LIMIT THEOREMS FOR RANK-ONE LIE GROUPS

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Abstract. We investigate asymptotic behaviour of averaging operators for actions of simple rank-one Lie groups. It was previously known that these averaging operators converge almost everywhere, and we establish a more precise asymptotic formula that describes their deviations from the limit.

1. Introduction

Let (X, μ) be a probability space and $\{T_t\}_{t\in\mathbb{R}}$ a measure preserving one-parameter flow on X. It is a fundamental problem in ergodic theory to analyse the statistical properties of "observables" $f(T_t x)$ defined for a measurable function $f : X \to \mathbb{C}$ and a point $x \in X$. In particular, one would like to understand asymptotic behaviour of the time averages

$$(\mathcal{A}_t f)(x) := \frac{1}{t}\int_0^t f(T_s x)\, ds.$$

It is natural to expect that for chaotic flows the quantities $f(T_t x)$ should behave similarly to independent identically distributed random variables. Indeed, assuming that the flow is ergodic, it follows from the Pointwise Ergodic Theorem that for every $f \in L^1(X)$, the average $\mathcal{A}_t f$ converges almost everywhere to $\int_X f\, d\mu$ as $t \to \infty$, which is reminiscent of the Law of Large Numbers, and assuming that the flow is aperiodic, there exists a function $f \in L^2(X)$ such that the normalised deviations $\frac{\mathcal{A}_t f - \int_X f\, d\mu}{\|\mathcal{A}_t f - \int_X f\, d\mu\|_2}$ converge in distribution to the standard normal law as $t \to \infty$ (see [BD87]), which is reminiscent of the Central Limit Theorem. The convergence of deviations to the normal law is a widespread phenomenon that holds, for instance, for all sufficiently smooth functions in the setting of the geodesic flows on compact manifolds of negative curvature (see [S60, R73]).

In this paper we are interested in the behaviour of deviations of averages for actions of an interesting class of large groups—the connected simple rank-one Lie groups with finite centre. For instance, our results apply to the groups of orientation preserving isometries of the hyperbolic spaces.

Let G be a connected simple rank-one Lie group with finite centre that acts measure-preservingly on a standard probability space (X, μ).

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We fix a maximal compact subgroup K in G and consider the sets

$$B_t := \{g \in G : \, d(gK, K) \leq t\}, \tag{1}$$

where d is the canonical Riemannian metric on the corresponding symmetric space G/K. Given a measurable function $f : X \to \mathbb{C}$ and point $x \in X$, we define the averaging operators

$$(\mathcal{A}_t f)(x) := \frac{1}{m(B_t)} \int_{B_t} f(g^{-1}x) \, dm(g), \tag{2}$$

where m denotes a Haar measure on G. A Pointwise Ergodic Theorem for these operators has been established in [N94, N97, NS97]. It shows that for every $f \in L^2(X)$,

$$\mathcal{A}_t f \to P_0 f \quad \text{as } t \to \infty, \tag{3}$$

in L^2-norm and almost everywhere, where P_0 denotes the orthogonal projection on the subspace of G-invariant functions. Our aim here is to investigate asymptotics of the deviations $\mathcal{A}_t f - P_0 f$. It turns out that they exhibit a very different behaviour compared with one-parameter flows. For instance, $\frac{\mathcal{A}_t f - P_0 f}{\|\mathcal{A}_t f - P_0 f\|_2}$ might converge almost everywhere (see Theorem 1.4 below).

