# LATTICE ACTION ON THE BOUNDARY OF $\mathrm{SL}(N,\mathbb{R})$

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Abstract. Let $\Gamma$ be a lattice in $G = \mathrm{SL}(n,\mathbb{R})$ and $X = G/S$ a homogeneous space of $G$, where $S$ is a closed subgroup of $G$ which contains a real algebraic subgroup $H$ such that $G/H$ is compact. We establish uniform distribution of orbits of $\Gamma$ in $X$ analogous to the classical equidistribution on torus. To obtain this result, we first prove an ergodic theorem along balls in the connected component of Borel subgroup of $G$.

## 1. Introduction

Let $G = \mathrm{SL}(n,\mathbb{R})$ and $\Gamma$ a lattice in $G$; that is, $\Gamma$ is a discrete subgroup of $G$ with finite covolume. We study the action of $\Gamma$ on a compact homogeneous space $X$ of algebraic origin. Namely, $X = G/S$ where $S$ is a closed subgroup of $G$ which contains the connected component of a real algebraic subgroup $H$ of $G$ such that $G/H$ is compact. An important example is provided by the Furstenberg boundary of $G$ [F63]. In this case, $X = G/B$ where $B$ is the subgroup of upper triangular matrices in $G$.

It is possible to deduce from a result of Dani [St, Theorem 13.1] that every orbit of $\Gamma$ in $X$ is dense. We will prove a quantitative estimate for the distribution of orbits.

Introduce a norm on $G$:

$$\|g\| = \left(\mathrm{Tr}({}^t gg)\right)^{1/2} = \left(\sum_{i,j} g_{ij}^2\right)^{1/2} \quad \text{for} \quad g = (g_{ij}) \in G. \tag{1}$$

For $T > 0$, $\Omega \subseteq X$, and $x_0 \in X$, define a counting function

$$N_T(\Omega, x_0) = |\{\gamma \in \Gamma : \|\gamma\| < T, \gamma \cdot x_0 \in \Omega\}|. \tag{2}$$

Let $m$ be a normalized $\mathrm{SO}(n)$-invariant measure on $X$. It follows from the Iwasawa decomposition (see (8) below) that $X$ is a homogeneous space of $\mathrm{SO}(n)$. Therefore, the measure $m$ is unique. The following theorem shows that orbits of $\Gamma$ in $X$ are uniformly distributed with respect to the measure $m$.

**Theorem 1.** *For a Borel set $\Omega \subseteq X$ such that $m(\partial\Omega) = 0$ and $x_0 \in X$,*

$$N_T(\Omega, x_0) \sim \frac{\gamma_n}{\bar{\mu}(\Gamma\backslash G)} m(\Omega) T^{n(n-1)} \quad \text{as} \quad T \to \infty, \tag{3}$$

*where $\gamma_n$ is a constant (defined in (14) below), and $\bar{\mu}$ is a finite $G$-invariant measure on $\Gamma\backslash G$ (defined in (10)).*

It would be interesting to obtain an estimate for the error term in (3). This, however, would demand introducing different techniques than those employed here.

This article is a part of author's PhD thesis at Ohio State University done under supervision of Prof. Bergelson.

Theorem 1 is analogous to the result of Ledrappier [L99] (see also [N00]) who investigated the distribution of dense orbits of a lattice $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ acting on $\mathbb{R}^2$. Ledrappier used the equidistribution property of the horocycle flow. Similarly, we deduce Theorem 1 from an equidistribution property of orbits of Borel subgroup.

Denote by $B^o$ the connected component of the upper triangular subgroup of $G$. To prove Theorem 1, we use the following ergodic theorem for the right action of $B^o$ on $\Gamma\backslash G$.

**Theorem 2.** *Let $\varrho$ be a right Haar measure on $B^o$, and $\nu$ the normalized $G$-invariant measure on $\Gamma\backslash G$. Then for any $\tilde{f} \in C_c(\Gamma\backslash G)$ and $y \in \Gamma\backslash G$,*

$$\frac{1}{\varrho(B^o_T)} \int_{B^o_T} \tilde{f}(yb^{-1})d\varrho(b) \to \int_{\Gamma\backslash G} \tilde{f} d\nu \quad as \quad T \to \infty,$$

*where $B^o_T = \{b \in B^o : \|b\| < T\}$.*

**Remarks.**

1. One can consider the analogous limit for a left Haar measure on $B^o$. In this case, it may happen that the limit is 0 for some $y \in \Gamma\backslash G$ and all $\tilde{f} \in C_c(\Gamma\backslash G)$ (see Proposition 8).
2. Since $B^o$ is solvable (hence, amenable), one might expect that convergence for a.e. $y \in \Gamma\backslash G$ follows from known ergodic theorems for amenable group actions. Moreover, since $\nu$ is the only normalized $B^o$-invariant measure on $\Gamma\backslash G$, one could expect that convergence holds for every $y$. However, this approach does not work because the sets $B^o_T$ do not form a Følner sequence, and the space $\Gamma\backslash G$ is not compact in general.
3. To prove Theorem 2, we use Ratner's classification of ergodic measures for unipotent flows [R91a]. In fact, we don't need the full strength of this result. Since the subgroup $U$ (defined in (37) below) is horospherical, it is enough to know classification of ergodic measures for horospherical subgroups. The situation is much easier in this special case (see [St, §13]).
4. We expect that analogs of Theorems 1 and 2 hold for a noncompact semisimple Lie group and its irreducible lattice with balls $B^o_T$ defined by the Riemann metric. The main difficulty here is to show that the measure of $B^o_T$ is "concentrated" on the "cone" $B^C_T$ (cf. Lemma 7).
5. It was pointed out by P. Sarnak that it might be possible to prove the results of this article using harmonic analysis on $\Gamma\backslash G$. In particular, Corollary 3 below can be deduced from the result of Good (Corollary on page 119 of [G]). Note that his method gives an estimate on the error term.

The paper is arranged as follows. In the next section, we give examples of applications of Theorem 1. In Section 3 we set up notations and prove some basic lemmas. Theorem 1 is deduced from Theorem 2 in Section 4. In Sections 5 and 6, we review results on the structure of unipotent flows and prove auxiliary facts about finite-dimensional representations of $\mathrm{SL}(n,\mathbb{R})$. Finally, Theorem 2 is proved in Section 7.

## 2. Examples

1. Let $X = \mathbb{R} \cup \{\infty\}$, which is considered as the boundary of the hyperbolic upper half plane. The group $G = \mathrm{SL}(2, \mathbb{R})$ acts on $X$ by fractional linear transformations:

$$g \cdot x = \frac{ax + b}{cx + d} \quad \text{for } x \in X,\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \tag{4}$$

Let $\Gamma$ be a lattice in $\mathrm{SL}(2, \mathbb{R})$. For $\Omega \subseteq X$ and $x_0 \in X$, define the counting function $N_T(\Omega, x_0)$ as in (2). Its asymptotics can be derived from Theorem 1. Note that the asymptotics of $N_T(X, x_0)$ as $T \to \infty$ provides a solution of the so-called hyperbolic circle problem (cf. [T, p. 266] and references therein).

**Corollary 3.** (of Theorem 1) *For $x_0 \in X$ and $-\infty \le a < b \le +\infty$,*

$$N_T((a, b), x_0) \sim c_\Gamma \left( \int_a^b \frac{dt}{1 + t^2} \right) T^2 \quad \textit{as} \quad T \to \infty,$$

*where $c_\Gamma = \frac{1}{2\bar{\mu}(\Gamma \backslash G)}$ ($\bar{\mu}$ is the $G$-invariant measure on $\Gamma \backslash G$ defined in (10)).*

*Proof.* It is easy to see from (4) that $G$ acts transitively on $X$, and the stabilizer of $\infty$ in $G$ is the group of upper triangular matrices $B$. Thus, Theorem 1 is applicable to the space $X$.

Note that

$$K = \mathrm{SO}(2) = \left\{ k_\theta = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix} : \theta \in [0, 1) \right\},$$

and the normalized Haar measure on $K$ is given by $dk = d\theta$. The measure $m$ on $X$ can be defined as the image of $dk$ under the map $K \to X : k \to k \cdot \infty$. By (4), $k_\theta \cdot \infty = -\mathrm{ctan}\ 2\pi\theta$. Then

$$m((a, b)) = \int_{k \cdot \infty \in (a,b)} dk = \int_{\substack{-\mathrm{ctan}\ 2\pi\theta \in (a,b) \\ \theta \in [0,1)}} d\theta = \frac{1}{\pi} \int_a^b \frac{dt}{1 + t^2}.$$

We have used the substitution $t = -\mathrm{ctan}\ 2\pi\theta$.

Finally, by Theorem 1,

$$N_T((a, b), x_0) \underset{T \to \infty}{\sim} \frac{\gamma_2 m((a, b))}{\bar{\mu}(\Gamma \backslash G)} T^2 = c_\Gamma \left( \int_a^b \frac{dt}{1 + t^2} \right) T^2.$$

Note that $\gamma_2 = \frac{\pi}{2}$ by (14) below. □

2. Let $X = \mathbb{P}^{n-1}$ be the projective space (more generally $X = \mathcal{G}_{n,k}$, Grassmann variety, or $X = \mathcal{F}_n$, flag variety), and $m$ be the rotation invariant normalized measure on $X$. Then the asymptotic estimate (3) holds for the standard action of $G = \mathrm{SL}(n, \mathbb{R})$ on $X$. This is a special case of Theorem 1 because $X$ can be identified with $G/S$ where $S$ is a closed subgroup of $G$ that contains $B$, the group of upper triangular matrices.

## 3. Basic facts

For $s = (s_1, \dots, s_n) \in \mathbb{R}^n$, define

$$a(s) = \mathrm{diag}(e^{s_1}, \dots, e^{s_n}).$$

For $t = (t_{ij} : 1 \le i < j \le n)$, $t_{ij} \in \mathbb{R}$, denote by $n(t)$ the unipotent upper triangular matrix with entries $t_{ij}$ above the main diagonal.

