note that for $v \in R_f$ the maps β_v and $\mathrm{ab}^1 \colon H^1(K_v, H) \to H^1_{\mathrm{ab}}(K_v, H)$ are bijections. Thus for $v \in R_f$ we have a canonical and functorial in H bijection $H^1(K_v, H) \xrightarrow{\sim} (\pi_1(\overline{H})_{\Gamma_v})_{\mathrm{tors}}$.

For any $v \in R$ we define a map μ_v as the composition

$$\mu_v \colon H^1_{\mathrm{ab}}(K_v, H) \xrightarrow{\lambda'_v} (\pi_1(\overline{H})_{\Gamma_v})_{\mathrm{tors}} \xrightarrow{\mathrm{cor}_v} (\pi_1(\overline{H})_{\Gamma})_{\mathrm{tors}} , \tag{A.1}$$

where cor_v is the obvious map.

We prove that (ii)⇔(iii). Since H is connected, using Lang's theorem and Hensel's lemma, we can prove easily that

$$X(\mathbb{A})/G(\mathbb{A}) = \bigoplus_v X(K_v)/G(K_v). \tag{A.2}$$

Here $\bigoplus$ means that we take the families of local orbits $(o_v \in X(K_v)/G(K_v))_{v \in R}$ with $o_v = x_0 \cdot G(K_v)$ for almost all v, where $x_0 \in X(K)$ is the image of the neutral element $e \in G(K)$. For any place v of K the set $X(K_v)/G(K_v)$ is finite (because $H^1(K_v, H)$ is finite, see [Se65, Ch. III §4.4, Thm. 5 and Ch. III §4.5, Thm. 6]). It follows that $X(\mathbb{A})/G(\mathbb{A})$ is finite if and only if $\#X(K_v)/G(K_v) = 1$ for almost all v. Thus (ii)⇔(iii).

We prove that (iv)⇒(iii). For almost all v the group $\mathfrak{g}_v$ is cyclic, hence by the assumption (iv) we have for such v

$$\ker[(\pi_1(\overline{H})_{\mathfrak{g}_v})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{g}_v})_{\mathrm{tors}}] = 0.$$

But for $v \in R_f$ we have canonical bijections

$$X(K_v)/G(K_v) \xrightarrow{\sim} \ker[H^1(K_v, H) \to H^1(K_v, G)] \xrightarrow{\sim} \ker[(\pi_1(\overline{H})_{\mathfrak{g}_v})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{g}_v})_{\mathrm{tors}}].$$

Thus for almost all v we have $\#(X(K_v)/G(K_v)) = 1$. This proves that (iv)⇒(iii).

We prove that (iii)⇒(iv). Indeed, assume that (iv) does not hold, i.e. there exists a cyclic subgroup $\mathfrak{h} \subset \mathfrak{g}$ such that

$$\ker[(\pi_1(\overline{H})_{\mathfrak{h}})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{h}})_{\mathrm{tors}}] \neq 0.$$

Then by Chebotarev's density theorem there exist infinitely many finite places v of K such that $\mathfrak{g}_v$ is conjugate to $\mathfrak{h}$. For all these places v we have

$$\ker[(\pi_1(\overline{H})_{\mathfrak{g}_v})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{g}_v})_{\mathrm{tors}}] \neq 0,$$

hence $\#(X(K_v)/G(K_v)) > 1$, which contradicts to (iii). Thus (iii)⇒(iv).

We prove that (ii)⇒(i). Indeed, by Borel's theorem

$$\ker[H^1(K, H) \to \prod_v H^1(K_v, H)]$$

is finite, see [Se65, Ch. III §4.6, Thm. 7]. It follows that all the fibers of the localization map

$$X(K)/G(K) \to X(\mathbb{A})/G(\mathbb{A})$$

are finite, Hence if the set $X(\mathbb{A})/G(\mathbb{A})$ is finite, then $X(K)/G(K)$ is finite as well. Thus (ii)⇒(i).

All what is left to prove is that (i)⇒(ii), i.e that if the set of K-orbits $X(K)/G(K)$ is finite, then the set of adelic orbits $X(\mathbb{A})/G(\mathbb{A})$ is finite. For this end we consider the group

$$\ker_{\mathrm{ab}}(K, H \to G) := \ker[H^1_{\mathrm{ab}}(K, H) \to H^1_{\mathrm{ab}}(K, G)].$$

Consider the following condition:

(v) The group $\ker_{\mathrm{ab}}(K, H \to G)$ *is finite.*

We shall prove that (i)⇒(v) and (v)⇒(ii). This will show that (i)⇒(ii).

We prove that (i)$\Rightarrow$(v). Write

$$H^1(K_\infty, H) = \prod_{v\in R_\infty} H^1(K_v, H),$$
$$H^1_{\mathrm{ab}}(K_\infty, H) = \prod_{v\in R_\infty} H^1_{\mathrm{ab}}(K_v, H).$$

Similarly we define

$$\ker(K_\infty, H\to G) = \ker[H^1(K_\infty, H)\to H^1(K_\infty, G)] = \prod_{v\in R_\infty} \ker(K_v, H\to G),$$
$$\ker_{\mathrm{ab}}(K_\infty, H\to G) = \ker[H^1_{\mathrm{ab}}(K_\infty, H)\to H^1_{\mathrm{ab}}(K_\infty, G)] = \prod_{v\in R_\infty} \ker_{\mathrm{ab}}(K_v, H\to G).$$

Set

$$k^f_{\mathrm{ab}} = \ker[\mathrm{loc}_\infty\colon \ker_{\mathrm{ab}}(K, H\to G)\to \ker_{\mathrm{ab}}(K_\infty, H\to G)].$$

Since for $v\in R_\infty$ we have $H^1_{\mathrm{ab}}(K_v, H)\simeq H^{-1}(\Gamma_v, \pi_1(\overline{H}))\subset (\pi_1(\overline{H})_{\Gamma_v})_{\mathrm{tors}}$, we see that $H^1_{\mathrm{ab}}(K_v, H)$ is finite for every $v\in R_\infty$, and therefore $\ker_{\mathrm{ab}}(K_\infty, H\to G)$ is a finite group. It follows that k^f_{ab} is a subgroup of finite index in $\ker_{\mathrm{ab}}(K, H\to G)$.

Consider the maps

$$\mathrm{ab}^1\colon H^1(K, H)\to H^1_{\mathrm{ab}}(K, H),$$
$$\mathrm{loc}_\infty\colon H^1(K, H)\to H^1(K_\infty, H).$$

By [Bo98, Thm. 5.12] these maps induce a canonical bijection

$$H^1(K, H)\xrightarrow{\sim} H^1_{\mathrm{ab}}(K, H)\underset{H^1_{\mathrm{ab}}(K_\infty, H)}{\times} H^1(K_\infty, H)$$

(with a fiber product in the right hand side). This bijection is functorial in H, hence we obtain a bijection

(A.3)
$$\ker(K, H\to G)\xrightarrow{\sim}\ker_{\mathrm{ab}}(K, H\to G)\underset{\ker_{\mathrm{ab}}(K_\infty, H\to G)}{\times}\ker(K_\infty, H\to G).$$

We define a map

$$k^f_{\mathrm{ab}}\to \ker_{\mathrm{ab}}(K, H\to G)\times\ker(K_\infty, H\to G)$$

by $x\mapsto (x, 1)$. Since $\mathrm{loc}_\infty(x) = 1$ for $x\in k^f_{\mathrm{ab}}\subset\ker_{\mathrm{ab}}(K, H\to G)$, we obtain from (A.3) an induced map $k^f_{\mathrm{ab}}\to\ker(K, H\to G)$, which is a section of the map

$$\mathrm{ab}\colon \ker(K, H\to G)\to\ker_{\mathrm{ab}}(K, H\to G)$$

over k^f_{ab}. Thus the group k^f_{ab} embeds as a subset into the set $\ker(K, H\to G)$.