Our arguments are based on representation-theoretic techniques. We denote by $\hat{G}$ the set of equivalence classes of irreducible unitary representations of G equipped with the Fell topology. The action of G on X defines a natural unitary representation of G on $L^2(X)$ which can be disintegrated as

$$L^2(X) = \int_{\hat{G}}^{\oplus} d_\sigma \mathcal{H}_\sigma \, d\nu(\sigma), \tag{4}$$

where $\sigma : G \to U(\mathcal{H}_\sigma)$ are the corresponding irreducible representations, $d_\sigma \in \mathbb{N} \cup \{\infty\}$ their multiplicities, and ν is a Borel measure on $\hat{G}$. Let $\mathrm{Spec}_G(X) \subset \hat{G}$ be the support of the measure ν. We denote by $\hat{G}^1$ the subset of $\hat{G}$ consisting of spherical representations, that is, representations $\sigma : G \to U(\mathcal{H}_\sigma)$ such that $\mathcal{H}_\sigma$ contains a nonzero K-invariant vector. Since the averaging operators $\mathcal{A}_t$ are bi-invariant under K, only the spherical representations will play an essential role in our analysis. The set $\hat{G}^1$ has been described for rank-one groups by Kostant [K69], and it is customary to parametrise it as

$$\hat{G}^1 \simeq \{\rho\} \cup (0, \rho'] \cup i\mathbb{R}^+, \tag{5}$$

where $0 < \rho' \leq \rho$. Here $\{\rho\}$ corresponds to the trivial representation, $(0, \rho']$ corresponds to the complementary series representations, and $i\mathbb{R}^+$ corresponds to the principle series representations. For $G = \mathrm{SO}^\circ(n, 1)$, we have $\rho = \rho' = \frac{n-1}{2}$.

A rich collection of examples of spaces equipped with actions of G, which play an important role in various number-theoretic applications, is provided by homogeneous spaces of algebraic groups. Let

$H \subset \mathrm{GL}_d(\mathbb{R})$ be a semisimple real algebraic group defined over $\mathbb{Q}$. Its congruence subgroup

$$\Gamma := \{\gamma \in H \cap \mathrm{GL}_d(\mathbb{Z}) : \gamma = \mathrm{id} \mod q\}$$

has finite covolume in H, and we take $X = H/\Gamma$ equipped with the normalised invariant measure μ. Let us also assume that G is the connected component of a real algebraic subgroup of H defined over $\mathbb{Q}$. Then we obtain a natural measure-preserving action $G \curvearrowright (X, \mu)$. We will call such actions *arithmetic*. Description of $\mathrm{Spec}_G(X)$ for arithmetic actions is the central problem in the theory of automorphic representations (see, for instance, [S05]).

The following conjecture has been formulated in [BLS92]:

Conjecture 1.1 (Purity). *The spectrum of arithmetic actions $G \curvearrowright (X, \mu)$ for $G = \mathrm{SO}^\circ(n, 1)$ satisfies*

$$\mathrm{Spec}_G(X) \cap \hat{G}^1 \subset \bigcup_{0 \le j < \frac{n-1}{2}} \left\{ \frac{n-1}{2} - j \right\} \cup i\mathbb{R}^+.$$

Although this conjecture was formulated only for homogeneous spaces of $\mathrm{SO}^\circ(n, 1)$, it follows from [BS91, Th. 1.1(a)] that it must also hold for general arithmetic actions of $\mathrm{SO}^\circ(n, 1)$.

It was demonstrated in [BLS92] that there are arithmetic actions for which representations corresponding to $\frac{n-1}{2} - j$ do occur in $\mathrm{Spec}_G(X)$. Conjecture 1.1 was partially proved by Bergeron and Clozel [BC10] who showed that for arithmetic actions of $\mathrm{SO}^\circ(n, 1)$,

$$\mathrm{Spec}_G(X) \cap \hat{G}^1 \subset \bigcup_{0 \le j < \frac{n-1}{2}} \left\{ \frac{n-1}{2} - j \right\} \cup \left(0, \frac{1}{2} - \frac{1}{N^2 + 1}\right] \cup i\mathbb{R}^+,$$

where $N = n$ if n is even and $N = n + 1$ if n is odd. We note that a similar purity phenomenon is expected to hold for other rank-one groups. In particular, it was shown in [BC05, Ch. 6] that Arthur's conjectures [A89] imply that such a property holds for arithmetic actions of $\mathrm{SU}(n, 1)$.