We use the following notations:

$$\begin{aligned} G &= \mathrm{SL}(n, \mathbb{R}), \\ K &= \mathrm{SO}(n), \\ A^o &= \{a(s) |\ s \in \mathbb{R}^n,\ \sum_i s_i = 0\}, \\ N &= \{n(t) |\ t_{ij} \in \mathbb{R}, 1 \le i < j \le n\}, \\ B^o &= A^o N = N A^o. \end{aligned}$$

For $s \in \mathbb{R}^n$, $\sum_i s_i = 0$, denote $\alpha_{ij}(s) = s_i - s_j$, where $i, j = 1, \dots, n$, $i < j$. These are the positive roots of the Lie algebra of $G$. Note that

$$\mathrm{Ad}_{a(s)} n(\{t_{ij}\}) = a(s) n(\{t_{ij}\}) a(s)^{-1} = n(\{e^{\alpha_{ij}(s)} t_{ij}\}). \tag{5}$$

Let

$$\delta(s) = \frac{1}{2} \sum_{i<j} \alpha_{ij}(s) = \sum_{1 \le k \le n} (n-k) s_k. \tag{6}$$

For $C \in \mathbb{R}$, define

$$A^C = \{a(s) \in A^o |\ s_i > C, i = 1, \dots, n-1\}.$$

Also put $B^C = A^C N$.

Let $dk$ be the normalized Haar measure on $K$. A Haar measure on $N$ is given by $dn = dt = \prod_{i<j} dt_{ij}$. A Haar measure on $A^o$ is $da = ds = ds_1 \dots ds_{n-1}$.

The product map $A^o \times N \to B^o$ is a diffeomorphism. The image of the product measure under this map is a left Haar measure on $B^o$. Denote this measure by $\lambda$. Then a right Haar measure $\varrho$ on $B^o$ can be defined by

$$\varrho(f) = \int_{B^o} f(b^{-1}) \lambda(b) = \int_{A^o \times N} f(a(s)n(t)) e^{2\delta(s)} ds dt, \quad f \in C_c(B^o). \tag{7}$$

The map corresponding to the Iwasawa decomposition

$$(k, a, n) \mapsto kan : K \times A^o \times N \to G \tag{8}$$

is a diffeomorphism. One can define a Haar measure $\mu$ on $G$ in terms of this decomposition:

$$\int_G f d\mu = \int_{K \times A^o \times N} f(ka(s)n(t)) e^{2\delta(s)} dk ds dt = \int_{K \times B^o} f(kb) dk d\varrho(b) \tag{9}$$

for $f \in C_c(G)$. For a lattice $\Gamma$ in $G$, there exists a finite measure $\bar{\mu}_\Gamma$ on $\Gamma \backslash G$ such that

$$\int_G f d\mu = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(\gamma g) d\bar{\mu}_\Gamma(g), \quad f \in C_c(G). \tag{10}$$

Let $\beta$ be an automorphism of $G$. Then $\beta(\Gamma)$ is a lattice too. Moreover, the following lemma holds.

**Lemma 4.** *Define a map*

$$\bar\beta : \Gamma\backslash G \to \beta(\Gamma)\backslash G : g\Gamma \mapsto \beta(g)\beta(\Gamma)$$

*Then* $\bar\beta(\bar\mu_\Gamma) = \bar\mu_{\beta(\Gamma)}$.

*Proof.* Since the automorphism group of $G$ is a finite extension of the group of inner automorphisms, and $G$ is unimodular, it follows that that the measure $\mu$ is $\beta$-invariant.

Every $\tilde f \in C_c(\beta(\Gamma)\backslash G)$ can be represented as $\sum_{\gamma\in\beta(\Gamma)} f(\gamma g)$ for some $f \in C_c(G)$ (see [R, Ch. 1]). Then

$$\begin{aligned}\bar\beta(\bar\mu_\Gamma)(\tilde f) &= \int_{\Gamma\backslash G} \sum_{\gamma\in\beta(\Gamma)} f(\gamma\beta(g))d\bar\mu_\Gamma(g) = \int_G f(\beta(g))d\mu(g) \\ &= \int_G f(g)d\mu(g) = \bar\mu_\Gamma(\tilde f).\end{aligned}$$

□

For a subset $D \subseteq G$ and $T > 0$, put

$$D_T = \{d \in D : \|d\| < T\}.$$

Note that

$$B^o_T = \left\{ a(s)n(t) : \sum_{1\le i\le n} e^{2s_i} + \sum_{1\le i<j\le n} e^{2s_i} t_{ij}^2 < T^2 \right\}. \tag{11}$$

For $s \in \mathbb{R}^n$, define

$$N(s) = \sum_i e^{2s_i}. \tag{12}$$

**Lemma 5.** *For* $C \in \mathbb{R}$,

$$\varrho(B^C_T) = c_n \int_{A^C_T} \left(T^2 - N(s)\right)^{\frac{n(n-1)}{4}} \exp\left(\sum_k (n-k)s_k\right) ds,$$

*where* $c_n = \pi^{n(n-1)/4}/\Gamma(1 + n(n-1)/4)$.

*Proof.* Use formulas (7), (6), (11), and make change of variables $t_{ij} \to e^{-s_i}t_{ij}$. Then the above formula follows from the fact that the volume of the unit ball in $\mathbb{R}^m$ is $\pi^{m/2}/\Gamma(1 + m/2)$. □

It follows from Lemma 5 that $\varrho(B^o_T) = O\left(T^{(n^2-n)}\right)$ as $T \to \infty$. In fact, more precise statement is true:

**Lemma 6.**

$$\varrho(B^o_T) \sim \gamma_n T^{(n^2-n)} \quad as \quad T \to \infty, \tag{13}$$

*where*

$$\gamma_n = \frac{\pi^{n(n-1)/4}}{2^{n-1}\Gamma\left(\frac{n^2-n+2}{2}\right)} \prod_{k=1}^{n-1} \Gamma\left(\frac{n-k}{2}\right). \tag{14}$$

*Proof.* Since the norm (1) is $K$-invariant, $G_T = KB^o_T$. By (9), $\mu(G_T) = \varrho(B^o_T)$. The asymptotics of $\mu(G_T)$ as $T \to \infty$ was computed in [DRS93, Appendix 1]. $\square$

**Lemma 7.** *For any $C \in \mathbb{R}$, $\varrho(B^C_T) \sim \varrho(B^o_T)$ as $T \to \infty$.*

*Proof.* For $i_0 = 1, \dots, n-1$, put

$$A^{i_0}_T = \{a(s) \in A^o_T : s_{i_0} \le C\} \quad \text{and} \quad B^{i_0}_T = \{a(s)n(t) \in B^o_T : s_{i_0} \le C\}.$$

We claim that $\varrho(B^{i_0}_T) = o(\varrho(B^o_T))$ as $T \to \infty$. It follows from (11) that if $a(s) \in A^o_T$, then $s_i < \log T$ for every $i = 1, \dots, n$. Then by Lemma 5,

$$\begin{aligned} \varrho(B^{i_0}_T) &\le c_n T^{\frac{n(n-1)}{2}} \int_{A^{i_0}_T} \exp\left(\sum_k (n-k)s_k\right) ds \\ &\ll T^{\frac{n(n-1)}{2}} \prod_{k<n, k\ne i_0} \int_{-\infty}^{\log T} e^{(n-k)s_k} ds_k \ll T^{n(n-1)-(n-i_0)}. \end{aligned}$$

(Here and later on $A \ll B$ means $A < c \cdot B$ for some absolute constant $c > 0$.) Now the claim follows from (13). Since $B^o_T - B^C_T = \cup_{i_0<n} B^{i_0}_T$, $\varrho(B^o_T - B^C_T) = o(\varrho(B^o_T))$ as $T \to \infty$. Therefore, $\varrho(B^C_T) \sim \varrho(B^o_T)$ as $T \to \infty$. $\square$

Next, we show that Theorem 2 fails for a left Haar measure on $B^o$.

**Proposition 8.** *Let $\Gamma$ be a lattice in $G = \mathrm{SL}(2, \mathbb{R})$, and $y \in \Gamma\backslash G$ be such that the orbit $yN$ is periodic. Then for every $\tilde{f} \in C_c(\Gamma\backslash G)$,*

$$\frac{1}{\lambda(B^o_T)} \int_{B^o_T} \tilde{f}(yb^{-1}) d\lambda(b) \to 0 \quad \text{as} \quad T \to \infty.$$

*Proof.* For $C \in \mathbb{R}$, put $\hat{A}^C = \{a(s) \in A^o : s_1 < C\}$ and $\hat{B}^C = \hat{A}^C N$. As in Lemma 7, one can show that $\lambda(B^o_T - \hat{B}^C_T) = o(\lambda(B^o_T))$ as $T \to \infty$. Therefore, for every $C \in \mathbb{R}$,

$$\frac{1}{\lambda(B^o_T)} \int_{B^o_T - \hat{B}^C_T} \tilde{f}(yb^{-1}) d\lambda(b) \to 0 \quad \text{as} \quad T \to \infty.$$

On the other hand, according to [St, Lemma 14.2], $yN^{-1}a(s)^{-1} \to \infty$ as $s_1 \to -\infty$. Thus, there exists $C \in \mathbb{R}$ such that $y(\hat{B}^C_T)^{-1} \cap \mathrm{supp}(\tilde{f}) = \emptyset$. Then

$$\int_{\hat{B}^C_T} \tilde{f}(yb^{-1}) d\lambda(b) = 0.$$

This proves the proposition. $\square$

## 4. Proof of Theorem 1

**Claim**. *It is enough to prove Theorem 1 for $X = G/B^o$.*

*Proof.* Suppose that the theorem is proved for $X = G/B^o$.