By the assumption (i) $X(K)/G(K)$ is a finite set. Since we have a canonical bijection

$$X(K)/G(K)\simeq\ker(K, H\to G),$$

we see that $\ker(K,H\to G)$ is finite. Since k^f_{ab} embeds into $\ker(K,H\to G)$, we see that k^f_{ab} is finite. Since k^f_{ab} is a subgroup of finite index of $\ker_{\mathrm{ab}}(K,H\to G)$, we conclude that $\ker_{\mathrm{ab}}(K,H\to G)$ is finite. Thus (i)⇒(v).

We prove that (v)⇒(ii). Here we use the abelian group structure in $\ker_{\mathrm{ab}}(K,H\to G)$. We write $\bigoplus_v$ for $\bigoplus_{v\in R}$.

We define a map $\mu\colon \bigoplus_v H^1_{\mathrm{ab}}(K_v,H)\to(\pi_1(\overline{H})_\Gamma)_{\mathrm{tors}}$ as the sum of the local maps μ_v defined in (A.1). Namely, if $\xi_{\mathbb{A}}=(\xi_v)\in\bigoplus_v H^1_{\mathrm{ab}}(K_v,H)$, we set $\mu(\xi_{\mathbb{A}})=\sum_v\mu_v(\xi_v)$. The sequence

$$H^1_{\mathrm{ab}}(k,H)\xrightarrow{\mathrm{loc}}\bigoplus_v H^1_{\mathrm{ab}}(K_v,H)\xrightarrow{\mu}(\pi_1(\overline{H})_\Gamma)_{\mathrm{tors}} \tag{A.4}$$

is exact, see [Bo98, Proof of Thm. 5.16].

The exact sequence (A.4) is functorial in H, hence the embedding $H\hookrightarrow G$ gives rise to a commutative diagram with exact rows

$$\begin{tikzcd} H^1_{\mathrm{ab}}(K,H) \arrow[r, "\mathrm{loc}"] \arrow[d] & \bigoplus_v H^1_{\mathrm{ab}}(K_v,H) \arrow[r, "\mu"] \arrow[d] & (\pi_1(\overline{H})_{\mathfrak{g}})_{\mathrm{tors}} \arrow[d] \\ H^1_{\mathrm{ab}}(K,G) \arrow[r, "\mathrm{loc}"] & \bigoplus_v H^1_{\mathrm{ab}}(K_v,G) \arrow[r, "\mu"] & (\pi_1(\overline{G})_{\mathfrak{g}})_{\mathrm{tors}} \end{tikzcd} \tag{A.5}$$

This diagram induces a homomorphism

$$\varkappa\colon \bigoplus_v \ker_{\mathrm{ab}}(K_v,H\to G)\to\ker[(\pi_1(\overline{H})_{\mathfrak{g}})_{\mathrm{tors}}\to(\pi_1(\overline{G})_{\mathfrak{g}})_{\mathrm{tors}}].$$

The group $\ker[(\pi_1(\overline{H})_{\mathfrak{g}})_{\mathrm{tors}}\to(\pi_1(\overline{G})_{\mathfrak{g}})_{\mathrm{tors}}]$ is clearly finite. Set $k_0=\ker\ \varkappa$.

We define

$$\text{Ш}^1_{\mathrm{ab}}(K,H):=\ker\left[H^1_{\mathrm{ab}}(K,H)\to\prod_v H^1_{\mathrm{ab}}(K_v,H)\right].$$

We construct a homomorphism

$$\psi\colon k_0\to\text{Ш}^1_{\mathrm{ab}}(K,G)/i_*(\text{Ш}^1_{\mathrm{ab}}(K,H))$$

as follows. Let

$$\xi_{\mathbb{A}}\in k_0\subset\bigoplus_v\ker_{\mathrm{ab}}(K_v,H\to G)\subset\bigoplus_v H^1_{\mathrm{ab}}(K_v,H).$$

Since the top row of the diagram (A.5) is exact, we see that $\xi_{\mathbb{A}}$ comes from some $\xi\in H^1_{\mathrm{ab}}(K,H)$, and this ξ is defined up to addition of $\xi'\in\text{Ш}^1_{\mathrm{ab}}(K,H)$. It is clear from the diagram that the image of ξ in $H^1_{\mathrm{ab}}(K,G)$ is contained in $\text{Ш}^1_{\mathrm{ab}}(K,G)$. Thus we obtain a map $\psi\colon k_0\to\text{Ш}^1_{\mathrm{ab}}(K,G)/i_*(\text{Ш}^1_{\mathrm{ab}}(K,H))$. It is easy to see that ψ is a homomorphism. By Lemma A.1.3 below, the group $\text{Ш}^1_{\mathrm{ab}}(K,G)$ is finite. Hence the group $\text{Ш}^1_{\mathrm{ab}}(K,G)/i_*(\text{Ш}^1_{\mathrm{ab}}(K,H))$ is finite. Set $k_{00}=\ker\ \psi$. Using diagram chasing, we see easily that k_{00} is the image of $\ker_{\mathrm{ab}}(K,H\to G)$ in k_0.

By the assumption (v) the group $\ker_{\mathrm{ab}}(K, H \to G)$ is finite, hence its image k_{00} is finite. Since we have a homomorphism of abelian groups ψ from k_0 to the finite group $Ш^1_{\mathrm{ab}}(K,G)/i_*(Ш^1_{\mathrm{ab}}(K,H))$ with finite kernel k_{00}, we see that k_0 is finite. Since we have a homomorphism of abelian groups $\varkappa$ from $\bigoplus_v \ker_{\mathrm{ab}}(K_v, H \to G)$ to the finite group $\ker[(\pi_1(\overline{H})_{\mathfrak{g}})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{g}})_{\mathrm{tors}}]$ with finite kernel k_0, we see that $\bigoplus_v \ker_{\mathrm{ab}}(K_v, H \to G)$ is finite. Since for all $v \in R_f$ we have bijections

$$\mathrm{ab}\colon \ker(K_v, H \to G) \xrightarrow{\sim} \ker_{\mathrm{ab}}(K_v, H \to G),$$

we see that the set $\bigoplus_v \ker(K_v, H \to G)$ is finite. This means that the set $X(\mathbb{A})/G(\mathbb{A})$ is finite. Thus (v)$\Rightarrow$(ii).