Motivated by Conjecture 1.1 and supporting results from [BC10], we investigate asymptotic behaviour of the averaging operators $\mathcal{A}_t$ under the following purity assumption (see Figure 1):

Assumption 1.2 (Purity). A measure-preserving action of G on (X, μ) satisfies

$$\mathrm{Spec}_G(X) \cap \hat{G}^1 = \{s_0, \ldots, s_k\} \cup \Omega$$

for $s_0 = \rho > s_1 > \cdots > s_k > r$ and a subset Ω of $(0, r] \cup i\mathbb{R}^+$.

Our first result concerns convergence in L^2-norm.

Theorem 1.3. *Let $G \curvearrowright (X, \mu)$ be a measure-preserving action on a standard probability space satisfying Assumption 1.2. Then there*

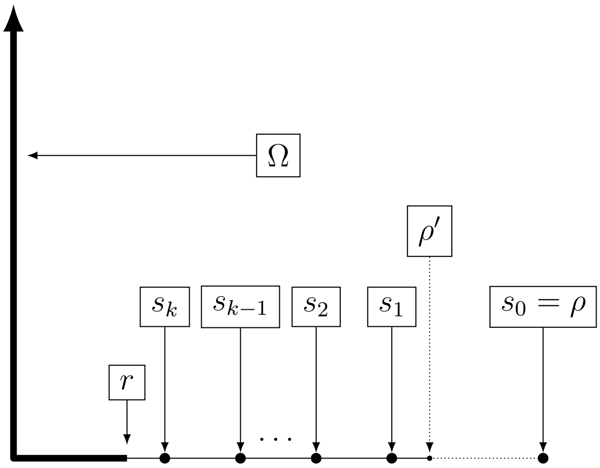

FIGURE 1. Purity.

exist pairwise orthogonal projections $P_0, \dots, P_k$ *on* $L^2(X)$ *and smooth functions* $\psi_0, \dots, \psi_k$ *such that*

$$\psi_0 = 1, \qquad \psi_j(t) \sim_{t\to\infty} c_j\, e^{-(\rho-s_j)t} \text{ with } c_j > 0,\ \ j = 1, \dots, k,$$

and for every $f \in L^2(X)$ *and* $t \geq 1$,

$$\left\| \mathcal{A}_t f - \sum_{j=0}^{k} \psi_j(t)\, P_j f \right\|_2 \ll t e^{-(\rho-r)t}\, \|f\|_2.$$

The projection P_0 in Theorem 1.3 is the orthogonal projection on the subspace of G-invariant functions in $L^2(X)$. When $k = 0$, Theorem 1.3 gives a quantitative Mean Ergodic Theorem for the operators $\mathcal{A}_t$ as in (3).

We also establish an almost everywhere asymptotic formula for the averaging operators at integral times.

Theorem 1.4. *Let* $G \curvearrowright (X, \mu)$ *be a measure-preserving action on a standard probability space satisfying Assumption 1.2. Then there exist pairwise orthogonal projections* $P_0, \dots, P_k$ *on* $L^2(X)$ *and functions* $\psi_0, \dots, \psi_k$ *as in Theorem 1.3 such that for every* $f \in L^2(X)$, *almost every* $x \in X$, *and* $n \in \mathbb{N}$,

$$\left| (\mathcal{A}_n f)(x) - \sum_{j=0}^{k} \psi_j(n)\, (P_j f)(x) \right| \leq C_\epsilon(x, f)\, n^{3/2+\epsilon} e^{-(\rho-r)n}, \quad \epsilon > 0,$$

where $\|C_\epsilon(\cdot, f)\|_2 \ll_\epsilon \|f\|_2$.

