Equidistribution on homogeneous spaces

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(joint work with Barak Weiss)

Problem

Let Γ be a discrete group acting on a compact topological space X. We study distribution of orbits of Γ in X:

- Fix a set of "balls" $\Gamma_T \subset \Gamma$, $T > 0$.

- For $f \in C(X)$ and $x_0 \in X$, what is the asymptotic behavior of

$$\sum_{\gamma \in \Gamma_T} f(\gamma \cdot x_0)?$$

Example (Bogolubov-Krylov + Birkhoff)

For a homeomorphism T of X, there exists an invariant measure μ such that for μ-a.e. $x_0 \in X$ and $f \in C(X)$,

$$\frac{1}{2N+1} \sum_{i=-N}^{N} f(T^i x_0) \to \int_X f \, d\mu$$

as $N \to \infty$.

Essentially, the same result has been extended to the class of amenable groups.

What about nonamenable groups?

Known results about equidistribution:

1. (Arnold-Krylov) Two isometries acting on S^2.
2. (Guivarc'h) Translations on homogeneous spaces of compact groups.
3. (Kazhdan, Guivarc'h, Vorobets) Groups acting by affine isometries on $\mathbb{R}^2$.
4. (Nevo-Stein, Grigorchuk, Bufetov) Pointwise ergodic theorem for free groups.
5. (Fujiwara-Nevo) Pointwise ergodic theorem for word-hyperbolic groups.

Let Γ be a lattice in a (noncompact) semisimple Lie group G (e.g., $\Gamma = \mathrm{SL}(n, \mathbb{Z})$). We prove equidistribution of all dense Γ-orbits for *algebraic* measure-preserving actions.

“Balls” in Γ

1. Let d be the Cartan-Killing Riemannian metric on the symmetric space $K\backslash G$ and
$$\Gamma_T = \{\gamma \in \Gamma : d(K\gamma, K) < \log T\}.$$

2. Let $\rho : G \to \mathrm{GL}(V)$ be a proper homomorphism, $\|\cdot\|$ a norm on $\mathrm{End}(V)$, and
$$\Gamma_T = \{\gamma \in \Gamma : \|\rho(\gamma)\| < T\}.$$

If G is not simple, we also need the following condition:
Balanced condition. For a decomposition $G = G_1 \cdot G_2$, where G_1 and G_2 are closed connected normal subgroups of G, and compact $C \subset G_1$,
$$\frac{\#(\Gamma_T \cap C \cdot G_2)}{\#\Gamma_T} \to 0 \quad \text{as } T \to \infty.$$

In case 1, this condition is equivalent to nonexistence of compact factors for G; in case 2, this condition may fail even for irreducible Γ.

Algebraic measure preserving actions

1. $\Gamma \subset G$ acts on L/Λ by left multiplication where

   - G is a closed subgroup of a Lie group L.

   - Λ is a lattice in L.

2. $\Gamma \subset G$ acts on L/Λ by automorphisms (e.g., $\mathrm{SL}(n, \mathbb{Z})$ action on $\mathbb{T}^n$) where

   - L is a Lie group.

   - Λ is a lattice in L.

   - G is a closed subgroup of $\mathrm{Aut}(L)$ such that Γ stabilizes Λ.

Main result

Theorem 1 *Consider an algebraic measure preserving action of* Γ *on a space* X *equipped with finite measure* μ*. Then for every* $x_0 \in X$ *such that* $\Gamma \cdot x_0$ *is dense in* X *and every* $f \in C_c(X)$*,*

$$\frac{1}{\#\Gamma_T} \sum_{\gamma \in \Gamma_T} f(\gamma \cdot x_0) \to \frac{1}{\mu(X)} \int_X f \, d\mu.$$

Previously, Hee Oh proved equidistribution of the action Γ by left multiplication on $\mathrm{SL}(n, \mathbb{R})/\Lambda$, where Γ and Λ are lattices in $\mathrm{SL}(n, \mathbb{R})$.