This completes the proof of Theorem A.1.2 modulo Lemma A.1.3. □

Lemma A.1.3. *Let G be a connected linear algebraic group over a number field K. Then the abelian group $Ш^1_{\mathrm{ab}}(K,G)$ is finite.*

Proof. We give two proofs.

First proof: by [Bo98, Thm. 5.12] we have a canonical bijection $Ш^1(K,G) \to Ш^1_{\mathrm{ab}}(K,G)$, and by Borel's theorem [Se65, Ch. III §4.6, Thm. 7] $Ш^1(K,G)$ is finite. Thus $Ш^1_{\mathrm{ab}}(K,G)$ is finite. (This short proof uses nonabelian cohomology.)

Second proof: We may and shall assume that G is reductive. Let

$$1 \to S \to G' \to G \to 1$$

be a flasque resolution of G, see [CT06, §3]. Here G' is a quasi-trivial reductive group and S is a torus. Let $P = (G')^{\mathrm{tor}}$, the biggest quotient torus of G'. Since G' is a quasi-trivial group, P is a quasi-trivial torus. For any field $F \supset K$ we have a canonical isomorphism

$$H^1_{\mathrm{ab}}(F,G) \xrightarrow{\sim} \ker[H^2(F,S) \to H^2(F,P)],$$

see [CT06, App. A]. Since P is quasi-trivial, we have $Ш^2(K,P) = 0$, and therefore

$$Ш^1_{\mathrm{ab}}(K,G) \simeq Ш^2(K,S).$$

It is known that the group $Ш^2(K,S)$ is finite for any K-group of multiplicative type S, see [Mi06, Ch. I, Thm. 4.20(a)]. Thus $Ш^1_{\mathrm{ab}}(K,G)$ is finite. (This proof is longer, but it is "abelian".) □

A.2. Corollaries of Theorem A.1.2.

Corollary A.2.1. *Let K, G, H, and X be as in A.1.1. If $\pi_1(\overline{H}) = 0$, then the set of orbits $X(K)/G(K)$ is finite.*

Proof. Indeed, then $(\pi_1(\overline{H})_{\mathfrak{h}})_{\mathrm{tors}} = 0$, hence the map $(\pi_1(\overline{H})_{\mathfrak{h}})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{h}})_{\mathrm{tors}}$ is injective (for any $\mathfrak{h}$). By Theorem A.1.2 the set $X(K)/G(K)$ is finite. □

Corollary A.2.2. *Let K, G, H, and X be as in A.1.1. If the map $\pi_1(\overline{H}) \to \pi_1(\overline{G})$ is an isomorphism, then the set of orbits $X(K)/G(K)$ is finite.*

Proof. Indeed, then the map $(\pi_1(\overline{H})_{\mathfrak{h}})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{h}})_{\mathrm{tors}}$ is an isomorphism, hence injective (for any $\mathfrak{h}$). By Theorem A.1.2 the set $X(K)/G(K)$ is finite. □

Corollary A.2.3. *Let K, G, H, X be as in A.1.1. Assume the set $X(K)/G(K)$ is finite. Then the induced homomorphism $i_*\colon \pi_1(\overline{H})_{\mathrm{tors}} \to \pi_1(\overline{G})_{\mathrm{tors}}$ is injective.*

Proof. Since the set $X(K)/G(K)$ is finite, by Theorem A.1.2 the map

$$i_*\colon (\pi_1(\overline{H})_{\mathfrak{h}})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{h}})_{\mathrm{tors}}$$

is injective for *any* cyclic subgroup $\mathfrak{h} \subset \mathfrak{g}$, in particular for $\mathfrak{h} = \{1\}$. Thus the map $\pi_1(\overline{H})_{\mathrm{tors}} \to \pi_1(\overline{G})_{\mathrm{tors}}$ is injective. □

Corollary A.2.4. *Let K, G, H, X be as in A.1.1. Assume the set $X(K)/G(K)$ is finite. If H has no $\overline{K}$-characters (e.g. semisimple), then the homomorphism $i_*\colon \pi_1(\overline{H}) \to \pi_1(\overline{G})$ is injective.*

Proof. Indeed, since H has no $\overline{K}$-characters, we see that $\pi_1(\overline{H})$ is finite, hence $\pi_1(\overline{H})_{\mathrm{tors}} = \pi_1(\overline{H})$, and we apply Corollary A.2.3. □

Corollary A.2.5. *Let K, G, H, X be as in A.1.1. Assume that both G and H have no $\overline{K}$-characters (e.g. they both are semisimple) and assume that $\pi_1(\overline{G}) = 0$. Then $X(K)/G(K)$ is finite if and only if $\pi_1(\overline{H}) = 0$.*

Proof. If $\pi(\overline{H}) = 0$, then by Corollary A.2.1 $X(K)/G(K)$ is finite. Conversely, assume that $X(K)/G(K)$ is finite. By Corollary A.2.4 the homomorphism $\pi_1(\overline{H}) \to \pi_1(\overline{G})$ is injective, hence $\pi_1(\overline{H}) = 0$. □

A.2.6. Let G be a connected semisimple K-group. We say that G is an inner form if G is an inner form of a K-split group. If G is an inner form, then the Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on $\pi_1(\overline{G})$ trivially. Indeed, for a K-split group G this follows from the definition of $\pi_1(\overline{G})$, and an inner twisting does not change the Galois module $\pi_1(\overline{G})$.

Corollary A.2.7. *Let K, G, H and X be as in A.1.1. Assume that the Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on $\pi_1(\overline{G})$ trivially. If the homomorphism $i_*\colon \pi_1(\overline{H}) \to \pi_1(\overline{G})$ is injective, then the set $X(K)/G(K)$ is finite.*

Proof. Since $\pi_1(\overline{H})$ injects into $\pi_1(\overline{G})$, we see that $\mathrm{Gal}(\overline{K}/K)$ acts also on $\pi_1(\overline{H})$ trivially. Thus $\mathfrak{g} = \{1\}$, hence the only cyclic subgroup $\mathfrak{h} \subset \mathfrak{g}$ is $\mathfrak{h} = \{1\}$. We see that the homomorphism

$$i_*\colon \pi_1(\overline{H})_{\mathfrak{h}} \to \pi_1(\overline{G})_{\mathfrak{h}}$$

is injective, hence the homomorphism

$$i_*\colon (\pi_1(\overline{H})_{\mathfrak{h}})_{\mathrm{tors}} \to (\pi_1(\overline{G})_{\mathfrak{h}})_{\mathrm{tors}}$$

is injective, and the corollary follows from Theorem A.1.2. □

A.3. Semisimple groups. In this section K is an *algebraically closed field* of characteristic 0. We consider pairs (G, H), where H is a connected semisimple K-subgroup of a connected semisimple K-group G. We find conditions under which the map $\pi_1(H) \to \pi_1(G)$ is injective.