Theorem 1.4, in particular, implies that for $f \in L^2(X)$ with $P_1 f \neq 0$,

$$\|\mathcal{A}_n f - P_0 f\|_2 \approx e^{-(\rho-s_1)t},$$

and $\frac{\mathcal{A}_n f - P_0 f}{\|\mathcal{A}_n f - P_0 f\|_2}$ converges almost everywhere to $\frac{P_1 f}{\|P_1 f\|_2}$ as $n \to \infty$.

Using an interpolation argument, we deduce an almost everywhere asymptotic formula for the averaging operators at continuous times.

Theorem 1.5. *Let $G \curvearrowright (X, \mu)$ be a measure-preserving action on a standard probability space satisfying Assumption 1.2. Then there exist pairwise orthogonal projections $P_0, \ldots, P_k$ on $L^2(X)$ and functions $\psi_0, \ldots, \psi_k$ as in Theorem 1.3 such that for every $f \in L^p(X)$, $p > 2$, almost every $x \in X$, and $t \geq 1$,*

$$\left| (\mathcal{A}_t f)(x) - \sum_{j=0}^{k} \psi_j(t)\, (P_j f)(x) \right| \leq C_p(x, f)\, e^{-(1/2-1/p)(\rho - r)t},$$

where $\|C_p(\cdot, f)\|_2 \ll_p \|f\|_p$.

We note that the interpolation argument in Theorem 1.5 inevitably diminishes the quality of the error term, and only the summands with

$$s_j > (1/2 - 1/p)r + (1/2 + 1/p)\rho$$

provide nontrivial information because the other summands are of smaller order than the error term.

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2. Preliminaries

Throughout the paper, G is a connected simple rank-one Lie group with finite centre, and K is a maximal compact subgroup of G. We fix a one-parameter subgroup $A = \{a_t\}_{t \in \mathbb{R}}$ such that the Cartan decomposition

$$G = K\, \{a_t\}_{t \geq 0}\, K$$

holds and $d(a_t K, K) = t$, where d denotes the canonical Riemannian metric on the symmetric space G/K.

Let m be a Haar measure on G. With respect to the Cartan decomposition, it can be given as [H, Ch. 10]

$$dm(k_1, t, k_2) = dm_K(k_1)\, \Delta(t) dt\, dm_K(k_2),$$

with the probability Haar measure m_K on K and

$$\Delta(t) = (\sinh t)^{n_1} (\sinh 2t)^{2n_1},$$

where n_1, n_2 denote the dimensions of the corresponding root spaces. In particular, for the Riemannian ball B_t, defined in (1), we have

$$m(B_t) = \int_0^t \Delta(t)\, dt \sim_{t \to \infty} c\, e^{2\rho t} \tag{6}$$

with $c > 0$ and $\rho = (n_1 + 2n_2)/2$. Note that the parameter ρ here is the same as in (5). It also follows from (6) that for every $t \geq 1$ and $\epsilon \in (0, 1)$,

$$m(B_{t+\epsilon} \backslash B_t) \ll \epsilon\, m(B_t). \tag{7}$$

We denote by β_t the uniform probability measure on G supported on the set B_t, namely,

$$d\beta_t(g) := \frac{\mathbf{1}_{B_t}(g)}{m(B_t)}\, dm(g).$$

Given a unitary representation $\sigma : G \to U(\mathcal{H})$, we define the corresponding averaging operator

$$\sigma(\beta_t) : \mathcal{H} \to \mathcal{H} : v \mapsto \int_G \sigma(g)v\, d\beta_t(g).$$

In particular, for the unitary representation

$$\pi(g)f(x) = f(g^{-1}x), \quad f \in L^2(X),$$

induced by a measure-preserving action of G on (X, μ), $\pi(\beta_t)$ is equal to the operator $\mathcal{A}_t$ defined in (2).