Sketch of the proof (inducing action)

Let

$$Y = (G \times X)/\sim,$$

where the equivalence relation is defined by

$$(g, x) \sim (g\gamma^{-1}, \gamma x) \qquad \text{for } \gamma \in \Gamma,$$

equipped with the measure

$$\nu = \mathrm{Vol}_{G/\Gamma} \otimes m.$$

The group G acts on Y by

$$g' \cdot [(g, x)] = [(g'g, x)].$$

We show that Theorem 1 is equivalent to equidistribution of G-orbits in Y:

For $y_0 = [(e, x_0)]$ and every $\phi \in C_c(Y)$,

$$(*) \quad \frac{1}{\mathrm{Vol}(G_T)} \int_{G_T} \phi(g \cdot y_0)\, dg \to \frac{1}{\nu(Y)} \int_Y \phi \, d\nu.$$

Sketch of the proof (proof of $(*)$)

We use Cartan decomposition $G = KA^+K$, where K is a maximal compact subgroup of G and A^+ is positive Weyl chamber. Then

$$dg = dk\, d\rho(a)\, dl, \quad g = kal.$$

Let $\alpha_i \in C(K)$, $i = 1, \ldots, N$, be a partition of unity and $l_i \in \mathrm{supp}(\alpha_i)$.

$$\begin{aligned}
&\int_{G_T} \phi(g \cdot y_0) dg \\
= \;& \int_K dk \int_K dl \int_{a:\|kal\|<T} \phi(kaly_0) d\rho(a) \\
= \;& \sum_{i=1}^N \int_K dk \int_K \alpha_i(l) dl \int_{a:\|kal\|<T} \phi(kaly_0) d\rho(a) \\
\simeq \;& \sum_{i=1}^N \int_K dk \int_K \alpha_i(l) dl \int_{a:\|kal_i\|<T} \phi(kaly_0) d\rho(a) \\
= \;& \sum_{i=1}^N \int_K dk \int_{a:\|kal_i\|<T} \left(\int_K \phi(kaly_0) \alpha_i(l) dl \right) d\rho(a).
\end{aligned}$$

Using Ratner theory (N. Shah), one shows that for $\phi \in C_c(Y)$ and $\alpha \in C(K)$,

$$\int_K \phi(aly_0)\alpha(l) dl \to \left(\int_K \alpha\, dl \right) \cdot \frac{1}{\nu(Y)} \int_Y \phi\, d\nu$$

as $a \to \infty$ in strong sense.

Spaces with infinite volume

Example (Ledrappier, Nogueira): Consider action of $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ on $\mathbb{R}^n$. If $x_0 \in \mathbb{R}^n$ is not a multiple of a rational vector, then $\Gamma \cdot x_0$ is dense in $\mathbb{R}^n$ and for every $f \in C_c(\mathbb{R}^n - \{0\})$,

$$\frac{1}{T^{(n-1)^2}} \sum_{\gamma\in\Gamma:\|\gamma\|<T} f(\gamma\cdot x_0) \to c_n \int_{\mathbb{R}^n-\{0\}} f(x)\,\frac{dx}{\|x\|}$$

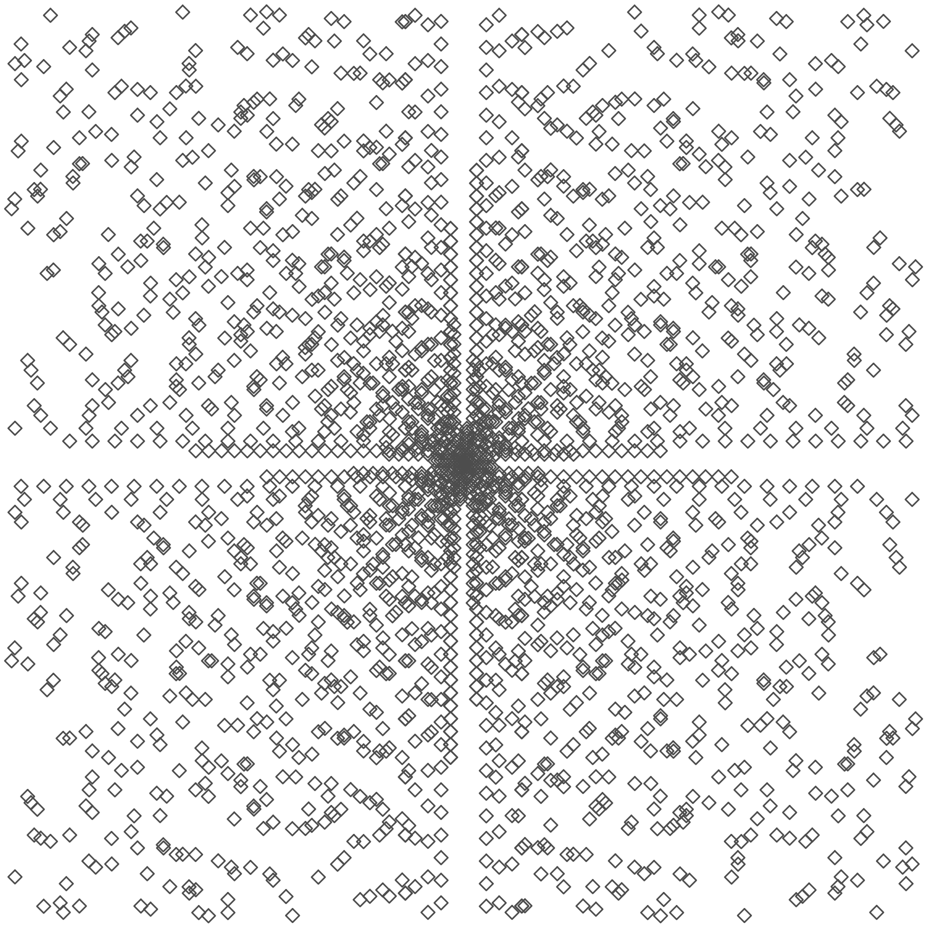
for some $c_n > 0$.