A.3.1. Let $H \subset G$ be connected semisimple K-groups. Let $i\colon H \to G$ be the inclusion homomorphism. Consider the map $i^{\mathrm{sc}}\colon H^{\mathrm{sc}} \to G^{\mathrm{sc}}$, where G^{sc} is the universal covering of G. Set $H' = i^{\mathrm{sc}}(H^{\mathrm{sc}}) \subset G^{\mathrm{sc}}$. Let $T_H \subset H$ be a maximal torus, and let $T_G \subset G$ be a maximal torus containing T_H. Let $T_{H^{\mathrm{sc}}} \subset H^{\mathrm{sc}}$, $T_{H'} \subset H'$, and $T_{G^{\mathrm{sc}}} \subset G^{\mathrm{sc}}$ be the maximal tori corresponding to T_H and T_G. For a K-torus T let $\mathsf{X}_*(T)$ denote the cocharacter group of T, i.e. $\mathsf{X}_*(T) = \mathrm{Hom}(\mathbf{G}_{m,K}, T)$. We have canonical homomorphisms $T_{H^{\mathrm{sc}}} \to T_{H'} \to T_{G^{\mathrm{sc}}}$ and the induced homomorphisms $\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_{H'}) \to \mathsf{X}_*(T_{G^{\mathrm{sc}}})$.

Lemma A.3.2. *Let $H \subset G$ be connected semisimple K-groups. With the notation of Subsection A.3.1 we have canonical isomorphisms*

$$\pi_1(H') \simeq \ker[\pi_1(H) \to \pi_1(G)] \simeq \mathrm{coker}[\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_{G^{\mathrm{sc}}})]_{\mathrm{tors}}.$$

where $_{\mathrm{tors}}$ *denotes the torsion subgroup (of the cokernel).*

Proof. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & k \arrow[d] & \\
0 \arrow[r] & \mathsf{X}_*(T_{H^{\mathrm{sc}}}) \arrow[r] \arrow[d] & \mathsf{X}_*(T_H) \arrow[r] \arrow[d] & \pi_1(H) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathsf{X}_*(T_{G^{\mathrm{sc}}}) \arrow[r] \arrow[d] & \mathsf{X}_*(T_G) \arrow[r] \arrow[d] & \pi_1(G) \arrow[r] & 0 \\
 & C^{\mathrm{sc}} & C & &
\end{tikzcd}$$

where k is the kernel and C^{sc} and C are the cokernels of the corresponding homomorphisms. By the snake lemma we have an exact sequence

$$0 \to k \to C^{\mathrm{sc}} \to C. \tag{A.6}$$

Since $\pi_1(H)$ is finite, clearly k is finite. Since T_H embeds into T_G, the group C has no torsion. From the exact sequence (A.6) we obtain an isomorphism $k \xrightarrow{\sim} (C^{\mathrm{sc}})_{\mathrm{tors}}$, i.e. an isomorphism

$$\ker[\pi_1(H) \to \pi_1(G)] \simeq \mathrm{coker}[\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_{G^{\mathrm{sc}}})]_{\mathrm{tors}},$$

Since the injective homomorphism $\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_{G^{\mathrm{sc}}})$ factorizes as a composition of injective homomorphisms $\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_{H'}) \to \mathsf{X}_*(T_{G^{\mathrm{sc}}})$, we obtain a short exact sequence

$$\begin{aligned} 0 \to \mathrm{coker}[\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_{H'})] &\to \mathrm{coker}[\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_{G^{\mathrm{sc}}})] \\ &\to \mathrm{coker}[\mathsf{X}_*(T_{H'}) \to \mathsf{X}_*(T_{G^{\mathrm{sc}}})] \to 0. \end{aligned}$$

Since $T_{H'}$ embeds into $T_{G^{\rm sc}}$, we have $\operatorname{coker}[\mathsf{X}_*(T_{H'}) \to \mathsf{X}_*(T_{G^{\rm sc}})]_{\rm tors} = 0$, and therefore we obtain an isomorphism

$$\operatorname{coker}[\mathsf{X}_*(T_{H^{\rm sc}}) \to \mathsf{X}_*(T_{H'})]_{\rm tors} \simeq \operatorname{coker}[\mathsf{X}_*(T_{H^{\rm sc}}) \to \mathsf{X}_*(T_{G^{\rm sc}})]_{\rm tors}.$$

But $\operatorname{coker}[\mathsf{X}_*(T_{H^{\rm sc}}) \to \mathsf{X}_*(T_{H'})]_{\rm tors} = \operatorname{coker}[\mathsf{X}_*(T_{H^{\rm sc}}) \to \mathsf{X}_*(T_{H'})] = \pi_1(H')$. Thus we obtain an isomorphism

$$\pi_1(H') \simeq \operatorname{coker}[\mathsf{X}_*(T_{H^{\rm sc}}) \to \mathsf{X}_*(T_{G^{\rm sc}})]_{\rm tors}.$$

This completes the proof of Lemma A.3.2. □

Corollary A.3.3. *With the assumptions and notation of Lemma A.3.2, the following assertions are equivalent:*

(i) H' is simply connected;

(ii) The homomorphism $\pi_1(H) \to \pi_1(G)$ is injective;

(iii) The group $\operatorname{coker}[\mathsf{X}_*(T_{H^{\rm sc}}) \to \mathsf{X}_*(T_{G^{\rm sc}})]$ *has no torsion.*

A.4. Symmetric pairs over an algebraically closed field. In this section we assume that K is an *algebraically closed* field of characteristic 0. We consider symmetric pairs (G, H) over K, where G is a simply connected almost simple K-group, and H is a symmetric semisimple subgroup. Recall that "symmetric" means that H is the group of invariants G^θ for some involutive automorphism θ of G. Symmetric pairs (G, H) (or (G, θ)) were classified by E. Cartan. We shall use the unified description of symmetric pairs due to V. Kac, see [He78] and [OV90]. A symmetric pair (G, H) with semisimple H corresponds to an affine Dynkin diagram D and a vertex s of D, see [OV90, Table 7]. We give a list of all symmetric pairs (G, H) with simply connected almost simple G, for which H is simply connected.

Theorem A.4.1. *Let G be a* **simply connected** *almost simple K-group over an algebraically closed field K of characteristic 0, and let $H \subset G$ a symmetric semisimple K-subgroup. Then H is simply connected for the symmetric pairs (G, H) in the list below, and $\pi_1(H) = \mathbb{Z}/2\mathbb{Z}$ for (G, H) not in the list.*

(**A II**) *$G = \mathrm{SL}_{2n}$, $H = \mathrm{Sp}_n$ $(n \geq 3)$.*

(**C II**) *$G = \mathrm{Sp}_{p+q}$, $H = \mathrm{Sp}_p \times \mathrm{Sp}_q$ $(1 \leq p \leq q)$.*

(**BD I**$(2l, 1)$) *$G = \mathrm{Spin}_{2l+1}$, $H = \mathrm{Spin}_{2l}$ $(l \geq 3)$.*

(**BD I**$(2l-1, 1)$) *$G = \mathrm{Spin}_{2l}$, $H = \mathrm{Spin}_{2l-1}$ $(l \geq 3)$.*

(**E IV**) *$G = E_6$, $H = F_4$.*

(**F II**) *$G = F_4$, $H = \mathrm{Spin}_9$.*

Proof. We consider two cases.

(i) θ is an inner automorphism.