If the representation σ is irreducible, the subspace $\mathcal{H}^K$ of K-invariant vectors has dimension at most one. If $\mathcal{H}^K$ has dimension one, σ is called spherical. In this case, we fix $v_\sigma \in \mathcal{H}^K$ with $\|v_\sigma\| = 1$. The corresponding spherical function is defined by

$$\phi_\sigma(g) := \langle \sigma(g)v_\sigma, v_\sigma \rangle .$$

Using the identification (5), we also write ϕ_s with $s \in \{\rho\} \cup (0, \rho'] \cup i\mathbb{R}^+$. An explicit integral formula for the spherical functions has been derived in [HC58a, HC58b], and in the case of rank-one groups they can be also expressed in terms of Jacobi functions [K84]. We refer to the monograph [GV88] for a comprehensive theory of spherical functions. In particular, we recall that by [GV88, 5.1.18],

$$|\phi_s(a_t)| \ll e^{-(\rho - \mathrm{Re}(s))t}(1 + t), \quad t \geq 0, \tag{8}$$

and by [K84, (2.19)], when $0 < s < \rho$

$$|\phi_s(a_t)| \sim_{t\to\infty} c(s)\, e^{-(\rho - s)t} \tag{9}$$

for some $c(s) > 0$.

3. Proofs

Proof of Theorem 1.3. We first investigate the behaviour of averages $\sigma(\beta_t)$ for irreducible representations $\sigma : G \to U(\mathcal{H})$. Since the measure β_t is left K-invariant, it follows that $\sigma(\beta_t)v$ is K-invariant for every $v \in \mathcal{H}$. In particular, $\sigma(\beta_t) = 0$ if σ is not spherical.

Now we assume that σ is spherical. Since β_t is right K-invariant, for every $v \in \mathcal{H}$,

$$\sigma(\beta_t)v = \sigma(\beta_t)\bar{v},$$

where

$$\bar{v} := \int_K \sigma(k)v \, dm_K(k).$$

Since $\dim(\mathcal{H}^K) = 1$,

$$\bar{v} = \langle \bar{v}, v_\sigma \rangle \, v_\sigma = \left(\int_K \langle \sigma(k)v, v_\sigma \rangle \; dm_K(k) \right) v_\sigma = \langle v, v_\sigma \rangle \, v_\sigma,$$

and

$$\sigma(\beta_t)v_\sigma = \langle \sigma(\beta_t)v_\sigma, v_\sigma \rangle \, v_\sigma = \psi_\sigma(t)v_\sigma,$$

where

$$\psi_\sigma(t) := \frac{1}{m(B_t)} \int_{B_t} \phi_\sigma(g) \, dm(g).$$

Hence, we conclude that for any irreducible representation σ,

$$\sigma(\beta_t) = \psi_\sigma(t) \, P_\sigma, \tag{10}$$

where P_σ is the orthogonal projection on $\mathcal{H}^K$. Clearly, the same formula also holds for representations which are direct sums of the representation σ.

Using the decomposition (4), every $f \in L^2(X)$ can be written as

$$f = \int_{\hat{G}}^{\oplus} f_\sigma \, d\nu(\sigma),$$

where $f_\sigma \in d_\sigma \mathcal{H}_\sigma$, and by (10),

$$\pi(\beta_t)f = \int_{\hat{G}}^{\oplus} \sigma(\beta_t) f_\sigma \, d\nu(\sigma) = \int_{\hat{G}^1}^{\oplus} \psi_\sigma(t) \, (P_{d_\sigma \sigma} f_\sigma) \, d\nu(\sigma).$$

Taking into account Assumption 1.2, we obtain

$$\pi(\beta_t)f = \sum_{j=0}^{k} \psi_j(t) \, P_j f + \int_{\Omega}^{\oplus} \psi_\sigma(t) \, (P_{d_\sigma \sigma} f_\sigma) \, d\nu(\sigma),$$

where

$$\psi_j(t) := \frac{1}{m(B_t)} \int_{B_t} \phi_{s_j}(g) \, dm(g),$$

and P_j are pairwise orthogonal projections.