At first, we consider a special case:

$$X = G/(B^o)^{g_0} \quad \text{for some } g_0 \in G, \tag{15}$$

where $(B^o)^{g_0} = g_0^{-1} B^o g_0$. By the Iwasawa decomposition (8), $(B^o)^{g_0} = (B^o)^{k_0}$ for some $k_0 \in K$.

The normalized $K$-invariant measure $m$ on $G/(B^o)^{k_0}$ is defined as

$$m(C) = \int_{k:k(B^o)^{k_0}\in C} dk \quad \text{for Borel set } C \subseteq G/(B^o)^{k_0}.$$

Similarly, one defines the normalized $K$-invariant measure $m^*$ on $G/B^o$. Consider a map

$$\beta : G/B^o \to G/(B^o)^{k_0} : gB^o \mapsto g^{k_0}(B^o)^{k_0}.$$

Clearly, $\beta$ is a diffeomorphism. Using that $K$ is unimodular, one proves that

$$m^*(\beta^{-1}(C)) = m(C) \quad \text{for Borel set } C \subseteq G/(B^o)^{k_0}. \tag{16}$$

Take

$$\Omega \subseteq G/(B^o)^{k_0} \quad \text{and} \quad x_0 = h_0(B^o)^{k_0} \in G/(B^o)^{k_0}$$

such that $m(\partial\Omega) = 0$. Set

$$\Omega^* = \beta^{-1}(\Omega) \subseteq G/B^o \quad \text{and} \quad x_0^* = h_0^{k_0^{-1}} B^o \in G/B^o.$$

By (16), $m^*(\partial\Omega^*) = 0$ too. For $\gamma \in \Gamma$, $\gamma \cdot x_0 \in \Omega$ iff $\gamma^{k_0^{-1}} \cdot x_0^* \in \Omega^*$. Note also that $\|\gamma^{k_0^{-1}}\| = \|\gamma\|$. Therefore,

$$N_T(\Omega, x_0) = |\{\gamma \in \Gamma^{k_0^{-1}} : \|\gamma\| < T, \gamma \cdot x_0^* \in \Omega^*\}|$$

Applying the assumption to the lattice $\Gamma^* = \Gamma^{k_0^{-1}}$, one gets

$$N_T(\Omega, x_0) \sim \frac{\gamma_n}{\bar{\mu}^*(\Gamma^*\backslash G)} m^*(\Omega^*) T^{n(n-1)} \quad \text{as} \quad T \to \infty,$$

where $\bar{\mu}^*$ is the measure on $\Gamma^*\backslash G$ defined in (10). Now the special case (15) follows from Lemma 4 and (16).

Let us consider the general case. Let $S$ be a closed subgroup of $G$ such that $S \supseteq H^o$, where $H$ is a real algebraic subgroup of $G$, and $G/H$ is compact. Since $H$ has finitely many connected components, $G/H^o$ is compact too. Recall that the homogeneous space $G/H^o$ is compact iff $H_{\mathbb{C}}$ contains a maximal connected $\mathbb{R}$-split solvable $\mathbb{R}$-subgroup of $G_{\mathbb{C}}$ (see, for example, [PR, Theorem 3.1]). Since maximal connected $\mathbb{R}$-split solvable $\mathbb{R}$-subgroups of $G_{\mathbb{C}}$ are conjugate over $G_{\mathbb{R}}$ (see [BT65], or Theorem 15.2.5 and Exercise 15.4.8 in [Sp]), for some $g_0 \in G$, $B^{g_0} \subseteq H$. Hence, $(B^o)^{g_0} \subseteq H^o \subseteq S$.

Denote by $\pi$ the projection map $G/(B^o)^{g_0} \to G/S$. Take

$$\Omega \subseteq G/S \quad \text{and} \quad x_0 \in G/S$$

such that $m(\partial\Omega) = 0$. Set

$$\Omega^* = \pi^{-1}(\Omega) \subseteq G/(B^o)^{g_0} \quad \text{and} \quad x_0^* \in \pi^{-1}(x_0).$$

Let $m^*$ be the $K$-invariant normalized measure on $G/(B^o)^{g_0}$. Then $m = \pi(m^*)$ is the $K$-invariant normalized measure on $G/S$. It is easy to check that $\pi(\partial\Omega^*) \subseteq \partial\Omega$. Hence, $m^*(\partial\Omega^*) \le m^*(\pi^{-1}(\partial\Omega)) = m(\partial\Omega) = 0$. Finally,

$$N_T(\Omega, x_0) = N_T(\Omega^*, x_0^*) \underset{T\to\infty}{\sim} \frac{\gamma_n}{\bar{\mu}(\Gamma\backslash G)} m^*(\Omega^*) T^{n(n-1)} = \frac{\gamma_n}{\bar{\mu}(\Gamma\backslash G)} m(\Omega) T^{n(n-1)}.$$

□

We need the following proposition that follows easily from Theorem 2.

**Proposition 9.** *Let $f$ be the characteristic function of a relatively compact Borel subset $Z \subseteq G$ such that $\mu(\partial Z) = 0$. Then for any $y \in \Gamma\backslash G$,*

$$\frac{1}{\varrho(B_T^o)} \int_{B_T^o} \tilde{f}(yb^{-1}) d\varrho(b) \longrightarrow \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_G f d\mu \quad \text{as } T \to \infty, \tag{17}$$

*where $\tilde{f}(\Gamma g) = \sum_{\gamma\in\Gamma} f(\gamma g) \in C_c(\Gamma\backslash G)$.*

*Proof.* The argument is quite standard. One chooses functions $\phi_n, \psi_n \in C_c(G)$, $n \geq 1$, such that $\phi_n \leq f \leq \psi_n$ and $\int_G (\psi_n - \phi_n) d\mu < \frac{1}{n}$. By Theorem 2 and (10),

$$\lim_{T\to\infty} \frac{1}{\varrho(B_T^o)} \int_{B_T^o} \tilde{\phi}_n(yb^{-1}) d\varrho(b) = \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_{\Gamma\backslash G} \tilde{\phi}_n d\bar{\mu} = \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_G \phi_n d\mu,$$

and

$$\lim_{T\to\infty} \frac{1}{\varrho(B_T^o)} \int_{B_T^o} \tilde{\psi}_n(yb^{-1}) d\varrho(b) = \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_G \psi_n d\mu$$

for every $n \geq 1$. This implies (17). ☐

The proof of Theorem 1 should be compared with similar arguments in [DRS93], [EM93], [EMS96], [EMM98] where other counting problems were also reduced to asymptotics of certain integrals.

*Proof of Theorem 1.* Let $\alpha : K \to G/B^o$ be a map defined by $\alpha(k) = kB^o$. By (8), $\alpha$ is a diffeomorphism. The measure $m$ is given by

$$m(C) = \int_{\alpha^{-1}(C)} dk \quad \text{for Borel set } C \subseteq G/B^o. \tag{18}$$

Since $\alpha$ is surjective, $x_0 = k_0^{-1} B^o$ for some $k_0 \in K$. It follows from the Iwasawa decomposition (8) that the product map

$$K \times B^o : (k, b) \mapsto k k_0^{-1} b k_0 \in G$$

is a diffeomorphism. For $g \in G$, define $k_g \in K$ and $b_g \in B^o$ such that

$$g = k_g k_0^{-1} b_g k_0.$$

Since $G$ and $K$ are unimodular, it follows from (9) that for $f \in C_c(G)$,

$$\int_G f d\mu = \int_{K\times B^o} f(k k_0^{-1} b k_0) dk d\varrho(b).$$

Let $\phi$ be the characteristic function of $\tilde{\Omega} \stackrel{def}{=} \alpha^{-1}(\Omega) k_0 \subseteq K$, and $\psi_{\mathcal{O}}$ the characteristic function of an open bounded symmetric neighborhood $\mathcal{O}$ of 1 in $B^o$ with boundary of measure 0 normalized so that $\int_{B^o} \psi_{\mathcal{O}} d\varrho = 1$. Then $\int_{B^o} \psi_{\mathcal{O}} d\lambda = 1$ too. Note that for $g \in G$, $gx_0 \in \Omega$ iff $k_g \in \tilde{\Omega}$. Put $f_{\mathcal{O}}(g) = \phi(k_g)\psi_{\mathcal{O}}(b_g)$. Let $\tilde{f}_{\mathcal{O}}(\Gamma g) = \sum_{\gamma\in\Gamma} f_{\mathcal{O}}(\gamma g)$. Now Proposition 9 can be applied to $\tilde{f}_{\mathcal{O}}$:

$$\begin{aligned} \frac{1}{\varrho(B_T^o)} \int_{B_T^o} \tilde{f}_{\mathcal{O}}(\Gamma k_0^{-1} b^{-1} k_0) d\varrho(b) &\underset{T\to\infty}{\longrightarrow} \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_G f_{\mathcal{O}}(g k_0) d\mu(g) \\ &= \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_K \phi dk \cdot \int_{B^o} \psi_{\mathcal{O}} d\varrho = \frac{1}{\bar{\mu}(\Gamma\backslash G)} \int_{\tilde{\Omega}} dk = \frac{m(\Omega)}{\bar{\mu}(\Gamma\backslash G)}. \end{aligned} \tag{19}$$

The last equality follows from (18).