In this case D is the extended Dynkin diagram of G. Let T_H be a maximal torus of H, and let T_G be a maximal torus of G containing T_H. Let $T_{H^{\rm sc}}$ be the corresponding maximal torus of the universal covering $H^{\rm sc}$ of H. Let $\mathsf{X}^*(T_G) := \operatorname{Hom}(T_G, \mathbf{G}_m)$ be the character group of T_G. Set

$V = \mathsf{X}^*(T_G) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $R = R(G, T_G) \subset \mathsf{X}^*(T_G) \subset V$ be the root system of G. Let $\alpha_0, \alpha_1, \ldots, \alpha_l \in R$ be the roots corresponding to the vertices of D, where α_0 is the lowest root. Then

$$\sum_{i=0}^{l} a_i \alpha_i = 0, \tag{A.7}$$

where $a_i \in \mathbb{Z}$, $a_0 = 1$. The distinguished vertex s corresponds to some root α_k, and $a_k = 2$ (see [He78, Ch. X §5],[OV90, Ch. 5 §1.5, Problem 38]).

Let $R^\vee \subset \mathsf{X}_*(T_G)$ denote the dual root system. For every $\alpha \in R$ let $\alpha^\vee \in R^\vee$ be the corresponding coroot. The coroots $\alpha_1^\vee, \ldots, \alpha_l^\vee$ constitute a basis of $\mathsf{X}_*(T_G)$, hence $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_l^\vee$ generate $\mathsf{X}_*(T_G)$. The diagram $D - \{s\}$ is the Dynkin diagram of H, and the coroots $\alpha_i^\vee$, $(i \neq k)$ constitute a basis of $\mathsf{X}_*(T_{H^{\mathrm{sc}}})$.

Let $W = W(R)$ denote the Weyl group. Choose a W-invariant scalar product $(\ ,\ )$ in V. We can embed $R^\vee$ into V by

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

We consider 4 subcases.

(i)(a) Suppose that D has no multiple edges. Then all the roots $\beta \in R$ are of the same length, and we can normalize the scalar product such that $(\beta, \beta) = 2$, hence $\beta^\vee = \beta$, for all $\beta \in R$. Now it follows from (A.7) that

$$\sum_{i=0}^{l} a_i \alpha_i^\vee = 0.$$

Recall that $a_i \in \mathbb{Z}$, $a_0 = 1$, and $a_k = 2$. We see that $\alpha_k^\vee \in \frac{1}{2}\mathsf{X}_*(T_{H^{\mathrm{sc}}})$, but $\alpha_k^\vee \notin \mathsf{X}_*(T_{H^{\mathrm{sc}}})$. Thus $\mathrm{coker}[\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_G)] = \mathbb{Z}/2\mathbb{Z}$, and by Lemma A.3.2 $\pi_1(H) = \mathbb{Z}/2\mathbb{Z}$.

(i)(b) Suppose that D has double edges and α_k is a *short* root. We can normalize the scalar product such that for any long root β we have $(\beta, \beta) = 2$, hence $\beta^\vee = \beta$. Then for any short root γ we have $(\gamma, \gamma) = 1$, hence $\gamma^\vee = 2\gamma$. Now it follows from (A.7) that

$$\sum_{i=0}^{l} a_i' \alpha_i^\vee = 0, \tag{A.8}$$

where $a_i' = a_i$ when α_i is long, and $a_i' = a_i/2$ when α_i is short.

Since α_0 is the lowest root, it is long. Since $a_0 = 1$, we obtain that $a_0' = 1$. Thus (A.8) gives

$$\alpha_0^\vee = \sum_{i=1}^{l} -a_i' \alpha_i^\vee.$$

Since $\alpha_0^\vee \in R^\vee$ and $(\alpha_i^\vee)_{i=1,\ldots,l}$ is a basis of $R^\vee$, we see that $a_i' \in \mathbb{Z}$ for all i. Since $a_k = 2$ and α_k is short, we see that $a_k' = 1$. Thus $\alpha_k^\vee$ is a linear combination of $\alpha_i^\vee$ $(i \neq k)$ with coefficients in $\mathbb{Z}$. We see that the map $\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_G)$ is surjective, and by Corollary A.3.3 the group H is simply connected.

(i)(c) Suppose that D has double edges and α_k is a *long* root. As in (i)(b), from (A.7) we obtain the relation (A.8) with $a_i' = a_i$ when α_i is long, and $a_i' = a_i/2$ when α_i is short. Again $a_0' = 1$ and $a_i' \in \mathbb{Z}$ for all i. Since $a_k = 2$ and now α_k is long, we see that $a_k' = 2$. As in (i)(a), we see that $\alpha_k^\vee \in \frac{1}{2}\mathsf{X}_*(T_{H^{\mathrm{sc}}})$, but $\alpha_k^\vee \notin \mathsf{X}_*(T_{H^{\mathrm{sc}}})$. Thus $\operatorname{coker}[\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_G)] = \mathbb{Z}/2\mathbb{Z}$, and by Lemma A.3.2 $\pi_1(H) = \mathbb{Z}/2\mathbb{Z}$.

(i)(d) Suppose that D has a triple edge (type G_2). We have $k = 1$,

$$\alpha_0 + 2\alpha_1 + 3\alpha_2 = 0, \tag{A.9}$$

(see [OV90, Table 7.I]), where

$$(\alpha_0, \alpha_0) = 3,\ (\alpha_1, \alpha_1) = 3,\ \text{and } (\alpha_2, \alpha_2) = 1.$$

Then

$$\alpha_0^\vee = \frac{2}{3}\alpha_0,\ \alpha_1^\vee = \frac{2}{3}\alpha_1,\ \alpha_2^\vee = 2\alpha_2.$$

From (A.9) we obtain

$$\alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee = 0.$$

Similarly to (i)(a), we see that $\alpha_1^\vee \in \frac{1}{2}\mathsf{X}_*(T_{H^{\mathrm{sc}}})$, but $\alpha_1^\vee \notin \mathsf{X}_*(T_{H^{\mathrm{sc}}})$. It follows that $\operatorname{coker}[\mathsf{X}_*(T_{H^{\mathrm{sc}}}) \to \mathsf{X}_*(T_G)] = \mathbb{Z}/2\mathbb{Z}$, and therefore $\pi_1(H) = \mathbb{Z}/2\mathbb{Z}$.

We obtain the following list of pairs (G, H) with simply connected H in the case when θ is inner:
(**C II**) $G = \mathrm{Sp}_{p+q}$, $H = \mathrm{Sp}_p \times \mathrm{Sp}_q$ $(1 \le p \le q)$.
(**BD I**$(2l,1)$) $G = \mathrm{Spin}_{2l+1}$, $H = \mathrm{Spin}_{2l}$ $(l \ge 3)$.
(**F II**) $G = F_4$, $H = \mathrm{Spin}_9$.

Case (ii): θ is an outer automorphism. We use case-by-case consideration.

When G is a classical group and θ is outer, we see from [OV90, Table 7.III] that H is simply connected only in the following cases:
(**A II**) $G = \mathrm{SL}_{2n}$, $H = \mathrm{Sp}_n$ $(n \ge 3)$.
(**BD I**$(2l-1,1)$) $G = \mathrm{Spin}_{2l}$, $H = \mathrm{Spin}_{2l-1}$ $(l \ge 3)$.
These are exactly the cases when D has a double edge and α_k is a short root.

