Ergodic theory of semisimple lattices

(joint work with *Amos Nevo*)

Alex Gorodnik

1. Basic problem: distribution of orbits

$$\begin{aligned} G &= \text{l.c. s.c. group,} \\ m &= \text{Haar measure on } G, \\ (X,\mu) &= \text{probability measure space.} \end{aligned}$$

For a measure-preserving action

$$G \curvearrowright (X,\mu)$$

we would like to understand distribution of G-orbits.

Take increasing sequence of compact subsets

$$G_t \subset G, \quad t > 0,$$

and for $f \in L^1(X,\mu)$, consider average

$$\mathcal{S}_t f(x) = \frac{1}{m(G_t)} \int_{G_t} f(g^{-1}\cdot x)\, dm(g).$$

Question. *What is the behavior of $\mathcal{S}_t f$ as $t \to \infty$?*

2. Properties of $\mathcal{S}_t f$

For *ergodic* G-actions, one hopes to have

- *(strong maximal inequality in L^p)* For every $f \in L^p(X, \mu)$,

$$\left\| \sup_{t \geq 1} |\mathcal{S}_t f| \right\|_p \leq C_p \|f\|_p,$$

- *(mean ergodic theorem in L^p)* For every $f \in L^p(X, \mu)$,

$$\left\| \mathcal{S}_t f - \int_X f \, d\mu \right\|_p \longrightarrow 0,$$

- *(pointwise ergodic theorem in L^p)* For every $f \in L^p(X, \mu)$,

$$\mathcal{S}_t f \longrightarrow \int_X f \, d\mu \quad \text{for } \mu\text{-a.e. } x \in X.$$

3. Ergodic theory of semisimple groups

Theorem (Margulis, Nevo, Stein).

$$\begin{aligned} G &= \textit{connected semisimple Lie group with no compact factors,} \\ G_t &= K\textit{-biinvariant Riemannian balls in } G, \\ \mathcal{S}_t f(x) &= \frac{1}{m(G_t)} \int_{G_t} f(g^{-1} \cdot x)\, dm(g). \end{aligned}$$

Then $\mathcal{S}_t$ satisfies

- *strong maximal inequality in L^p, $p > 1$.*
- *pointwise ergodic theorem in L^p, $p > 1$.*

We extend this theorem to a general class of increasing compact sets $\{G_t\}$ satisfying some “continuity” conditions.

4. Conditions on G_t's

$B_R(g) =$ ball of radius R with respect to a right invariant Riemannian metric on G.

We assume that there exists $c > 0$ such that

(1) For every $t \geq 1$,

$$m(G_{t+1}) \leq c \cdot m(G_t).$$

(2) For every small $\varepsilon > 0$ and sufficiently large t,

$$B_\varepsilon(e) \cdot G_t \cdot B_\varepsilon(e) \subset G_{t+c\varepsilon}.$$

(3) For every small $\varepsilon > 0$ and sufficiently large t,

$$m(G_{t+\varepsilon}) \leq (1 + c\varepsilon) m(G_t).$$

Note that (2) implies that

$$G_t \supset B_{\delta t}(g)$$

for some $\delta > 0$ and sufficiently large t.

Assume that G is simple.

Theorem. *If G_t's satisfy (1)–(3), then $\mathcal{S}_t$ satisfies*

- *strong maximal inequality in L^p, $p > 1$.*
- *pointwise ergodic theorem in L^p, $p > 1$, on a G-invariant set.*

Examples of G_t's:

$$\begin{array}{ccl} L & = & \text{semisimple Lie group}, \\ G & \subset & L, \\ K\backslash L & = & \text{symmetric space of } L, \\ d & = & \text{Cartan-Killing metric } d \text{ on } K\backslash L, \\ & & \\ G_t & = & \{g \in G : \, d(u \cdot g, v) \le t\}, \quad u, v \in K\backslash L. \end{array}$$

$$\begin{array}{ccl} \rho : G \to \mathrm{GL}(V) & - & \text{proper homomorphism}, \\ \|\cdot\| & - & \text{norm on } \mathrm{End}(V), \\ & & \\ G_t & = & \{g \in G : \, \|\rho(g)\| \le e^t\}. \end{array}$$

5. Proof of pointwise convergence

Let

$$\nu_t = \frac{\chi_{G_t}}{m(G_t)} dm.$$

For $f \in L^2(X)$ and $\pi(g) f(x) = f(g^{-1}x)$,

$$\mathcal{S}_t f(x) = \pi(\nu_t) f(x).$$

“Perturbing” sets G_t, we may assume that

$G_t = \{g \in G : \ D(g) \le T\}, \quad D : G \to \mathbb{R}^+$ — absolutely continuous.