Take $r > 1$. There exists a bounded open symmetric neighborhood $\mathcal{O}$ of identity in $B^o$ (with boundary of measure 0) such that for any $b \in \mathcal{O} = \mathcal{O}^{-1}$ and $x \in M(n, \mathbb{R})$,

$$r^{-1}\|x\| \le \|b^{-1}x\| \le r\|x\|. \tag{20}$$

Then for $\mathcal{O}$ as above,

$$\sum_{\gamma\in\Gamma} \int_{B_T^o} f_{\mathcal{O}}(\gamma k_0^{-1} b^{-1} k_0) d\varrho(b) = \sum_{\gamma\in\Gamma} \int_{(B_T^o)^{-1}} f_{\mathcal{O}}(k_\gamma k_0^{-1} b_\gamma b k_0) d\lambda(b)$$
$$= \sum_{\gamma\in\Gamma} \int_{b_\gamma (B_T^o)^{-1}} f_{\mathcal{O}}(k_\gamma k_0^{-1} b k_0) d\lambda(b) = \sum_{\gamma\in\Gamma} \int_{\|b^{-1}b_\gamma\|<T} \phi(k_\gamma)\psi_{\mathcal{O}}(b) d\lambda(b)$$
$$= \sum_{\gamma : k_\gamma \in \tilde{\Omega}} \int_{\|b^{-1}b_\gamma\|<T} \psi_{\mathcal{O}}(b) d\lambda(b) = \sum_{\gamma:\gamma\cdot x_0 \in \Omega} \int_{\|b^{-1}b_\gamma\|<T} \psi_{\mathcal{O}}(b) d\lambda(b).$$

The integral

$$I_\gamma \overset{def}{=} \int_{\|b^{-1}b_\gamma\|<T} \psi_{\mathcal{O}}(b) d\lambda(b)$$

is not greater than 1. By (20), $I_\gamma = 0$ for $\gamma \in \Gamma$ such that $\|\gamma\| = \|b_\gamma\| \ge rT$. Therefore,

$$N_{rT}(\Omega, x_0) \ge \sum_{\gamma\in\Gamma} \int_{B_T^o} f_{\mathcal{O}}(\gamma k_0^{-1} b^{-1} k_0) d\varrho(b). \tag{21}$$

By (20), $I_\gamma = 1$ for $\gamma \in \Gamma$ such that $\|\gamma\| = \|b_\gamma\| < r^{-1}T$. Thus,

$$N_{r^{-1}T}(\Omega, x_0) \le \sum_{\gamma\in\Gamma} \int_{B_T^o} f_{\mathcal{O}}(\gamma k_0^{-1} b^{-1} k_0) d\varrho(b). \tag{22}$$

It follows from (13) that $\varrho(B^o_{r^{-1}T}) \sim \gamma_n r^{n-n^2} T^{n^2-n}$ as $T \to \infty$. Then using (21) and (19), we get

$$\begin{aligned}\liminf_{T\to\infty} \frac{N_T(\Omega, x_0)}{T^{n^2-n}} &\ge \liminf_{T\to\infty} \frac{\gamma_n}{r^{n^2-n}\varrho(B^o_{r^{-1}T})} \int_{B^o_{r^{-1}T}} \tilde{f}_{\mathcal{O}}(\gamma k_0^{-1} b^{-1} k_0) d\varrho(b) \\ &= \frac{\gamma_n m(\Omega)}{r^{n^2-n}\bar{\mu}(\Gamma\backslash G)}.\end{aligned}$$

This inequality holds for any $r > 1$. Hence,

$$\liminf_{T\to\infty} \frac{N_T(\Omega, x_0)}{T^{n^2-n}} \ge \frac{\gamma_n m(\Omega)}{\bar{\mu}(\Gamma\backslash G)}.$$

Similarly, using (22) and (19), one can prove that

$$\limsup_{T\to\infty} \frac{N_T(\Omega, x_0)}{T^{n^2-n}} \le \frac{\gamma_n m(\Omega)}{\bar{\mu}(\Gamma\backslash G)}.$$

This finishes the proof of Theorem 1. □

## 5. Behavior of unipotent flows

In this section we review some deep results on equidistribution of unipotent flows, which are crucial for the proof of Theorem 2. Note that there are two different approaches available: Ratner [R91b] and Dani, Margulis [DM93]. Both of these methods rely on Ratner's measure rigidity [R91a]. We follow the method of Dani and Margulis. The results below were proved in [DM91, DM93] for the case of one-dimensional flows and extended to higher dimensional flows and even polynomial trajectories in [Sh94, EMS97, Sh96]. See [St, §19] and [KSS02] for more detailed exposition.

Appropriate adjustments are made for the right $G$-action on $\Gamma\backslash G$ instead of left $G$-action on $G/\Gamma$.

Notations: Let $G$ be a connected semisimple Lie group without compact factors, and $\Gamma$ a lattice in $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$. For positive integers $d$ and $n$, denote by $\mathcal{P}_{d,n}(G)$ the set of functions $q : \mathbb{R}^n \to G$ such that for any $a, b \in \mathbb{R}^n$, the map

$$t \in \mathbb{R} \mapsto \mathrm{Ad}(q(at+b)) \in \mathfrak{g}$$

is a polynomial of degree at most $d$ with respect to some basis of $\mathfrak{g}$.

Let $V_G = \oplus_{i=1}^{\dim \mathfrak{g}} \wedge^i \mathfrak{g}$. There is a natural action of $G$ on $V_G$ induced from the adjoint representation. Fix a norm on $V_G$. For a Lie subgroup $H$ of $G$ with Lie algebra $\mathfrak{h}$, take a unit vector $p_H \in \wedge^{\dim \mathfrak{h}} \mathfrak{h}$.

The following theorem allows us to estimate divergence of polynomial trajectories. For its proof, see [Sh96, Theorems 2.1–2.2].

**Theorem 10.** *There exist closed subgroups $U_i$ $(i = 1, \dots, r)$ such that each $U_i$ is the unipotent radical of a maximal parabolic subgroup, $\Gamma U_i$ is compact in $\Gamma\backslash G$, and for any $d, n \in \mathbb{N}$, $\varepsilon, \delta > 0$, there exists a compact set $C \subseteq \Gamma\backslash G$ such that for any $q \in \mathcal{P}_{d,n}(G)$ and a bounded open convex set $D \subseteq \mathbb{R}^n$, one of the following holds:*

1. *There exist $\gamma \in \Gamma$ and $i = 1, \dots, r$ such that $\sup_{t\in D} \|q(t)^{-1}\gamma \cdot p_{U_i}\| \le \delta$.*
2. *$\ell(t \in D : \Gamma q(t) \notin C) < \varepsilon \ell(D)$, where $\ell$ is the Lebesgue measure on $\mathbb{R}^n$.*

Denote by $\mathcal{H}_\Gamma$ the family of all proper closed connected subgroups $H$ of $G$ such that $\Gamma \cap H$ is a lattice in $H$, and $\mathrm{Ad}(H \cap \Gamma)$ is Zariski-dense in $\mathrm{Ad}(H)$.

**Theorem 11.** *The set $\mathcal{H}_\Gamma$ is countable. For any $H \in \mathcal{H}_\Gamma$, $\Gamma \cdot p_H$ is discrete.*

For proofs, see [R91a, Theorem 1.1] and [DM93, Theorems 2.1 and 3.4].

Fix a subgroup $U$ generated by 1-parameter unipotent subgroups. For a closed subgroup $H$ of $G$, denote

$$X(H, U) = \{g \in G : gU \subseteq Hg\}.$$

Define a singular set

$$Y = \bigcup_{H\in\mathcal{H}_\Gamma} \Gamma X(H, U) \subseteq \Gamma\backslash G. \tag{23}$$

It follows from Dani's generalization of Borel density theorem and Ratner's topological rigidity that $y \in Y$ iff $yU$ is not dense in $\Gamma\backslash G$ (see [St, Lemma 19.4]).

One needs to estimate behavior of polynomial trajectories near the singular set $Y$. The following result can be deduced from [Sh94, Proposition 5.4]. It is formulated in [Sh96] and [KSS02]. Note that it is analogous to Theorem 10 with the point at infinity being replaced by the singular set.

**Theorem 12.** *Let $d, n \in \mathbb{N}$, $\varepsilon > 0$, $H \in \mathcal{H}_\Gamma$. For any compact set $C \subseteq \Gamma X(H,U)$, there exists a compact set $F \subseteq V_G$ such that for any neighborhood $\Phi$ of $F$ in $V_G$, there exists a neighborhood $\Psi$ of $C$ in $\Gamma\backslash G$ such that for any $q \in \mathcal{P}_{d,n}(G)$ and a bounded open convex set $D \subseteq \mathbb{R}^n$, one of the following holds:*

1. *There exists $\gamma \in \Gamma$ such that $q(D)^{-1}\gamma \cdot p_H \subseteq \Phi$.*
2. *$\ell(t \in D : \Gamma q(t) \in \Psi) < \varepsilon \ell(D)$, where $\ell$ is the Lebesgue measure on $\mathbb{R}^n$.*

## 6. Representations of $\mathrm{SL}(n,\mathbb{R})$

In order to be able to use the results from the previous section, we collect here some information about representations of $\mathrm{SL}(n,\mathbb{R})$.

The next lemma is essentially Lemma 5.1 from [Sh96]. We present its proof because more precise information about dependence on $\beta$ in the inequality (24) is needed.

**Lemma 13.** *Let $V$ be a finite dimensional vector space with a norm $\|\cdot\|$, $\mathfrak{n}$ be a nilpotent Lie subalgebra of $\mathrm{End}(V)$ with a basis $\{b_i : i = 1,\dots,m\}$, and $N = \exp(\mathfrak{n}) \subseteq \mathrm{GL}(V)$ be the Lie group of $\mathfrak{n}$. Define a map $\Theta : \mathbb{R}^m \to N$:*

$$\Theta(t_1,\dots,t_m) = \exp\left(\sum_{i=1}^m t_i b_i\right), \quad (t_1,\dots,t_m) \in \mathbb{R}^m.$$

*For $\beta > 0$, put $D(\beta) = \Theta\left([0,\beta] \times \cdots \times [0,\beta]\right)$. Let*

$$W = \{v \in V : \mathfrak{n} \cdot v = 0\}.$$

*Denote by $\mathrm{pr}_W$ the orthogonal projection on $W$ with respect to some scalar product on $V$.*

*Then there exists a constant $c_0 > 0$ such that for any $\beta \in (0,1)$ and $v \in V$,*

$$\sup_{n \in D(\beta)} \|\mathrm{pr}_W(n \cdot v)\| \geq c_0 \beta^d \|v\|, \tag{24}$$

*where $d$ is the degree of the polynomial map $\Theta$.*

*Proof.* Let

$$\mathcal{I} = \left\{(i_1,\dots,i_m) \in \mathbb{Z}^m : i_k \geq 0, \sum_k i_k \leq d\right\}.$$

For $t \in \mathbb{R}^m$ and $I = (i_1,\dots,i_m) \in \mathcal{I}$, denote $t^I = \prod_k t_k^{i_k}$, and $|I| = \sum_k i_k$.