We list all the other classical cases with θ outer:
(**A I**) $G = \mathrm{SL}_n$, $H = \mathrm{SO}_n$ $(n \ge 3,\ n \neq 4)$.
(**BD I**$(2p+1, 2q+1)$) $G = \mathrm{Spin}_{2p+2q+2}$, $H = (\mathrm{Spin}_{2p+1} \times \mathrm{Spin}_{2q+1})/\mu_2$ $(1 \le p \le q)$.
In these cases $\pi_1(H) = \mathbb{Z}/2\mathbb{Z}$.

We must treat the case $D = E_6^{(2)}$, see [OV90, Table 7.III] (this diagram has a double edge). Then either $H = F_4$ or $H = C_4$.

When $H = F_4$, clearly H is simply connected. In this case α_k is a short root.

When H is of the type C_4, the restriction of the adjoint representation of G to H is the direct sum $\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus \mathfrak{p}$ of two irreducible representations. Here the the representation of $H = C_4$ in $\mathfrak{p}$ is a subrepresentation of the representation of C_4 in $\bigwedge^4 R$, where R is the standard 8-dimensional representation of C_4. We see that the central element $-1 \in H^{\mathrm{sc}}(K) = \mathrm{Sp}_4(K)$ acts trivially in $\mathrm{Lie}(H)$ and in $\mathfrak{p}$, hence the image of -1 in the adjoint group G^{ad} is 1. Since $\ker[G \to G^{\mathrm{ad}}]$ is of order 3, we see that the image of this element in G is 1. Thus $\pi_1(H) = \mathbb{Z}/2\mathbb{Z}$. In this case α_k is a long root.

We obtain the following list of symmetric pairs (G, H) with simply connected H in the case E_6:
(**E IV**) $G = E_6$, $H = F_4$.

This completes the proof of Theorem A.4.1. $\square$

Corollary A.4.2. *Let G be a simply connected almost simple K-group over an algebraically closed field K of characteristic 0, and let $H \subset G$ be a symmetric semisimple K-subgroup. Assume that the symmetric pair (G, H) corresponds to (D, s) as above. If D has a double edge and s corresponds to a short root, then H is simply connected; otherwise $\pi_1(H) = \mathbb{Z}/2\mathbb{Z}$.*

A.5. Symmetric pairs over a number field. First we consider symmetric homogeneous spaces $X = H\backslash G$ with G *simply connected*.

Theorem A.5.1. *Let K be a number field. A symmetric homogeneous space $X = H\backslash G$ over K with semisimple H and* **simply connected** *absolutely almost simple G has finitely many $G(K)$-orbits in the following cases (and only in these cases):*
(**A II**) *G is a K-form of SL_{2n}, H is a K-form of Sp_n $(n \geq 3)$.*
(**C II**) *G is a K-form of Sp_{p+q}, H is a K-form of $\mathrm{Sp}_p \times \mathrm{Sp}_q$ $(1 \leq p \leq q)$.*
(**BD I**$(2l, 1)$) *G is a K-form of Spin_{2l+1}, H is a K-form of Spin_{2l} $(l \geq 3)$.*
(**BD I**$(2l-1, 1)$) *G is a K-form of Spin_{2l}, H is a K-form of Spin_{2l-1} $(l \geq 3)$.*
(**E IV**) *G is a K-form of E_6 (simply connected), H is a K-form of F_4.*
(**F II**) *G is a K-form of F_4, H is a K-form of Spin_9.*

Proof. We have $\pi_1(\overline{G}) = 0$. By Corollary A.2.5 the set of orbits $X(K)/G(K)$ is finite if and only if $\pi_1(\overline{H}) = 0$, i.e H is simply connected. The symmetric pairs $(\overline{G}, \overline{H})$ over $\overline{K}$ with simply connected $\overline{H}$ were listed in Theorem A.4.1. The list of Theorem A.5.1 is exactly the list of Theorem A.4.1. $\square$

Now we consider symmetric homogeneous spaces $X = H\backslash G$ with G *adjoint*.

Theorem A.5.2. *Let K be a number field. A symmetric homogeneous space $X = H\backslash G$ over K with semisimple H and* **adjoint** *absolutely simple G has finitely many $G(K)$-orbits in the following cases (and only in these cases):*
(**A II**) *G is a form of PSL_{2n}, H is a form of PSp_n $(n \geq 3)$, where either n is odd or G is an* **inner** *form.*
(**C II**) *G is a form of PSp_{p+q}, H is a form of $(\mathrm{Sp}_p \times \mathrm{Sp}_q)/\mu_2$ $(1 \leq p \leq q)$.*
(**BD I**$(2l, 1)$) *G is a form of SO_{2l+1}, H is a form of SO_{2l} $(l \geq 3)$.*

(**BD I**$(2l-1,1)$) *G is an* **inner** *form of* PSO_{2l}*, H is a form of* SO_{2l-1} $(l \geq 3)$.
(**E IV**) *G is a form of E_6 (adjoint), H is a form of F_4.*
(**F II**) *G is a form of F_4, H is a form of* Spin_9.

Proof. First assume that $X(K)/G(K)$ is finite. Since H is semisimple, by Corollary A.2.4 the homomorphism $\pi_1(\overline{H}) \to \pi_1(\overline{G})$ is injective. Let H' denote the image of $i^{\mathrm{sc}}\colon H^{\mathrm{sc}} \to G^{\mathrm{sc}}$. By Corollary A.3.3 H' is simply connected. Thus $(\overline{G}^{\mathrm{sc}}, \overline{H}')$ is a symmetric pair with simply connected groups $\overline{G}^{\mathrm{sc}}$ and $\overline{H}'$. Such pairs were listed in Theorem A.4.1. Thus we obtain that the pair $(\overline{G}, \overline{H}')$ is from the list of Theorem A.4.1, hence (G, H) is from the following list:
(**A II**) G is a form of PSL_{2n}, H is a form of PSp_n $(n \geq 3)$
(**C II**) G is a form of PSp_{p+q}, H is a form of $(\mathrm{Sp}_p \times \mathrm{Sp}_q)/\mu_2$ $(1 \leq p \leq q)$.
(**BD I**$(2l,1)$) G is a form of SO_{2l+1}, H is a form of SO_{2l} $(l \geq 3)$.
(**BD I**$(2l-1,1)$) G is a form of PSO_{2l}, H is a form of SO_{2l-1} $(l \geq 3)$.
(**E IV**) G is a form of E_6 (adjoint), H is a form of F_4.
(**F II**) G is a form of F_4, H is a form of Spin_9.

Conversely, let us check, for which (G, H) from this list the set of orbits $X(K)/G(K)$ is finite.

If G is an *inner* form, then $\mathrm{Gal}(\overline{K}/K)$ acts on $\pi_1(\overline{G})$ trivially, see Subsection A.2.6, and by Corollary A.2.7 the set of orbits $X(K)/G(K)$ is finite. Thus in the cases (**C II**), (**BD I**$(2l,1)$), and (**F II**) the set $X(K)/G(K)$ is finite, because any form of G is inner in these cases.