Then Haar measure

$$m = \int_0^\infty m_t \, dt, \qquad \text{supp}(m_t) \subset \partial G_t$$

Since

$$\nu_t = \frac{1}{m(G_t)} \int_0^t m_t \, dt,$$

the function $t \mapsto \nu_t$ is absolutely continuous and for $f \in L^2(X)$ and a.e. $x \in X$,

$$\pi(\nu_t)f(x) - \pi(\nu_s)f(x) = \int_s^t \frac{d}{dr}\pi(\nu_r)f(x)\, dr.$$

We have

$$\frac{d}{dt}\nu_t = \ldots = \frac{m_t(\partial G_t)}{m(G_t)}(\partial\nu_t - \nu_t)$$

where

$$\partial\nu_t = \frac{m_t}{m_t(\partial G_t)}.$$

By (3),

$$\frac{m_t(\partial G_t)}{m(G_t)} = \lim_{\varepsilon\to 0^+} \frac{m(G_{t+\varepsilon}) - m(G_t)}{\varepsilon m(G_t)} \leq c.$$

It suffices to prove pointwise convergence on a dense subspace of

$$\mathcal{H} = L^2_0(X)$$

(assuming strong maximal inequality and mean ergodic theorem).

Set

$$\begin{aligned}
\mathcal{H} &= \int_{\hat{G}}^{\oplus} m_z \mathcal{H}_z \, dE(z), \\
\mathcal{H}_p &= \int_{\hat{G}_p}^{\oplus} m_z \mathcal{H}_z \, dE(z), \\
\hat{G}_p &= \left\{ \pi \in \hat{G} : \begin{array}{l} \langle \pi(g)u, v\rangle \in L^q(G) \text{ for } q > p \\ \text{and } u, v \in \text{dense subspace of } \mathcal{H}_\pi \end{array} \right\}, \\
\mathcal{H}_{p,n} &= \{u \in \mathcal{H}_p : \dim \langle \pi(K)u \rangle \le n\}.
\end{aligned}$$

Then

$$\bigcup_{p>2,n>1} \mathcal{H}_{p,n} \qquad \text{is dense in } \mathcal{H}.$$

We prove pointwise convergence for $u \in \mathcal{H}_{p,n}$.

For $f \in \mathcal{H}_{p,n}$

$$\|\pi(g)f\| \leq a(p,n)e^{-\delta_p\|g\|}\|f\|.$$

Then for some $a, \delta > 0$,

$$\|\pi(\nu_t)f\| \leq ae^{-\delta t}\|f\|, \qquad \|\pi(\partial\nu_t)f\| \leq ae^{-\delta t}\|f\|.$$

For a.e. $x \in X$ and $M > 0$,

$$\limsup_{s,t\to\infty} |\pi(\nu_t)f(x) - \pi(\nu_s)f(x)| \leq \int_M^\infty \left|\frac{d}{dr}\pi(\nu_r)f\right| dr.$$

Finally,

$$\begin{aligned}
&\mu\left(\left\{x \in X : \limsup_{s,t\to\infty} |\pi(\nu_t)f(x) - \pi(\nu_s)f(x)| > \varepsilon\right\}\right) \\
&\leq \varepsilon^{-1}\int_X\int_M^\infty \left|\frac{d}{dr}\pi(\nu_r)f(x)\right| dr d\mu(x) \leq \varepsilon^{-1}\int_M^\infty \left\|\frac{d}{dr}\pi(\nu_r)f(x)\right\| dr \\
&\leq \varepsilon^{-1}c\int_M^\infty \|\pi(\nu_r)f + \pi(\partial\nu_r)f\|\ dr \\
&\leq \varepsilon^{-1}ce^{-M\delta/2}\int_M^\infty e^{\delta r/2}\left(\|\pi(\nu_r)f\| + \|\pi(\partial\nu_r))f\|\right) dr \\
&\to 0 \quad \text{as } M \to \infty.
\end{aligned}$$

6. Proof of G-invariance

Assume $f \geq 0$.