Since by (8),

$$|\phi_\sigma(a_\tau)| \ll e^{-(\rho - r)\tau}(1 + \tau) \quad \text{for all } \sigma \in \Omega \text{ and } \tau \geq 0,$$

we deduce that for all $\sigma \in \Omega$ and $t \geq 1$,

$$\begin{aligned} |\psi_\sigma(t)| &\ll e^{-2\rho t} \int_0^t e^{2\rho\tau} |\phi_\sigma(a_\tau)| \, d\tau \\ &= e^{-2\rho t} \int_0^t e^{(\rho + r)\tau} (1 + \tau) \, d\tau \ll t e^{-(\rho - r)t}. \end{aligned}$$

This implies that

$$\left\|\pi(\beta_t)f - \sum_{j=0}^{k} \psi_j(t)\, P_j f\right\|_2^2 = \int_\Omega^\oplus |\psi_\sigma(t)|^2\, \|P_{d_\sigma \sigma} f_\sigma\|^2\, d\nu(\sigma)$$
$$\ll \left(te^{-(\rho-r)t}\right)^2 \|f\|_2^2,$$

which is the main estimate of the theorem.

Finally, we observe that since $\phi_\rho = 1$, we have $\psi_0 = 1$, and it follows from l'Hôpital's rule and (9) that

$$\psi_j(t) = \frac{\int_0^t \phi_{s_j}(a_\tau)\Delta(\tau)\, d\tau}{\int_0^t \Delta(\tau)\, d\tau} \sim_{t\to\infty} c_j\, e^{-(\rho - s_j)t}$$

for some $c_j > 0$. This completes the proof of the theorem. □

Proof of Theorem 1.4. Let $\epsilon > 0$. For $f \in L^2(X)$, we set

$$C_\epsilon(\cdot, f) := \left(\sum_{n=1}^{\infty} n^{-3-2\epsilon} e^{2(\rho-r)n} \left|\pi(\beta_n)f - \sum_{j=0}^{k} \psi_j(n)\, P_j f\right|^2\right)^{1/2}.$$

Then by Theorem 1.3,

$$\|C_\epsilon(\cdot, f)\|_2^2 \le \sum_{n=1}^{\infty} n^{-3-2\epsilon} e^{2(\rho-r)n} \left\|\pi(\beta_n)f - \sum_{j=0}^{k} \psi_j(n)\, P_j f\right\|_2^2$$
$$\ll \left(\sum_{n=1}^{\infty} n^{-1-2\epsilon}\right) \|f\|_2^2 < \infty.$$

This shows that $\|\, C_\epsilon(\cdot, f)\,\|_2 \ll_\epsilon \|f\|_2$. In particular, $C_\epsilon(\cdot, f)$ is finite almost everywhere. For almost every $x \in X$,

$$\left|\pi(\beta_n)f(x) - \sum_{j=0}^{k} \psi_j(n)\, (P_j f)(x)\right| \le C_\epsilon(x, f)\, n^{3/2+\epsilon} e^{-(\rho-r)n},$$

as desired. □

We start the proof of Theorem 1.5 with two lemmas:

Lemma 3.1. *For all $t \ge 1$ and $0 < \epsilon < 1$,*

$$|\psi_j(t+\epsilon) - \psi_j(t)| \ll \epsilon.$$

Proof. Since $|\phi_s| \le 1$, we have

$$|\psi_j(t+\epsilon) - \psi_j(t)| = \left|\int_G \phi_{s_j}(g) \left(\frac{\mathbf{1}_{B_{t+\epsilon}}(g)}{m(B_{t+\epsilon})} - \frac{\mathbf{1}_{B_t}(g)}{m(B_t)}\right) dm(g)\right|$$
$$\ll \left\|\frac{\mathbf{1}_{B_{t+\epsilon}}(g)}{m(B_{t+\epsilon})} - \frac{\mathbf{1}_{B_t}(g)}{m(B_t)}\right\|_1,$$