One can write $\Theta(t) = \sum_{I \in \mathcal{I}} t^I B_I$ for some $B_I \in \mathrm{End}(V)$. Then

$$\mathrm{pr}_W(\Theta(t)v) = \sum_{I \in \mathcal{I}} t^I \mathrm{pr}_W(B_I v).$$

Consider a map $T : V \to \oplus_{I \in \mathcal{I}} W$ defined by

$$Tv = \sum_{I \in \mathcal{I}} \mathrm{pr}_W(B_I v),$$

and a map $A_t : \oplus_{I \in \mathcal{I}} W \to W$ for $t \in \mathbb{R}^m$ defined by

$$A_t\left(\bigoplus_{I \in \mathcal{I}} w_I\right) = \sum_{I \in \mathcal{I}} t^I w_I.$$

For $I \in \mathcal{I}$, take fixed $s_I \in \mathbb{R}^m$ such that $0 < s_{I,k} < 1$ and $s_{I_1,k} \neq s_{I_2,k}$ for $I_1 \neq I_2$, and put $t_I = \beta s_I$. Let

$$A = \bigoplus_{I\in\mathcal{I}} A_{t_I} : \bigoplus_{I\in\mathcal{I}} W \to \bigoplus_{I\in\mathcal{I}} W.$$

The map $A$ has a matrix form $\left(t_I^J\right)_{I,J\in\mathcal{I}}$. This matrix is a Kronecker product of Vandermonde matrices which implies that $A$ is invertible. Using elementary row and column operations, one can write

$$\left(t_I^J\right)_{I,J\in\mathcal{I}} = B \cdot \mathrm{diag}\left(\beta^{|I|} : I \in \mathcal{I}\right) \cdot C \tag{25}$$

for some $B, C \in \mathrm{GL}\left(\oplus_{I\in\mathcal{I}} W\right)$, which are independent of $\beta$. It is convenient to use a norm on $\oplus_{I\in\mathcal{I}} W$ defined by

$$\left\| \bigoplus_{I\in\mathcal{I}} w_I \right\| = \max_{I\in\mathcal{I}} \|w_I\|, \quad w_I \in W.$$

Then by (25), for $w \in \oplus_{I\in\mathcal{I}} W$,

$$\|Aw\| \geq \|B^{-1}\|^{-1} \cdot \beta^d \cdot \|C^{-1}\|^{-1} \cdot \|w\|. \tag{26}$$

It follows from Lie-Kolchin theorem that $T$ is injective (see [Sh96, Lemma 5.1]). Therefore, there exists a constant $c_1 > 0$ such that $\|Tv\| \geq c_1\|v\|$ for $v \in V$. Then using (26), we get

$$\sup_{n\in D(\beta)} \|\mathrm{pr}_W(n \cdot v)\| = \sup_{t:0<t_i<\beta} \|A_t T v\| \geq \|ATv\| \geq c_0 \beta^d \|v\|,$$

where $c_0 = \|B^{-1}\|^{-1} \cdot \|C^{-1}\|^{-1} \cdot c_1$. □

We will need the following elementary observation:

**Lemma 14.** *Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a $C^1$ bijection such that $F(0) = 0$, and $F^{-1}$ is $C^1$ too. Fix a norm on $\mathbb{R}^m$ and denote by $D(r)$ a ball of radius $r$ centered at the origin. Then there are $c_1, c_2 > 0$ such that*

$$D(c_1 r) \subseteq F(D(r)) \subseteq D(c_2 r)$$

*for every $r \in (0, 1)$.*

Let $\mathfrak{g}$ be the Lie algebra of $G = \mathrm{SL}(n, \mathbb{R})$, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Recall the root space decomposition of $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \oplus \sum_{i\neq j} \mathfrak{g}_{ij},$$

where $\mathfrak{g}_0$ is the subalgebra of diagonal matrices of $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{g}_{ij} = \mathbb{C}E_{ij}$ ($E_{ij}$ is the matrix with 1 in position $(i, j)$ and 0's elsewhere).

Introduce *fundamental weights* of $\mathfrak{g}_{\mathbb{C}}$:

$$\omega_i(s) = s_1 + \cdots + s_i, \quad 1 \leq i \leq n - 1, \tag{27}$$

where $s \in \mathbb{C}^n$ and $\sum_i s_i = 0$. *Dominant weights* are defined as linear combinations with non-negative integer coefficients of the fundamental weights $\omega_i$, $1 \leq i \leq n - 1$.

A *highest weight* of a representation of $\mathfrak{g}_{\mathbb{C}}$ is a weight that is maximal with respect to the ordering on the dual space of $\mathfrak{g}_0$. Recall that irreducible representations of $\mathfrak{g}_{\mathbb{C}}$ are classified by their highest weights (see, for example, [GW, Ch. 5]). The highest weights are precisely the dominant weights defined above.

**Lemma 15.** *Let $\pi$ be a representation of $G$ on a finite dimensional complex vector space $V$. Let $x \in V - \{0\}$ be such that $\pi(N)x = x$. Then $x$ is a sum of weight vectors with dominant weights. Moreover, if $V$ does not contain non-zero $G$-fixed vectors, every weight in this sum is not zero.*

*Proof.* Consider a representation $\tilde{\pi}$ of $\mathfrak{g}_{\mathbb{C}}$ on $V$ induced by the representation $\pi$. Since this representation is completely reducible, it is enough to consider the case when it is irreducible.

We claim that in this case, $x$ is a vector with the highest weight. Write $x = \sum_k x_k$ where each $x_k \in V$ is a weight vector with a weight $\lambda_k$. We may assume that $\lambda_k \neq \lambda_l$ for $k \neq l$. Since $\pi(N)x = x$, $\tilde{\pi}(E_{ij})x = 0$ for $i < j$. Thus, $\sum_k \tilde{\pi}(E_{ij})x_k = 0$. Since $\tilde{\pi}(E_{ij})x_k$ is either 0 or a weight vector with the weight $\lambda_k + \alpha_{ij}$, the non-zero terms in the sum are linearly independent. Hence, $\tilde{\pi}(E_{ij})x_k = 0$ for $i < j$. Suppose that $\lambda_k$ is not the highest weight. Note that $\tilde{\pi}(\mathfrak{g}_0)x_k = \mathbb{C}x_k$, and $\tilde{\pi}(E_{ji})x_k$ has weight $\lambda_k - \alpha_{ij} < \lambda_k$ for $i < j$. By Poincaré-Birkhoff-Witt theorem, the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{b}^-) \oplus \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}$, where $\mathfrak{b}^-$ is the space of lower triangular matrices, and $\mathfrak{n}$ is the Lie algebra of $N$. Therefore, the space $\tilde{\pi}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))x_k = \tilde{\pi}(\mathcal{U}(\mathfrak{b}^-))x_k$ does not contain a vector with the highest weight. This contradicts irreducibility of $\tilde{\pi}$. Thus, each $x_k$ is of the highest weight, and $x$ is a highest weight vector. Since every highest weight is dominant, the lemma is proved. □

For a fixed $g_0 \in G$, define $q_s(t) = g_0 n(t)^{-1} a(s)^{-1}$. We are going to study $q_s$ using techniques from Section 5. The next lemma guarantees that the first possibility in Theorems 10 and 12 does not occur.

For $\beta > 0$, define

$$D(\beta) = \left\{ n(t) \in N : \sum_{i<j} t_{ij}^2 < \beta^2 \right\}. \tag{28}$$

**Lemma 16.** *Let $\pi$ be a nontrivial representation of $G$ on a real vector space $V$ equipped with a norm $\|\cdot\|$. Let $V_0 = \{v \in V : \pi(G)v = v\}$, and $V_1$ be a $G$-invariant complement for $V_0$. (Note that $V_1$ exists because $\pi$ is completely reducible.) Denote by $\Pi$ the projection on $V_1$.*

*Then for any relatively compact set $K \subseteq V$ and $r > 0$, there exist $\alpha \in (0,1)$ and $C_0 > 0$ such that for any $s$ with $a(s) \in A_T^{C_0}$ and $x \in V$ with $\|\Pi(x)\| > r$,*

$$\pi\left(q_s\left(D(e^{-\alpha s_1})\right)\right)^{-1} \cdot x \nsubseteq K. \tag{29}$$

*Proof.* It is convenient to extend $\pi$ to $V_{\mathbb{C}} = V \otimes \mathbb{C}$. $(V_0)_{\mathbb{C}}$ is the space of $G$-fixed vectors in $V_{\mathbb{C}}$. Thus, we may assume $V$ to be complex. Also dealing with projections on $V_1$, we may assume that $V$ has no $G$-fixed vectors.

Since $\{g_0^{-1} \cdot x : \|x\| > r\} \subseteq \{x : \|x\| > r_1\}$ for some $r_1 > 0$, we may assume that $g_0 = 1$.