Let Ω_g be the set of full measure such that for $y \in \Omega_g$,

$$\frac{1}{m(gG_t)} \int_{gG_t} f(h^{-1}y)\, dm(h) \to \int_X f\, d\mu.$$

Take a countable dense set $\{g_i\} \subset G$ and consider

$$\Omega = \bigcap_i \Omega_{g_i}.$$

For any $\varepsilon > 0$ and $g \in G$, there exists g_i such that

$$g_i \in B_{\varepsilon/c}(g).$$

Then

$$g_i G_{t-\varepsilon} \subset gG_t \subset g_i G_{t+\varepsilon}$$

$$\begin{aligned}\frac{1}{m(G_t)}\int_{G_t} f(h^{-1}g^{-1}y)dm(h) &= \frac{1}{m(G_t)}\int_{gG_t} f(h^{-1}y)dm(h) \\ &\le \frac{(1+c\varepsilon)}{m(g_iG_{t+\varepsilon})}\int_{g_iG_{t+\varepsilon}} f(h^{-1}y)dm(h).\end{aligned}$$

This implies that for every $g \in G$ and $y \in \Omega$,

$$\limsup_{t\to\infty} \mathcal{S}_t f(g^{-1}y) \le (1+c\varepsilon)\int_X f\,d\mu.$$

Estimate for $\liminf$ is similar, and

$$\lim_{t\to\infty} \mathcal{S}_t f(y) = \int_X f\,d\mu$$

for $y \in G\cdot\Omega$.

7. Ergodic theory of lattices

G = connected Lie group,
m = Haar measure on G,
Γ = a lattice in G,
$\left.\begin{array}{c}\Gamma \curvearrowright (X,\mu) \\ G \curvearrowright (Y,\nu)\end{array}\right\}$ = probability measure spaces.

For increasing sequence of compact sets

$$G_t \subset G, \quad t > 0,$$

define averages

$$\mathcal{R}_t\phi(x) = \frac{1}{|\Gamma \cap G_t|} \sum_{\gamma \in \Gamma \cap G_t} \phi(\gamma^{-1} \cdot x), \quad \phi \in L^p(X,\mu)$$

$$\mathcal{S}_t\psi(y) = \frac{1}{m(G_t)} \int_{G_t} \psi(g^{-1} \cdot y)\, dm(g), \quad \psi \in L^p(Y,\nu).$$

Theorem. If G_t's satisfy (1)-(3), then for $\mathcal{S}_t$,

- strong maximal inequality for $\mathcal{S}_t$ $\Rightarrow$ strong maximal inequality for $\mathcal{R}_t$
- mean ergodic theorem for $\mathcal{S}_t$ $\Rightarrow$ mean ergodic theorem for $\mathcal{R}_t$
- pointwise ergodic theorem for $\mathcal{S}_t$ $\Rightarrow$ pointwise ergodic theorem for $\mathcal{R}_t$

Corollary. Let Γ be a lattice in a connected simple Lie group and $G_t \subset G$ satisfy (1)–(3). Then

- strong maximal inequality for $\mathcal{R}_t$ in L^p, $p > 1$.
- mean ergodic theorem for $\mathcal{R}_t$ in L^p, $p \geq 1$.
- pointwise ergodic theorem for $\mathcal{R}_t$ in L^p, $p > 1$.

Example:

Γ = lattice in a simple Lie group G

Λ = finite index subgroup in Γ

$A \subset \Gamma/\Lambda$

$G_t \subset G$ satisfying (2)–(3)

Then

$$|\{\gamma \in \Gamma \cap G_t :\; \gamma\Lambda \in A\}| \sim \frac{|A|}{|\Gamma : \Lambda|} \cdot |\Gamma \cap G_t|$$

8. Kazhdan groups

Assume that G has property (T).

Theorem. If G_t's satisfy (1)-(3), then there exists $\delta > 0$ such that for

$$p > 1, \quad 2p/(p+1) < r < p, \quad \phi \in L^p(X,\mu),$$

we have

$$\begin{aligned} \left\| \sup_{t \geq 1} e^{\delta t} \left| \mathcal{R}_t \phi - \int_Y \phi \, d\nu \right| \right\|_r &\leq C \|\phi\|_p, \\ \left| \mathcal{R}_t \phi(x) - \int_Y \phi \, d\nu \right| &\leq C(x,\phi) e^{-\delta t}, \\ \|C(\cdot,\phi)\|_r &\leq C \|\phi\|_p. \end{aligned}$$

9. Idea of the proof: induced action

Let

$$(Y,\nu) = ((G \times X)/\sim, \nu),$$

with equivalence relation

$$(g,x) \sim (g\gamma, \gamma^{-1}x) \quad \text{for } \gamma \in \Gamma.$$

The group G acts on Y by

$$g' \cdot [(g,x)] = [(g'g,x)].$$

For

$$\phi \in L^p(X,\mu),$$

$$\chi : G \to \mathbb{R} \quad \text{– bump function,} \quad \chi = \frac{\chi_{B_\varepsilon(e)}}{m(B_\varepsilon(e))}$$

set

$$\psi(g,x) = \sum_{\gamma\in\Gamma} \chi(g\gamma)\phi(\gamma^{-1}x).$$

10. Proof of mean ergodic theorem for $\mathcal{R}_t$

It suffices to prove mean ergodic theorem for $p > \dim G$ and $\phi \geq 0$.