Note that

1. $\#\{\gamma \in \Gamma : \|\gamma\| < T\} \sim c \cdot T^{n^2-n}$.

2. The measure $\frac{dx}{\|x\|}$ is *not* invariant under Γ.

Example



Infinite volume homogeneous spaces

Let G be a connected Lie group, H a closed subgroup of G, and Γ a lattice in G. We investigate distribution of dense orbits for the action of Γ on G/H by left multiplication.

Asymptotics for discrete Γ-orbits in G/H (for reductive H) was obtained by Eskin, Mozes, Shah.

Skew balls: For $g_1, g_2 \in G$, define

$$H_T[g_1, g_2] = g_1^{-1} G_T g_2^{-1} \cap H.$$

Moderate volume growth condition: for any bounded $B \subset G$ and $\varepsilon > 0$, there exists a neighborhood of e in H such that for any $g_1, g_2 \in B$,

$$\mathrm{Vol}(\mathcal{O} \cdot H_T[g_1, g_2]) \leq (1+\varepsilon)\mathrm{Vol}(H_T[g_1, g_2]).$$

Asymptotics

Idea:

$$\begin{array}{ccc} & G & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\tau} \\ \Gamma\backslash G & & G/H \end{array}$$

Theorem 2 *For* $g_0H \in G/H$ *such that*

$$\overline{\Gamma g_0 H} = G,$$

the following is equivalent:

- *For any* $g \in G$ *and* $\phi \in C_c(\Gamma\backslash G)$,

$$\frac{1}{\mathrm{Vol}(H_T[g,g_0])}\int_{H_T[g,g_0]} \phi(\Gamma g_0 h^{-1})dh \to \int_{\Gamma\backslash G} \phi.$$

- *For every* $f \in C_c(G/H)$,

$$\sum_{\gamma\in\Gamma_T} f(\gamma\cdot g_0 H) \sim \int_{G_T} f(g\cdot g_0H)\,dg.$$

The first condition (equidistribution of skew-balls in $\Gamma\backslash G$) can be proved for semisimple groups H that satisfies *balanced condition* and for some unipotent groups H.

Balanced condition. For a decomposition $H = H_1 \cdot H_2$, where H_1 and H_2 are closed connected normal subgroups of H, and compact $C \subset H_1$,

$$\frac{\mathrm{Vol}(H_T \cap C \cdot H_2)}{\mathrm{Vol}(H_T)} \to 0 \quad \text{as } T \to \infty.$$

Equidistribution

Let $Y \subset G$ be measurable section of the projection map $\tau : G \to G/H$, i.e., the product map $Y \times H \to G$ is a Borel isomorphism. Then

$$dg = d\nu_Y(y) \otimes dh, \quad g = yh,$$

where dg and dh are right Haar measures on G and H, and ν_Y is a measure on Y. We assume existence of the limit

$$(+) \qquad \lim_{T\to\infty} \frac{\mathrm{Vol}(H_T[g_1, g_2])}{\mathrm{Vol}(H_T[e, e])} \stackrel{def}{=} \alpha(g_1, g_2).$$

Let

$$\nu_{g_0}(y) = \alpha(y, g_0^{-1}) d\nu_Y(y).$$

Main result

Theorem 3 *Assuming the moderate volume growth condition, (+), and equidistribution for the H-orbit in $\Gamma\backslash G$ (as in Theorem 2), we have*

$$\frac{1}{\mathrm{Vol}(H_T)} \sum_{\gamma \in \Gamma_T} f(\gamma \cdot g_0 H) \to \int_{G/H} f \, d\nu_{g_0}.$$

Proof.