For some $R > 0$, $K \subseteq \{x \in V : \|x\| < R\}$. If (29) fails for some $s \in \mathbb{R}^n$ and $x \in V$, then

$$\sup_{n \in D(e^{-\alpha s_1})} \|\pi(a(s)n)x\| < R. \tag{30}$$

Let $W = \{x \in V : \pi(N)x = x\}$. Clearly, the statement of the lemma is independent of the norm used. It is convenient to use the max-norm with respect to a basis $\{v_i\}$ of $V$

consisting of weight vectors, i.e.

$$\left\|\sum_i u_i v_i\right\| = \max_i |u_i|, \quad u_i \in \mathbb{C},$$

and each $v_i$ is an eigenvector of $A$. Moreover we can choose the basis $\{v_i\}$ so that it contains a basis of $W$. Let $\mathrm{pr}_W$ be the projection onto $W$ with respect to this basis. Then $\mathrm{pr}_W$ commutes with $a(s)$. Note that there exists $C' > 0$ such that for any $v \in V$,

$$\|v\| \geq C' \|\mathrm{pr}_W(v)\|. \tag{31}$$

Let $\mathcal{K} \subseteq \mathbb{N}$ be such that $k \in \mathcal{K}$ iff $v_k$ has a non-zero dominant weight. Denote this weight by $\lambda_k$. By Lemma 15, $W \subseteq \langle v_k : k \in \mathcal{K}\rangle$. In particular, for $n \in N$,

$$\mathrm{pr}_W(\pi(n)x) = \sum_{k\in\mathcal{K}} u_k(n) v_k \quad \text{for some } u_k(n) \in \mathbb{C}. \tag{32}$$

Therefore,

$$\|\mathrm{pr}_W(\pi(n)x)\| = \max_{k\in\mathcal{K}} |u_k(n)|. \tag{33}$$

Using the fact that $\mathrm{pr}_W$ and $\pi(a(s))$ commute, (31), (32), and (33), we have that for any $n \in N$,

$$\begin{aligned} \|\pi(a(s)n)x\| \geq C'\|\mathrm{pr}_W(\pi(a(s))\pi(n)x)\| &= C' \left\|\pi(a(s))\left(\sum_{k\in\mathcal{K}} u_k(n)v_k\right)\right\| \\ &= C' \max_{k\in\mathcal{K}} \left(|u_k(n)| e^{\lambda_k(s)}\right) \geq C' \exp\left(\min_{k\in\mathcal{K}} \lambda_k(s)\right) \|\mathrm{pr}_W(\pi(n)x)\|. \end{aligned} \tag{34}$$

Let $\mathfrak{n}$ be the Lie algebra of $N$. Denote by $\tilde{\pi}$ the representation of $\mathfrak{g}$ induced by $\pi$. Since $\mathfrak{g}$ is simple, $\tilde{\pi}$ is faithful. Thus, $\tilde{\pi}$ defines an isomorphism between $\mathfrak{n}$ and $\tilde{\pi}(\mathfrak{n})$. Since the exponential map $\mathfrak{n} \to N$ is a polynomial isomorphism, the coordinates on $N$ used in Lemma 13 and the coordinates $\{t_{ij}\}$ are connected by a polynomial isomorphism too. By Lemma 14, (24) holds for the set $D(\beta)$ defined in (28). Therefore,

$$\sup_{n\in D(e^{-\alpha s_1})} \|\mathrm{pr}_W(\pi(n)x)\| \geq c_0 (e^{-\alpha s_1})^d \|x\| \geq c_0 r e^{-\alpha d s_1} \tag{35}$$

for some positive integer $d$. It follows from (34) and (35) that if (30) holds, then

$$\exp\left(\min_{k\in\mathcal{K}} \lambda_k(s) - \alpha d s_1\right) \leq \frac{R}{c_0 C' r}.$$

Take $\alpha < d^{-1}$. Since each $\lambda_k$ is a non-zero dominant weight, it follows from (27) that that $\lambda_k(s) - \alpha d s_1 \to \infty$ as $C \to \infty$ for $s$ such that $a(s) \in A^C$. Hence, there exists $C_0 > 0$ such that (30) does not hold for $s$ with $a(s) \in A_T^{C_0}$. Since (30) fails, (29) holds. □

**Lemma 17.** *Use notations from Lemma 16. Let $H$ be a subgroup of $G$, and $x \in V$ such that $\pi(H)x$ is discrete in $V$. Then $\Pi(\pi(H)x)$ is discrete in $V_1$.*

*Proof.* Denote by $x_0 \in V_0$ and $x_1 \in V_1$ the components of $x$ with respect to the decomposition $V = V_0 \oplus V_1$. Then $\Pi(\pi(H)x) = \pi(H)x_1$. Suppose that for some $\{h_n\} \subseteq H$, $\pi(h_n)x_1 \to y$ for some $y \in V_1$. Then $\pi(h_n)x$ converges to $x_0 + y$. It follows that $\pi(h_n)x$ is constant for large $n$. Therefore, $\pi(h_n)x_1 = \pi(h_n)x - x_0$ is constant for large $n$ too. □

## 7. Proof of Theorem 2

Let $\mathcal{Z} = (\Gamma\backslash G) \cup \{\infty\}$ be the 1-point compactification of $\Gamma\backslash G$. For $T > 0$, define a normalized measure on $\mathcal{Z}$ by

$$\nu_T(\tilde{f}) = \frac{1}{\varrho(B_T^o)} \int_{B_T^o} \tilde{f}(yb^{-1}) d\varrho(b), \quad \tilde{f} \in C_c(\Gamma\backslash G).$$

To prove Theorem 1, we need to show that $\nu_T \to \nu$ in weak-$*$ topology as $T \to \infty$. Since the space of normalized measures on $\mathcal{Z}$ is compact, it is enough to prove that if $\nu_{T_i} \to \eta$ as $T_i \to \infty$ for some normalized measure $\eta$ on $\mathcal{Z}$, then $\eta$ is $G$-invariant, and $\eta(\{\infty\}) = 0$.

It follows from Lemma 7 that for any $C \in \mathbb{R}$,

$$\eta(\tilde{f}) = \lim_{T_i\to\infty} \frac{1}{\varrho(B_{T_i}^o)} \int_{B_{T_i}^C} \tilde{f}(yb^{-1}) d\varrho(b), \quad \tilde{f} \in C_c(\Gamma\backslash G). \tag{36}$$

Let

$$U = \{n(t) \in N : t_{ij} = 0 \text{ for } i < j < n\}. \tag{37}$$

**Lemma 18.** *The measure $\eta$ is $U$-invariant.*

*Proof.* For $\tilde{f} \in C_c(\Gamma\backslash G)$, and $g_0 \in G$, define $\tilde{f}^{g_0}(\Gamma g) = \tilde{f}(\Gamma g g_0) \in C_c(\Gamma\backslash G)$.

Let $\tilde{f} \in C_c(\Gamma\backslash G)$. Take $M > 0$ such that $|\tilde{f}| < M$.

For $T > 0$ and $s \in \mathbb{R}^{n-1}$, define a set

$$D_{s,T} = \{n \in N : \|a(s)n\| < T\}. \tag{38}$$

Denote by $\chi_{s,T}(n)$ the characteristic function of the set $D_{s,T}$. Then we can rewrite (36) as

$$\eta(\tilde{f}) = \lim_{T_i\to\infty} \frac{1}{\varrho(B_{T_i}^o)} \int_{A_{T_i}^C} \int_N \tilde{f}(yn^{-1}a(-s))\chi_{s,T_i}(n) e^{2\delta(s)} dn ds. \tag{39}$$

Let

$$A_T'^C = \{a(s) \in A_T^C : T^2 - N(s) > T\}, \tag{40}$$

where $N(s)$ is defined in (12), and $B_T'^C = A_T'^C N \cap B_T^o$. We claim that the equality (39) holds when $A_{T_i}^C$ is replaced by $A_{T_i}'^C$. By Lemma 5,

$$\begin{aligned} &\int_{A_{T_i}^C - A_{T_i}'^C} \int_N \tilde{f}(yn^{-1}a(-s))\chi_{s,T_i}(n) e^{2\delta(s)} dn ds \\ \ll\ & \int_{A_{T_i}^C - A_{T_i}'^C} \left(T_i^2 - N(s)\right)^{\frac{n(n-1)}{4}} \exp\left(\sum_k (n-k)s_k\right) ds \\ =\ & O\left(T_i^{\frac{3n(n-1)}{4}}\right) \quad \text{as} \quad T_i \to \infty. \end{aligned}$$

Now the claim follows from (13).

Take $u \in U$. Let $u(s) = \mathrm{Ad}_{a(-s)}(u)$. Then

$$\begin{aligned} |\eta(\tilde{f}^u) - \eta(\tilde{f})| \le \limsup_{T_i\to\infty} \frac{1}{\varrho(B_{T_i}^o)} \int_{A_{T_i}'^C} \int_N &|\tilde{f}(yn^{-1}u(s)a(-s))\chi_{s,T_i}(n) \\ &- \tilde{f}(yn^{-1}a(-s))\chi_{s,T_i}(n)| dn e^{2\delta(s)} ds. \end{aligned} \tag{41}$$

We estimate the last integral:

$$\begin{aligned}\int_N \left|\tilde{f}(yn^{-1}u(s)a(-s))\chi_{s,T_i}(n) - \tilde{f}(yn^{-1}a(-s))\chi_{s,T_i}(n)\right| dn \\ = \int_N \left|\tilde{f}(yn^{-1}a(-s))\right| \cdot |\chi_{s,T_i}(u(s)n) - \chi_{s,T_i}(n)|\, dn \\ \le M \int_N |\chi_{s,T_i}(u(s)n) - \chi_{s,T_i}(n)|\, dn. \end{aligned} \tag{42}$$

Recall that $\alpha_{i,n}(-s) = -s_i + s_n$. Therefore, by (5), $u(s)_{in} = e^{-s_i+s_n} u_{in}$ for $i = 1, \dots, n-1$. It follows from the triangle inequality that

$$\|a(s)u(s)n\| \le \|a(s)n\| + \sqrt{\sum_{i=1}^{n-1} e^{2s_i} u(s)_{in}^2} \le \|a(s)n\| + e^{s_n}\|u\|,$$

and similarly,

$$\|a(s)n\| \le \|a(s)u(s)n\| + e^{s_n}\|u^{-1}\| = \|a(s)u(s)n\| + e^{s_n}\|u\|.$$