1. Mean ergodic theorem for $\mathcal{S}_t$ in $L^p(G/\Gamma)$ implies that

$$|\Gamma \cap G_t| \sim m(G_t) \quad \text{as } t \to \infty.$$

2. For $(g, x) \in B_\varepsilon(e) \times X$ and large t,

$$(1 - c\varepsilon)\mathcal{S}_{t-c\varepsilon}\psi(g, x) \leq \mathcal{R}_t\phi(x) \leq (1 + c\varepsilon)\mathcal{S}_{t+c\varepsilon}\psi(g, x).$$

This implies that

$$\begin{aligned} \left\| \mathcal{R}_t\psi - \int_X \psi \, d\mu \right\|_{L^p(\mu)} &= m(B_\varepsilon(e))^{-1/p} \left\| \mathcal{R}_t\psi - \int_X \psi \, d\mu \right\|_{L^p(m\otimes\mu|_{B_\varepsilon(e)\times X})} \\ \leq & m(B_\varepsilon(e))^{-1/p} \left\| (1 + c\varepsilon)\mathcal{S}_{t+c\varepsilon}\phi - (1 - c\varepsilon)\mathcal{S}_{t-c\varepsilon}\phi \right\|_{L^p(m\otimes\mu|_{B_\varepsilon(e)\times X})} \\ + & m(B_\varepsilon(e))^{-1/p} \left\| (1 - c\varepsilon)\mathcal{S}_{t-c\varepsilon}\phi - \int_X \psi \, d\mu \right\|_{L^p(m\otimes\mu|_{B_\varepsilon(e)\times X})} . \end{aligned}$$

3. By mean ergodic theorem for $\mathcal{S}_t$,

$$\left\| \mathcal{S}_t\phi - \int_X \psi \, d\mu \right\|_{L^p(\nu)} \to 0 \quad \text{as } t \to \infty,$$
$$\limsup_{t\to\infty} \|\mathcal{S}_t\phi\|_{L^p(\nu)} \le \|\psi\|_{L^1(\mu)}.$$

4. Since

$$\|\cdot\|_{L^p(m\otimes\mu|_{B_\varepsilon(e)\times X})} \ll \|\cdot\|_{L^p(\nu)},$$

we have

$$\begin{aligned}\left\| \mathcal{R}_t\psi - \int_X \psi \, d\mu \right\|_p &\ll m(B_\varepsilon(e))^{-1/p} \cdot \varepsilon \cdot \|\phi\|_{L^1(\nu)} \\ &\ll \varepsilon^{-\dim G/p} \cdot \varepsilon \cdot \|\phi\|_{L^1(\nu)}.\end{aligned}$$

This implies mean ergodic theorem for $p > \dim G$.

11. Equidistribution

$\Gamma =$ lattice in simple Lie group G,
$\Gamma_t = \Gamma \cap G_t$, with $G_t \subset G$ satisfying (2)-(3).

Theorem (G., Weiss). Consider algebraic measure preserving action:

$$\Gamma \curvearrowright (X, \mu)$$

Then for every $x \in X$ such that $\overline{\Gamma \cdot x} = X$ and every $\phi \in C_c(X)$,

$$\frac{1}{|\Gamma_t|} \sum_{\gamma \in \Gamma_t} \phi(\gamma^{-1} \cdot x) \to \int_X \phi \, d\mu.$$

12. Algebraic measure-preserving actions

Examples:

$$\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$$
$$\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$$

Notations:

G = simple Lie group,
Γ = lattice in G,
L = Lie group,
Λ = lattice in L.

$$G \subset L$$

$\Gamma \curvearrowright L/\Lambda$ by left multiplication

$$\begin{array}{ccc} G & \subset & \mathrm{Aut}(L) \\ \cup & & \\ \Gamma & \subset & \mathrm{Aut}(L/\Lambda) \end{array}$$

$\Gamma \curvearrowright L/\Lambda$ by automorphisms