$$\begin{aligned}
& \int_{G_T} f(g \cdot g_0 H)\, dg \\
= \ & \int_{g:\|g g_0^{-1}\|<T} f(gH)\, dg \\
= \ & \int_Y d\nu_Y(y) \int_{h:\|y h g_0^{-1}\|<T} f(yhH)\, dh \\
= \ & \int_Y f(yH) \mathrm{Vol}(H_T[y, g_0^{-1}])\, d\nu_Y(y) \\
\sim \ & \mathrm{Vol}(H_T) \cdot \int_{G/H} f \, d\nu_{g_0}.
\end{aligned}$$

Lattice points in affine symmetric spaces

G/H – affine symmetric space,

S – symmetric spaces of G,

d – Cartan-Killing metric on S,

Γ – lattice in G.

For $u, v \in S$,

$$\Gamma_t = \{\gamma \in \Gamma : d(u \cdot \gamma, v) \le t\},$$
$$H_t = \{h \in H : d(u \cdot h, v) \le t\}.$$

Corollary 4 *For*

$$x \in G/H \text{ such that } \overline{\Gamma \cdot x} = G/H,$$

we have

$$\frac{1}{\mathrm{Vol}(H_t)} \sum_{\gamma \in \Gamma_t} \delta_{\gamma x} \to c_x \cdot \mu$$

for $c_x > 0$ *and a smooth measure* μ *on* G/H*.*

Measure μ

$K \subset H$ – maximal compact subgroup,

$s_0 \in S$ – fixed point of K,

$\mathfrak{a} \subset \mathrm{Lie}(H)$ – Cartan subalgebra,

$a_0 \in \mathfrak{a}$ – barycenter (direction of maximal volume growth),

$$\delta = \lim_{t\to\infty} \frac{\log \mathrm{Vol}(G_t)}{t},$$

ν – G-invariant measure on G/H.

For $v \in S$ and $a \in \mathfrak{a}$, *Busemann* function:

$$\beta(v,a) = \lim_{t\to\infty} (d(s_0 \exp(ta), v) - t).$$

$$d\mu(y) = \left(\int_K \exp(-\delta\beta(vyk, a_0))dk\right) d\nu(y)$$

Oppenheim conjecture

Let

$$Q(x) = \sum_{i,j=1}^{d} a_{ij} x_i x_j, \quad x \in \mathbb{R}^d,$$

be a real nondegenerate indefinite quadratic form. Assume that $d \geq 3$ and Q is not a multiple of a rational form.

Theorem 5 (Margulis) *$Q(\mathbb{Z}^d)$ is dense in $\mathbb{R}$.*

For example, the set

$$\{m^2 + n^2 - \sqrt{2}k^2 : m, n, k \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$.

Theorem 6 (Eskin,Margulis,Mozes) *If*

$$\text{sign}(Q) \neq (2,1), (2,2),$$

then for any $(a,b) \subset \mathbb{R}$,

$$\#\{x \in \mathbb{Z}^d : a < Q(x) < b, \|x\| < T\} \sim c_Q (b-a) T^{d-2}$$

for some $c_Q > 0$.

Oppenheim conjecture for frames

Denote by B the bilinear form that corresponds to the quadratic form Q. Let

$\mathcal{F}^d = \{(f_1, \ldots, f_d) : f_i \in \mathbb{R}^d, \mathrm{Vol}(f_1, \ldots, f_d) = 1\}$

be the spaces of integral unimodular bases of $\mathbb{R}^d$. For $f \in \mathcal{F}^d$, the matrix

$$B(f) \stackrel{def}{=} (B(f_i, f_j))_{1 \leq i,j \leq d}$$

belongs to the variety

$$\mathcal{V}_Q = \left\{ g \in \mathrm{GL}(d, \mathbb{R}) : \begin{array}{l} \det(g) = \det(Q) \\ \mathrm{sign}(g) = \mathrm{sign}(Q) \end{array} \right\}$$

One can show that $B(\mathcal{F}^d_{\mathbb{Z}})$ is dense in $\mathcal{V}_Q$, and moreover,

Theorem 7 *Let* $\mathrm{sign}(B) = (p, q)$, $p \leq q$. *For a "nice" bounded* $\Omega \subset \mathcal{V}_Q$,

$$\#\{f \in \mathcal{F}^d_{\mathbb{Z}} : B(f) \in \Omega, \|f_i\| < T\}$$
$$\sim \; m_Q(\Omega) \cdot \begin{cases} T^{p(q-1)}, & p < q, \\ T^{p(p-1)}(\log T), & p = q, \end{cases}$$

where m_Q *is an (explicit) smooth measure on* $\mathcal{V}_Q$.