Hence,

$$\chi_{s,T_i - e^{s_n}\|u\|}(n) \le \chi_{s,T_i}(u(s)n) \le \chi_{s,T_i + e^{s_n}\|u\|}(n)$$

for $n \in N$. Therefore,

$$\int_N |\chi_{s,T_i}(u(s)n) - \chi_{s,T_i}(n)|\, dn \le \int_N \left(\chi_{s,T_i + e^{s_n}\|u\|} - \chi_{s,T_i - e^{s_n}\|u\|}\right) dn. \tag{43}$$

Let $\varepsilon > 0$. We claim that there exists $C_0 > 0$ such that for $C > C_0$ and $a(s) \in A'^C_{T_i}$,

$$\int_N \left(\chi_{s,T_i + e^{s_n}\|u\|} - \chi_{s,T_i - e^{s_n}\|u\|}\right) dn \le \varepsilon \int_N \chi_{s,T_i} dn. \tag{44}$$

Similarly to Lemma 5,

$$\int_N \chi_{s,T_i} dn = c_n \left(T_i^2 - N(s)\right)^{\frac{n(n-1)}{4}} \exp\left(\sum_k (n-k)s_k\right).$$

Also $e^{s_n} \to 0$ for $a(s) \in A'^C_{T_i}$ as $C \to \infty$. Therefore, the equation (44) will follow from the following.

*Claim.* There exists $d_0 = d_0(\varepsilon) > 0$ such that for any $d \in (0, d_0)$ and $a(s) \in A'^C_{T_i}$,

$$\Big((T_i + d)^2 - N(s)\Big)^{\frac{n(n-1)}{4}} - \Big((T_i - d)^2 - N(s)\Big)^{\frac{n(n-1)}{4}} < \varepsilon\Big((T_i - d)^2 - N(s)\Big)^{\frac{n(n-1)}{4}}. \tag{45}$$

Note that by (40), $(T_i - d)^2 - N(s) > 0$ for $a(s) \in A'^C_{T_i}$ and $d < 1/2$. The inequality (45) is equivalent to

$$(T_i + d)^2 - N(s) < (1 + \bar{\varepsilon})\Big((T_i - d)^2 - N(s)\Big),$$

where $\bar{\varepsilon} = (1 + \varepsilon)^{\frac{4}{n(n-1)}} - 1$. By (40), the last inequality follows from

$$(T_i + d)^2 + \bar{\varepsilon}(T_i^2 - T_i) < (1 + \bar{\varepsilon})(T_i - d)^2,$$

or equivalently,

$$T_i(4d - \bar{\varepsilon} + 2d\bar{\varepsilon}) < \bar{\varepsilon} d^2.$$

If one takes $d < d_0 = \bar{\varepsilon}/(2\bar{\varepsilon}+4)$, the left hand side is negative. This proves the claim (45).

Thus, (44) holds. Then by (41), (42), (43), and (44),

$$\begin{aligned} |\eta(\tilde{f}^u) - \eta(\tilde{f})| &\leq (M\varepsilon) \limsup_{T_i \to \infty} \frac{1}{\varrho(B^o_{T_i})} \int_{A'^C_{T_i}} \int_N \chi_{s,T_i}(n) e^{2\delta(s)} dn ds \\ &= (M\varepsilon) \limsup_{T_i \to \infty} \frac{\varrho(B'^C_{T_i})}{\varrho(B^o_{T_i})} \leq M\varepsilon. \end{aligned}$$

This shows that $\eta(\tilde{f}^u) = \eta(\tilde{f})$. ☐

**Lemma 19.** *Let $\alpha \in (0,1)$. Let*

$$\tilde{A}^C_T = \left\{ a(s) \in A^C_T | \, (T^2 - N(s))^{1/2} > \exp\left( \max_{1 \leq j \leq n-1} \{s_j\} - \alpha s_1 \right) \right\}, \tag{46}$$

*where $N(s)$ is as in (12), and $\tilde{B}^C_T = \tilde{A}^C_T N \cap B^o_T$. Then for $C > 0$,*

$$\eta(\tilde{f}) = \lim_{T_i \to \infty} \frac{1}{\varrho(B^o_{T_i})} \int_{\tilde{B}^C_{T_i}} \tilde{f}(yb^{-1}) d\varrho(b), \quad \tilde{f} \in C_c(\Gamma \backslash G).$$

*Proof.* By (36), it is enough to show that

$$\frac{\varrho\left(B^C_{T_i} - \tilde{B}^C_{T_i}\right)}{\varrho\left(B^o_{T_i}\right)} \to 0 \quad \text{as} \quad T_i \to \infty. \tag{47}$$

As in Lemma 5,

$$\varrho\left(B^C_{T_i} - \tilde{B}^C_{T_i}\right) = c_n \int_{A^C_{T_i} - \tilde{A}^C_{T_i}} \left(T_i^2 - N(s)\right)^{\frac{n(n-1)}{4}} \exp\left(\sum_k (n-k)s_k\right) ds.$$

Therefore,

$$\begin{aligned} \varrho\left(B^C_{T_i} - \tilde{B}^C_{T_i}\right) &\leq c_n \int_{A^C_{T_i}} \exp\Big(\frac{n(n-1)}{2} \max_{1 \leq j \leq n-1} \{s_j\} \\ -\frac{\alpha n(n-1)}{2} s_1 + \sum_k (n-k)s_k\Big) ds &\leq c_n \sum_{1 \leq j \leq n-1} \int_{A^C_{T_i}} \exp\Big(\frac{n(n-1)}{2} s_j \\ &\quad -\frac{\alpha n(n-1)}{2} s_1 + \sum_k (n-k)s_k\Big) ds \end{aligned}$$

Then as in the proof of Lemma 7, for $j \neq 1$

$$\int_{A^C_{T_i}} \exp\left(\frac{n(n-1)}{2}s_j - \frac{\alpha n(n-1)}{2}s_1 + \sum_k (n-k)s_k\right) ds$$
$$\leq \int_{-\infty}^{\log T_i} \exp\left(\left(-\frac{\alpha n(n-1)}{2} + n - 1\right) s_1\right) ds_1$$
$$\cdot \int_{-\infty}^{\log T_i} \exp\left(\left(\frac{n(n-1)}{2} + n - j\right) s_j\right) ds_j \cdot \prod_{k<n, k\neq 1, j} \int_{-\infty}^{\log T_i} e^{(n-k)s_k} ds_k$$
$$\ll T_i^{n(n-1)-\alpha n(n-1)/2}$$

as $T_i \to \infty$. For $j = 1$, the same estimate can be obtained by a similar calculation. Now (47) follows from (13). □

Let $y = \Gamma g_0$ for $g_0 \in G$. Define $q_s(t) = g_0 n(t)^{-1} a(s)^{-1}$. We apply the results of Section 5 to the map $q_s$.

**Lemma 20.** *$\eta(\{\infty\}) = 0$.*

*Proof.* Let $\varepsilon, \delta > 0$. Apply Theorem 10 to the map $q_s(t)$. By Theorem 11, the set $\Gamma \cdot p_{U_i} \subseteq V_G$ is discrete. Write $V_G = V_0 \oplus V_1$, where $V_0$ is the space of vectors fixed by $G$, and $V_1$ is its $G$-invariant complement. Denote by $\Pi$ the projection on $V_1$. By Lemma 17, $\Pi(\Gamma \cdot p_{U_i})$ is discrete. Also $0 \notin \Pi(\Gamma \cdot p_{U_i})$. Otherwise $p_{U_i}$ is fixed by $G$, and it would follow that $U_i$ is normal in $G$ which is a contradiction. Therefore, there exists $r > 0$ such that

$$\|\Pi(x)\| > r \quad \text{for} \quad x \in \bigcup_i \Gamma \cdot p_{U_i}.$$

Now we can apply Lemma 16. Let

$$K = \{x \in V_G : \|x\| \leq \delta\}.$$

By Lemma 16, there exist $\alpha \in (0, 1)$ and $C_0 > 0$ such that the first case of Theorem 10 fails for $q_s$ when $D$ is a bounded open convex set which contains $D(e^{-\alpha s_1})$ (it is defined in (28)), and $s$ is such that $a(s) \in A^{C_0}_{T_i}$. Therefore, for some compact set $C \subseteq \Gamma\backslash G$,

$$\omega\left(\{n(t) \in D : \Gamma q_s(t) \notin C\}\right) < \varepsilon \omega(D) \tag{48}$$

when $D \supseteq D(e^{-\alpha s_1})$ and $a(s) \in A^{C_0}_{T_i}$, where $\omega = dt$ denotes the Lebesgue measure on $N$.