In the case (**E IV**) we have $\pi_1(\overline{H}) = 0$, and by Corollary A.2.1 the set $X(K)/G(K)$ is finite (when G is an inner form or an outer form).

What is left is to consider the cases (**A II**) and (**BD I**$(2l-1,1)$) with outer forms of G.

We consider the case (**A II**). Then $\overline{G} = \mathrm{PSL}_{2n}$, $\pi_1(\overline{G}) = \mathbb{Z}/2n\mathbb{Z}$, $\overline{H} = \mathrm{PSp}_n$, $\pi_1(\overline{H}) = \mathbb{Z}/2\mathbb{Z}$. The embedding $\pi_1(\overline{H}) \hookrightarrow \pi_1(\overline{G})$ is given by

$$\overline{1} \mapsto \overline{n}, \quad \text{where } \overline{1} = 1 + 2\mathbb{Z} \in \mathbb{Z}/2\mathbb{Z},\ \overline{n} = n + 2n\mathbb{Z} \in \mathbb{Z}/2n\mathbb{Z}.$$

Since G is an outer form, $\mathfrak{g} = \{1, \sigma\}$, where the nontrivial element σ of $\mathfrak{g}$ is of order 2 and acts on $\pi_1(\overline{G}) = \mathbb{Z}/2n\mathbb{Z}$ by ${}^\sigma x = -x$. We see that ${}^\sigma x - x = -2x$. Thus the kernel of the canonical map $\pi_1(\overline{G}) \to \pi_1(\overline{G})_{\mathfrak{g}}$ is the subset

$$\{\overline{2k} \subset \mathbb{Z}/2n\mathbb{Z} \mid k \in \mathbb{Z}\}.$$

We see that the element $\overline{n}$ lies in this kernel if and only if n is even.

If n is even, then the map $\pi_1(\overline{H})_{\mathfrak{g}} \to \pi_1(\overline{G})_{\mathfrak{g}}$ is the zero map. In other words, for $\mathfrak{h} = \mathfrak{g}$ the map $\pi_1(\overline{H})_{\mathfrak{h}} \to \pi_1(\overline{G})_{\mathfrak{h}}$ is not injective. By Theorem A.1.2 the set $X(K)/G(K)$ is infinite.

If n is odd, then the map $\pi_1(\overline{H})_{\mathfrak{g}} \to \pi_1(\overline{G})_{\mathfrak{g}}$ is injective. In other words, for $\mathfrak{h} = \mathfrak{g}$ the map $\pi_1(\overline{H})_{\mathfrak{h}} \to \pi_1(\overline{G})_{\mathfrak{h}}$ is injective. On the other hand, the map $\pi_1(\overline{H}) \to \pi_1(\overline{G})$ is injective (for any n). In other words, for $\mathfrak{h} = \{1\}$ the map $\pi_1(\overline{H})_{\mathfrak{h}} \to \pi_1(\overline{G})_{\mathfrak{h}}$ is injective as well. By Theorem A.1.2 the set $X(K)/G(K)$ is finite.

We consider the case (**BD I**$(2l-1,1)$) when G is an outer form. We show that $X(K)/G(K)$ is infinite in this case.

In this case G is a form of D_l and H is a form of B_l. We have $\pi_1(\overline{H}) = \mathbb{Z}/2\mathbb{Z}$, and $\pi_1(\overline{G})$ is $\mathbb{Z}/4\mathbb{Z}$ when l is odd and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ when l is even. The group $\pi_1(\overline{H})$ embeds into $\pi_1(\overline{G})$, and the image is a $\mathrm{Gal}(\overline{K}/K)$-invariant subgroup of order 2.

We observe that in the case $l = 4$, G does not come from triality. Indeed, if G comes from triality, then $\mathrm{Gal}(\overline{K}/K)$ acts transitively on the set of nonzero elements of $\pi_1(\overline{G})$, and therefore $\pi_1(\overline{G})$ cannot have a $\mathrm{Gal}(\overline{K}/K)$-invariant subgroup of order 2.

We see that for any l, the group $\mathfrak{g}$ is of order 2. We write $\mathfrak{g} = \{1, \sigma\}$.

Assume that l is odd. Then $\pi_1(\overline{G}) = \mathbb{Z}/4\mathbb{Z}$, and σ acts on $\pi_1(\overline{G})$ by ${}^{\sigma}x = -x$. Arguing as in the case (**A II**) with even n, we see that $X(K)/G(K)$ is infinite.

Assume that l is even. Denote the elements of $\pi_1(\overline{G}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by $0, a, b, c$. We may assume that σ permutes a and b and fixes c. Then clearly the image of $\pi_1(\overline{H})$ is $\{0, c\}$. Since $a - {}^{\sigma}a = a + b = c$, we see that the map $\pi_1(\overline{H})_{\mathfrak{g}} \to \pi_1(\overline{G})_{\mathfrak{g}}$ is the zero map, hence it is not injective. By Theorem A.1.2 the set of orbits $X(K)/G(K)$ is infinite.

This completes the proof of Theorem A.5.2. □

Theorem A.5.3. *Let $X = H\backslash G$, where (G, H) is a symmetric pair as in Theorem A.5.1 or as in Theorem A.5.2. Then:*

(i) For any finite place v of K, the group $G(K_v)$ acts on $X(K_v)$ transitively.

(ii) Write $K_\infty = \prod_{v \in R_\infty} K_v$ (so that $X(K_\infty) = \prod_{v \in R_\infty} X(K_v)$). Then every obit of $G(K_\infty)$ in $X(K_\infty)$ contains exactly one orbit of $G(K)$ in $X(K)$. In particular, any two K-points in the same connected component of $X(K_\infty)$ are $G(K)$-conjugate.

Proof. (i) Recall that $\mathfrak{g}$ is the image of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{Aut}\, \pi_1(\overline{H}) \times \mathrm{Aut}\, \pi_1(\overline{G})$. Since $\pi_1(\overline{H})$ embeds into $\pi_1(\overline{G})$, we can say that $\mathfrak{g}$ is the image of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{Aut}\, \pi_1(\overline{G})$. We have seen in the proof of Theorem A.5.2 that G does not come from triality. Thus either $\mathfrak{g} = 1$ or $\mathfrak{g} = \mathbb{Z}/2\mathbb{Z}$. We see that $\mathfrak{g}$ is cyclic. It follows that all the decomposition groups $\mathfrak{g}_v$ are cyclic. The condition (iv) of Theorem A.1.2 shows now that $\ker[\pi_1(\overline{H})_{\mathfrak{g}_v} \to \pi_1(\overline{G})_{\mathfrak{g}_v}] = 0$ for any v (because $\mathfrak{g}_v$ is cyclic for any v). It follows that $\ker[H^1(K_v, H) \to H^1(K_v, G)] = 1$ for $v \in R_f$, hence there is only one orbit of $G(K_v)$ in $X(K_v)$ for such v, which proves (i).

(ii) G is an absolutely almost simple K-group. By [Sa81, Cor. 5.4] G satisfies the Hasse principle and has the weak approximation property. Since H is a connected K-subgroup of G, by [Bo99, Cor. 1.7] X has the real approximation property, i.e. $X(K)$ is dense in $X(K_\infty)$. Any orbit of $G(K_\infty)$ in $X(K_\infty)$ is open, hence it contains a K-point.