and by (7),

$$\left\| \frac{\mathbf{1}_{B_{t+\epsilon}}(g)}{m(B_{t+\epsilon})} - \frac{\mathbf{1}_{B_t}(g)}{m(B_t)} \right\|_1 = \frac{m(B_{t+\epsilon}\backslash B_t)}{m(B_{t+\epsilon})} \ll \epsilon, \tag{11}$$

which completes the proof. □

Lemma 3.2. *For $f \in L^\infty(X)$, $t \geq 1$, $0 < \epsilon < 1$, and almost every $x \in X$,*

$$|\pi(\beta_{t+\epsilon})f(x) - \pi(\beta_t)f(x)| \ll \|f\|_\infty \, \epsilon.$$

Proof. Since for almost every $x \in X$,

$$\begin{aligned} |\pi(\beta_{t+\epsilon})f(x) - \pi(\beta_t)f(x)| &= \left| \int_G f(g^{-1}x) \left(\frac{\mathbf{1}_{B_{t+\epsilon}}(g)}{m(B_{t+\epsilon})} - \frac{\mathbf{1}_{B_t}(g)}{m(B_t)} \right) dm(g) \right| \\ &\leq \|f\|_\infty \left\| \frac{\mathbf{1}_{B_{t+\epsilon}}(g)}{m(B_{t+\epsilon})} - \frac{\mathbf{1}_{B_t}(g)}{m(B_t)} \right\|_1, \end{aligned}$$

the lemma follows from (11). □

Proof of Theorem 1.5. Let $\delta = \rho - r$ and $\{t_n\} \subset [1, \infty)$ be an increasing sequence such that each interval $[m, m+1]$, $m \in \mathbb{N}$, is divided in to $\lfloor e^{\delta m/2} + 1 \rfloor$ sub-intervals of equal length. Then,

$$\sum_{n=1}^{\infty} t_n^2 e^{-\delta\, t_n} \leq \sum_{m=1}^{\infty} (m+1)^2 \lfloor e^{\delta m/2} + 1 \rfloor e^{-\delta m} < \infty. \tag{12}$$

We first show that the claim of the theorem holds along times t_n. For $f \in L^2(X)$, we set

$$C(\cdot, f) := \left(\sum_{n=1}^{\infty} e^{\delta t_n} \left| \pi(\beta_{t_n})f - \sum_{j=0}^{k} \psi_j(t_n)\, P_j f \right|^2 \right)^{1/2}.$$

By Theorem 1.3,

$$\begin{aligned} \|C(\cdot, f)\|_2^2 &\leq \sum_{n=1}^{\infty} e^{\delta t_n} \left\| \pi(\beta_{t_n})f - \sum_{j=0}^{k} \psi_j(t_n)\, P_j f \right\|_2^2 \\ &\ll \left(\sum_{n=1}^{\infty} t_n^2 e^{-\delta t_n} \right) \|f\|_2^2 < \infty, \end{aligned}$$

Hence, for almost all $x \in X$, we have $C(x, f) < \infty$, and

$$\left| \pi(\beta_{t_n})f(x) - \sum_{j=0}^{k} \psi_j(t_n)\, (P_j f)(x) \right| \leq C(x, f)\, e^{-\delta t_n/2}.$$

Now, let $t \geq 1$, and suppose that $t_n \leq t < t_{n+1}$. Then by Lemmas 3.1–3.2,

$$\begin{aligned}\left|\pi(\beta_t)f - \sum_{j=0}^{k} \psi_j(t)\, P_j f\right| &\leq \left|\pi(\beta_{t_n})f - \sum_{j=0}^{k} \psi_j(t_n)\, P_j f\right| \\ &\quad + |\pi(\beta_t)f - \pi(\beta_{t_n})f| \\ &\quad + \sum_{j=0}^{k} |P_j f||\psi_j(t) - \psi_j(t_n)| \\ &\ll C(\cdot, f)\, e^{-\delta t_n/2} + \lfloor e^{\delta \lfloor t_n \rfloor/2} + 1 \rfloor^{-1} \|f\|_\infty \\ &\quad + \sum_{j=0}^{k} |P_j f|\, \lfloor e^{\delta \lfloor t_n \rfloor/2} + 1 \rfloor^{-1} \\ &\ll \left(C(\cdot, f) + \|f\|_\infty + \sum_{j=0}^{k} |P_j f| \right) e^{-\delta t/2}.\end{aligned}$$