Let $D_{s,T_i}$ be as in (38). By Lemma 19,

$$\eta(\tilde{f}) = \lim_{T_i \to \infty} \frac{1}{\varrho(B^o_{T_i})} \int_{\tilde{A}^C_{T_i}} \int_{D_{s,T_i}} \tilde{f}(\Gamma q_s(t)) e^{2\delta(s)} dt ds, \quad \tilde{f} \in C_c(\Gamma\backslash G). \tag{49}$$

Note that

$$D_{s,T_i} = \left\{ n(t) \in N : \sum_{i<j} e^{2s_i} t_{ij}^2 < T_i^2 - N(s) \right\},$$

where $N(s)$ is defined in (12) (cf. (11)). It follows that $D_{s,T_i}$ contains $D(\beta)$ for

$$\beta < (T_i^2 - N(s))^{1/2} \exp\left(-\max_{1\leq i\leq n-1} \{s_i\}\right).$$

When $a(s) \in \tilde{A}^{C_0}_{T_i}$, the right hand side is greater then $e^{-\alpha s_1}$ (see (46)). Therefore, $D_{s,T_i} \supseteq D(e^{-\alpha s_1})$ when $a(s) \in \tilde{A}^{C_0}_{T_i}$. By (48),

$$\omega\left(\{n(t) \in D_{s,T_i} : \Gamma q_s(t) \notin C\}\right) < \varepsilon\omega(D_{s,T_i}) \quad \text{for } \; a(s) \in \tilde{A}^{C_0}_{T_i}. \tag{50}$$

Let $\chi_C$ be the characteristic function of the set $C$. Take $\tilde{f} \in C_c(\Gamma\backslash G)$ such that $\chi_C \leq \tilde{f} \leq 1$. Then using (49) and (50), we get

$$\begin{aligned} \eta(\mathrm{supp}(\tilde{f})) &\geq \lim_{T_i\to\infty} \frac{1}{\varrho(B^o_{T_i})} \int_{\tilde{A}^{C_0}_{T_i}} \int_{D_{s,T_i}} \chi_C(\Gamma q_s(t)) e^{2\delta(s)} dt ds \\ &\geq \lim_{T_i\to\infty} \frac{1}{\varrho(B^o_{T_i})} \int_{\tilde{A}^{C_0}_{T_i}} (1-\varepsilon)\omega(D_{s,T_i}) e^{2\delta(s)} ds \\ &= (1-\varepsilon) \lim_{T_i\to\infty} \frac{\varrho(\tilde{B}^{C_0}_{T_i})}{\varrho(B^o_{T_i})} = 1-\varepsilon. \end{aligned}$$

Hence, $\eta(\{\infty\}) \leq \eta(\mathrm{supp}(\tilde{f})^c) \leq \varepsilon$ for every $\varepsilon > 0$. □

Recall that the singular set $Y$ of $U$ was defined in (23).

**Lemma 21.** $\eta(Y) = 0$.

*Proof.* By (23) and Theorem 11, it is enough to show that $\eta(\Gamma X(H,U)) = 0$ for any $H \in \mathcal{H}_\Gamma$. Moreover, since $\Gamma X(H,U)$ is $\sigma$-compact, we just need to show that $\eta(C) = 0$ for any compact set $C \subseteq \Gamma X(H,U)$.

Take $\varepsilon > 0$. Apply Theorem 12 to the map $q_s(t)$. Let $F \subseteq V_G$ be as in Theorem 12. Fix a relatively compact neighborhood $\Phi$ of $F$ in $V_G$. Take $\Psi \supseteq C$ as in Theorem 12. By Theorem 11, the set $\Gamma \cdot p_H$ is discrete. Let $\Pi$ be as in the proof of Lemma 20. By Lemma 17, $\Pi(\Gamma \cdot p_H)$ is discrete. If $0 \in \Pi(\Gamma \cdot p_H)$, the vector $p_H$ is fixed by $G$, and $H$ is normal in $G$, which is impossible. Therefore, for some $r > 0$, $\|\Pi(x)\| > r$ for every $x \in \Gamma \cdot p_H$. Applying Lemma 16 with $K = \Phi$, one gets that there exist $\alpha \in (0,1)$ and $C_0 > 0$ such that the first case of Theorem 10 fails for $q_s$ when $D$ is a bounded open convex set containing $D(e^{-\alpha s_1})$, and $s$ is such that $a(s) \in A^{C_0}_{T_i}$. Therefore, the second case should hold:

$$\omega\left(\{n(t) \in D : \Gamma q_s(t) \in \Psi\}\right) < \varepsilon\omega(D) \tag{51}$$

when $D \supseteq D(e^{-\alpha s_1})$ and $a(s) \in A^{C_0}_{T_i}$.

Let $D_{s,T_i}$ be as in (38). It was shown in the proof of Lemma 20 that $D_{s,T_i} \supseteq D(e^{-\alpha s_1})$ when $a(s) \in \tilde{A}^{C_0}_{T_i}$. It follows from (51) that

$$\omega\left(\{n(t) \in D_{s,T_i} : \Gamma q_s(t) \in \Psi\}\right) < \varepsilon\omega(D_{s,T_i}) \quad \text{for } \; a(s) \in \tilde{A}^{C_0}_{T_i}. \tag{52}$$

Take a function $\tilde{f} \in C_c(\Gamma\backslash G)$ such that $\tilde{f} = 1$ on $C$, $\mathrm{supp}(\tilde{f}) \subseteq \Psi$, and $0 \leq \tilde{f} \leq 1$. Let $\chi_\Psi$ be the characteristic function of $\Psi$. Then using (49) and (52), we get

$$\eta(C) \leq \lim_{T_i\to\infty} \frac{1}{\varrho(B^o_{T_i})} \int_{\tilde{A}^{C_0}_{T_i}} \int_{D_{s,T_i}} \chi_\Psi(\Gamma q_s(t)) e^{2\delta(s)} dt ds$$

$$\leq \lim_{T_i\to\infty} \frac{1}{\varrho(B^o_{T_i})} \int_{\tilde{A}^{C_0}_{T_i}} \varepsilon\omega(D_{s,T_i}) e^{2\delta(s)} ds = \varepsilon \lim_{T_i\to\infty} \frac{\varrho(\tilde{B}^{C_0}_{T_i})}{\varrho(B^o_{T_i})} = \varepsilon.$$

This shows that $\eta(C) = 0$. Hence, $\eta(Y) = 0$. $\square$

Recall that by Lemma 18, $\eta$ is $U$-invariant. By Ratner's measure classification [R91a], an ergodic component of $\eta$ is either $G$-invariant or supported on $Y \cup \{\infty\}$. By Lemmas 20 and 21, the set of ergodic components of the second type has measure 0. Therefore, $\eta$ is $G$-invariant, and $\eta = \nu$. This finishes the proof of Theorem 2.

## Acknowledgments

The author wishes to thank his advisor V. Bergelson for constant support, encouragement, and help with this paper, N. Shah and Barak Weiss for insightful comments about the paper, and H. Furstenberg and G. Margulis for interesting discussions.

## References

[BT65] A. Borel and J. Tits. Groupes réductifs. *Inst. Hautes tudes Sci. Publ. Math.* **27** (1965), 55–150.

[DM91] S. G. Dani and G. A. Margulis. Asymptotic behavior of trajectories of unipotent flows on homogeneous spaces. *Proc. Indian Acad. Sci. Math. Sci.* **101** (1991), 1–17.

[DM93] S. G. Dani and G. A. Margulis. Limit distributions of orbits of unipotent flows and values of quadratic forms. I. M. Gelfand Seminar, 91–137, *Adv. Soviet Math.* **16**, Part 1, AMS, Providence, RI, 1993.

[DRS93] W. Duke, Z. Rudnick and P. Sarnak. Density of integer points on affine homogeneous varieties. *Duke Math. J.* **71** (1993), 143–179.

[EM93] A. Eskin and C. McMullen. Mixing, counting, and equidistribution in Lie groups. *Duke Math. J.* **71** (1993), 181–209.

[EMM98] A. Eskin, G. Margulis and S. Mozes. Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture. *Ann. Math.* **147** (1998), 93–141.

[EMS96] A. Eskin, S. Mozes and N. Shah. Nimish Unipotent flows and counting lattice points on homogeneous varieties. *Ann. Math.* **143** (1996), 253–299.

[EMS97] A. Eskin, S. Mozes and N. Shah. Non-divergence of translates of certain algebraic measures. *Geom. Funct. Anal.* **7** (1997), 48–80.

[F63] H. Furstenberg. A Poisson formula for semi-simple Lie groups. *Ann. Math.* **77** (1963), 335–386.

[G] A. Good, *Local analysis of Selberg's trace formula*, Lecture Notes in Mathematics, 1040, Springer-Verlag, Berlin, 1983.

[GW] R. Goodman and N. R. Wallach. *Representations and invariants of the classical groups.* Cambridge University Press, Cambridge, 1998.

[KSS02] D. Kleinbock, N. Shah and A. Starkov. Dynamics of subgroup actions on homogeneous spaces of Lie groups and applications to number theory. in *Handbook of dynamical systems*, Vol. 1A, 813–930, North-Holland, Amsterdam, 2002.

[L99] F. Ledrappier, Distribution des orbites des réseaux sur le plan réel. *C. R. Acad. Sci. Paris Sr. I Math.* **329** (1999), 61–64.

[N00] A. Nogueira, Orbit distribution on $\mathbb{R}^2$ under the natural action of $\mathrm{SL}(2, \mathbb{Z})$, preprint, 2000.

[PR] V. Platonov and A. Rapinchuk. *Algebraic groups and number theory.* Academic Press, Inc., Boston, MA, 1994.

[R] M. S. Raghunathan. *Discrete Subgroups of Lie Groups.* Springer-Verlag, New York, 1972.

[R91a] M. Ratner. On Raghunathan's measure conjecture. *Ann. Math.* 134 (1991), 545–607.

[R91b] M. Ratner. Raghunathan's topological conjecture and distributions of unipotent flows. *Duke Math. J.* **63** (1991), 235–280.

[Sh94] N. A. Shah. Limit distributions of polynomial trajectories on homogeneous spaces. *Duke Math. J.* **75** (1994), 711–732.

[Sh96] N. Shah. Limit distributions of expanding translates of certain orbits on homogeneous spaces. *Proc. Indian Acad. Sci. Math. Sci.* **106** (1996), 105–125.

[Sp] T. A. Springer. *Linear algebraic groups.* Birkhäuser, Boston, 1998.

[St] A. N. Starkov. *Dynamical systems on homogeneous spaces.* Translations of Mathematical Monographs 190, AMS, Providence, RI, 2000.

[T] A. Terras. *Harmonic analysis on symmetric spaces and applications I.* Springer-Verlag, New York, 1985.

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