Now let $x, y \in X(K)$ lie in the same $G(K_\infty)$-orbit. We wish to prove that they lie in the same $G(K)$-orbit. Our homogeneous space $X = H\backslash G$ has a distinguished K-point x_0, the image of the unit element $e \in G(K)$. The stabilizer of x_0 in G is H. Let H_y denote the stabilizer of y in G. Clearly the pair (G, H_y) is a symmetric pair satisfying the hypotheses of Theorem A.5.2 (or Theorem A.5.1). We may and shall assume that $y = x_0$.

So let $x \in X(K)$ lie in the $G(K_\infty)$-orbit of x_0. We wish to prove that x lies in the $G(K)$-orbit of x_0. Let $c(x)$ denote the class of the $G(K)$-orbit of x in $\ker(K, H \to G) := \ker[H^1(K, H) \to H^1(K, G)]$ (we use the notation of the proof of Theorem A.1.2). For any place v of K let

$$\mathrm{loc}_v\colon \ker(K, H \to G) \to \ker(K_v, H \to G)$$

be the localization map. By (i) for any finite place v of K we have $\ker(K_v, H \to G) = 1$, hence $\mathrm{loc}_v(c(x)) = 1$. Since x lies in the $G(K_\infty)$-orbit of x_0, we have $\mathrm{loc}_v(c(x)) = 1$ for all infinite places v of K.

Set $B = \ker[H^{\mathrm{sc}} \to H]$. By [Sa81, Cor. 4.4] there is a canonical bijection $Ш^1(K, H) \xrightarrow{\sim} Ш^2(K, B)$. From the lists of Theorems A.5.1 and A.5.2 we see that in our case either $B = 0$ or $B = \mathbb{Z}/2\mathbb{Z}$. Since in both cases $Ш^2(K, B) = 0$, we conclude that $Ш^1(K, H) = 1$. This means that $\ker\left[\mathrm{loc}\colon H^1(K, H) \to \prod_v H^1(k_v, H)\right] = 1$. We have seen that $\mathrm{loc}(c(x)) = 1$. Hence $c(x) = 1$. This means that x lies in the $G(K)$-orbit of x_0. This completes the proof of Theorem A.5.3. □

A.6. Addendum: Further examples. In this addendum we give examples of homogeneous spaces satisfying assumptions (i–iii) of Theorem 1.1 but not covered by Theorems A.5.1 and A.5.2.

A.6.1. *Example with G not absolutely simple.* Let K be a number field, K'/K a quadratic extension, D/K' a central simple algebra of dimension r^2 with an involution of second kind σ (i.e. σ induces the nontrivial automorphism σ_0 of K' over K). Let m be a natural number and let Φ be a σ-Hermitian form on D^m. Set

$$G = \mathrm{PSL}_D(D^m), \quad H = \mathrm{PSU}(D^m, \Phi),$$

where we regard G and H as K-groups. Then G is adjoint and H is a symmetric subgroup of G. An easy calculation shows that $\pi_1(\overline{G}) = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ and $\pi_1(\overline{H}) = \mathbb{Z}/n\mathbb{Z}$, where $n = mr$. The group $\mathrm{Gal}(\overline{K}/K')$ acts trivially on $\pi_1(\overline{G})$ and $\pi_1(\overline{H})$. The non-identity element $\sigma_0 \in \mathrm{Gal}(K'/K)$ acts on $\pi_1(\overline{H})$ by multiplication by -1. Thus $\mathfrak{g} = \mathrm{Gal}(K'/K)$ and $\pi_1(\overline{H})_{\mathfrak{g}} = (\mathbb{Z}/n\mathbb{Z})/2(\mathbb{Z}/n\mathbb{Z})$.

Now assume that n is odd (i.e. both r and m are odd). Then $(\mathbb{Z}/n\mathbb{Z})/2(\mathbb{Z}/n\mathbb{Z}) = 0$, hence $\pi_1(\overline{H})_{\mathfrak{g}} = 0$ and the homomorphism

$$\pi_1(\overline{H})_{\mathfrak{g}} \to \pi_1(\overline{G})_{\mathfrak{g}}$$

is injective. Since H^{sc} embeds into G^{sc}, by Corollary A.3.3 the homomorphism $\pi_1(\overline{H}) \to \pi_1(\overline{G})$ is also injective. By Theorem A.1.2 the set $X(K)/G(K)$ is finite and for almost all v the group $G(K_v)$ acts transitively on $X(K_v)$.

Similar examples can be constructed for G and H of type E_6 (then $\pi_1(\overline{H}) = \mathbb{Z}/3\mathbb{Z}$) and for G and H of types E_8, F_4 and G_2.

A.6.2. *Examples with spherical non-symmetric H.* We are interested in examples of (G, H), where G and H are connected semisimple, H is a spherical non-symmetric subgroup of G, and H is a maximal connected subgroup of G. From [42, Tab. 1] and [Vi01, Ch. I §3, Table 1] one can see that there are only two such examples:

(a) G is a form of SO_7, H is a form of G_2;

(b) G is a form of G_2, H is a form of SL_3.

In both cases G is adjoint and $\pi_1(\overline{H}) = 0$. By Corollary A.2.1 the set $X(K)/G(K)$ is finite and for almost all v the group $G(K_v)$ acts transitively on $X(K_v)$. Note that in case (a) any K-form H of G_2 appears in such a pair (G, H).

A.6.3. *Examples with non-spherical H.* There are lots of pairs (G, H) satisfying assumptions (i) and (ii) of Theorem 1.1, that is such that H is a semisimple maximal connected subgroup of a connected semisimple group G. They were classified by Dynkin [Dy52a], [Dy52b]. In particular, if H is a an almost simple group over $\overline{K}$ and V an irreducible representation of H, then almost always H is a maximal connected subgroup in $G = \mathrm{SL}(V)$, $\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$, with a small number of exceptions, see [Dy52a, Thm. 1.5].

We list such pairs (G, H) satisfying also the assumption (iii) of Theorem 1.1, in the simplest case when H is PSL_2. The pairs are:

(a) $\mathrm{PSL}_2 \subset \mathrm{PSp}_m$, $m \geq 2$; we set $n = 2m$;

(b) $\mathrm{PSL}_2 \subset \mathrm{SO}_{2m+1}$, $m \geq 5$, $m \equiv 1, 2 \mod 4$; we set $n = 2m + 1$.

In each case (a) and (b) the embedding is given by the standard n-dimensional irreducible representation of SL_2. Dynkin [Dy52a, Thm. 1.5] proved that H is maximal in G. Under the chosen conditions on m the map $H^{\mathrm{sc}} \to G^{\mathrm{sc}}$ is an embedding, and the map $\pi_1(\overline{H}) \to \pi_1(\overline{G})$ is an isomorphism. By Corollary A.2.2 the set $X(K)/G(K)$ is finite and for almost all v the group $G(K_v)$ acts transitively on $X(K_v)$. Note that, using twisting, we can obtain such a pair (G, H) with any K-form H of PSL_2.

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