Since

$$\|C(\cdot, f)\|_2 \ll \|f\|_2 \leq \|f\|_\infty$$

and

$$\|P_j f\|_2 \leq \|f\|_2 \leq \|f\|_\infty,$$

we deduce that for every $f \in L^\infty(X)$,

$$\left\| \sup_{t \geq 1} e^{\delta t/2} \left| \pi(\beta_t)f - \sum_{j=0}^{k} \psi_j(t)\, P_j f \right| \right\|_2 \ll \|f\|_\infty. \tag{13}$$

By [NS97, Th. 4], for every $f \in L^2(X)$,

$$\left\| \sup_{t \geq 1} |\pi(\beta_t)f| \right\|_2 \ll \|f\|_2.$$

Therefore, for any $f \in L^2(X)$, we also have an estimate

$$\begin{aligned}&\left\| \sup_{t \geq 1} \left| \pi(\beta_t)f - \sum_{j=0}^{k} \psi_j(t)\, P_j f \right| \right\|_2 \\ &\leq \left\| \sup_{t \geq 1} |\pi(\beta_t)f| \right\|_2 + \left\| \sup_{t \geq 1} \left| \sum_{j=0}^{k} \psi_j(t)\, P_j f \right| \right\|_2 \ll \|f\|_2.\end{aligned} \tag{14}$$

Now we combine estimates (13) and (14) using Stein's Complex Interpolation Theorem [SW, Sec. V.4]. For a measurable function $\tau : X \to [1, \infty)$ and complex parameter z, we define a family of operators

$$U_z^\tau f(x) := e^{z\delta\tau(x)/2}\left(\pi(\beta_{\tau(x)})f(x) - \sum_{j=0}^{k}\psi_j(\tau(x))\,(P_jf)(x)\right).$$

It follows from (13) that when $\mathrm{Re}(z) = 1$, the operator

$$U_z^\tau : L^\infty(X) \to L^2(X)$$

is a bounded, and when $\mathrm{Re}(z) = 0$, the operator

$$U_z^\tau : L^2(X) \to L^2(X)$$

is bounded, with bounds independent of τ. Therefore, by the Complex Interpolation Theorem, for every $u \in (0,1)$, the operator

$$U_u^\tau : L^p(X) \to L^q(X)$$

with

$$\frac{1}{p} = \frac{1-u}{2} \quad \text{and} \quad \frac{1}{q} = \frac{1-u}{2} + \frac{u}{2}$$

is bounded as well with a bound independent of τ. By taking a supremum over all τ, we deduce that for every $u \in (0,1)$,

$$\left\|\sup_{t\ge 1} e^{u\delta t/2}\left|\pi(\beta_t)f - \sum_{j=0}^{k}\psi_j(t)\,P_jf\right|\right\|_2 \ll_u \|f\|_{2/(1-u)}.$$

Let

$$C_u(x,f) := \sup_{t\ge 1} e^{u\delta t/2}\left|\pi(\beta_t)f(x) - \sum_{j=0}^{k}\psi_j(t)\,(P_jf)(x)\right|.$$

Then for almost every $x \in X$, we have $C_u(x,f) < \infty$, and

$$\left|\pi(\beta_t)f(x) - \sum_{j=0}^{k}\psi_j(t)\,(P_jf)(x)\right| \le C_u(x,f)\,e^{-u\delta t/2},$$

proving the theorem. □

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