Recall

We have reduced the proof to Koopman–von-Neumann decomposition:

$$f = f_U + f_{U^\perp} + E.$$

Multiple averages are controled by Gowers uniformity norms.

Generalized von Neumann theorem

Theorem

For a k-pseudorandom measure $\nu : \mathbb{Z}_N \to \mathbb{R}^+$,
$\lambda_0, \ldots, \lambda_{k-1} \in \mathbb{Z}_N$, $\lambda_i \neq \lambda_j$,
functions $f_0, \ldots, f_{k-1} : \mathbb{Z}_N \to \mathbb{R}$ *such that* $|f_i| \leq \nu + 1$, *we have*

$$\mathbb{E}\left(\prod_{i=0}^{k-1} f_i(x + \lambda_i h) \,|\, x, h \in \mathbb{Z}_N\right) = O\left(\min_{0 \leq i \leq k-1} \|f_i\|_{U^{k-1}}\right) + o(1).$$

We may assume that

$$\min_{0 \leq j \leq k-1} \|f_j\|_{U^{k-1}} = \|f_0\|_{U^{k-1}}, \quad \lambda_{k-1} = 0, \quad \lambda_0 = 1.$$

Sketch of the proof (for $|f_i| \leq 1$)

Induction on k.
We have to estimate

$$\left|\mathbb{E}\left(f_{k-1}(x)\mathbb{E}\left(\prod_{j=0}^{k-2} T^{\lambda_j r} f_j(x) | r \in \mathbb{Z}_N\right) \ | \ x \in \mathbb{Z}_N\right)\right|,$$

and by the Cauchy-Schwartz inequality,

$$\leq \mathbb{E}\left(\left|\mathbb{E}\left(\prod_{j=0}^{k-2} T^{\lambda_j r} f_j(x) | r \in \mathbb{Z}_N\right)\right|^2 \ | \ x \in \mathbb{Z}_N\right)^{1/2}.$$

Sketch of the proof

By Van der Corput Lemma,

$$\begin{aligned}
&\mathbb{E}\left(\left|\mathbb{E}\left(\prod_{j=0}^{k-2} T^{\lambda_j r} f_j(x) \mid r \in \mathbb{Z}_N\right)\right|^2 \mid x \in \mathbb{Z}_N\right) \\
=&\mathbb{E}\left(\left(\prod_{j=0}^{k-2} T^{\lambda_j r} f_j(x)\right)\left(\prod_{j=0}^{k-2} T^{\lambda_j (r+h)} f_j(x)\right) \mid x, h, r \in \mathbb{Z}_N\right) \\
=&\mathbb{E}\left(\mathbb{E}\left(\prod_{j=0}^{k-2} T^{\lambda_j r}(f_j T^{\lambda_j h} f_j)(x) | x, r \in \mathbb{Z}_N\right) \mid h \in \mathbb{Z}_N\right).
\end{aligned}$$

By the induction hypothesis,

$$\left|\mathbb{E}\left(\prod_{j=0}^{k-2} T^{\lambda_j r}(f_j T^{\lambda_j h} f_j)(x) \mid x, r \in \mathbb{Z}_N\right)\right| \ll \min_j \|f_j T^h f_j\|_{U^{k-2}}.$$

Sketch of the proof

Finally, by Hölder inequality

$$\left|\mathbb{E}\left(\left|\mathbb{E}\left(\prod_{j=0}^{k-2} T^{\lambda_j r} f_j(x) | r \in \mathbb{Z}_N\right)\right|^2 \quad | \; x \in \mathbb{Z}_N\right)\right|$$
$$\leq \min_j \mathbb{E}(\|f_j T^h f_j\|_{U^{k-2}} | h \in \mathbb{Z}_N)$$
$$\leq \min_j \|f_j\|^2_{U^{k-1}}.$$

When $|f_i| \leq \nu$, we get multiple averages for ν.
The estimate follows from the linear form condition.

Generalized Koopman–von Neumann theorem

Theorem

Let ν be a k-pseudorandom measure,
$f : \mathbb{Z}_N \to \mathbb{R}$ such that $0 \leq f \leq \nu$.
Then we have decomposition

$$f = f_U + f_{U^\perp} + E$$

such that

$$f_U + f_{U^\perp} \geq 0, \quad E \geq 0,$$
$$\mathbb{E}(E) = o(1),$$
$$0 \leq f_{U^\perp} \leq 1 + o(1),$$
$$\|f_U\|_{U^{k-1}} \text{ is small} \quad \text{(uniformity).}$$

Outline of the proof

- We set

  $$f_U = (1 - 1_\Omega)(f - \mathbb{E}(f|\mathcal{B})),\ f_{U^\perp} = (1 - 1_\Omega)\mathbb{E}(f|\mathcal{B}),\ E = 1_\Omega f$$

  for a suitably constructed σ-algebra $\mathcal{B}$ of $\mathbb{Z}_N$ and $\Omega \in \mathcal{B}$.
- The σ-algebra $\mathcal{B}$ is constructed inductively.
- First, we start with $\mathcal{B} = \{\emptyset, \mathbb{Z}_N\}$. Then all properties hold except possibly "uniformity".
- Obstruction to uniformity is detected by dual function $\mathcal{D}(f_U)$.
- We refine $\mathcal{B}$ by adding "level sets" of $\mathcal{D}(f_U)$.
- In finitely many steps, we achieve uniformity
  because $\|f_{U^\perp}\|_{L^2}$ increases (energy increment property),
  but $\|f_{U^\perp}\|_\infty$ is bounded.

Obstructions to uniformity

Let $f : \mathbb{Z}_N \to \mathbb{R}$ such that $|f| \leq \nu$.

- ($k = 3$) If for some $\delta > 0$,

  $$\|f\|_{U^2} \geq \delta,$$

  then for some $r \in \mathbb{Z}_N$ and $c = c(\delta) > 0$,

  $$\mathbb{E}(f(x)e^{2\pi i r x/N} | x \in \mathbb{Z}_N) \geq c.$$

- (k general, conjecturally) If for some $\delta > 0$,

  $$\|f\|_{U^{k-1}} \geq \delta,$$

  then for some $F : \mathbb{Z}_N \to \mathbb{R}$ and $c = c(\delta) > 0$,

  $$\mathbb{E}(f(x)F(x) | x \in \mathbb{Z}_N) \geq c$$

  where $F(n) = \phi(g^n u_0)$ and $\phi : G/\Gamma \to \mathbb{R}$ is a smooth function on a nilmanifold G/Γ of degree $k - 2$.

Dual functions

For a function $F : \mathbb{Z}_N \to \mathbb{R}$, define the dual function of F:

$$\mathcal{D}F(x) = \mathbb{E}\left(\prod_{\omega \in \{0,1\}^{k-1} : \omega \neq 0^{k-1}} F(x + \omega h) \mid h \in \mathbb{Z}_N^{k-1} \right).$$

Note that

$$\langle F, \mathcal{D}F \rangle = \|F\|_{U^{k-1}}^{2^{k-1}}.$$

If $\|F\|_{U^{k-1}}$ is large, F correlates with its dual function.

The dual functions provide obstructions to uniformity!

Properties of dual functions

Proposition

For a function $F : \mathbb{Z}_N \to \mathbb{R}$ *such that* $|F| \leq \nu + 1$, *we have*

$$\|\mathcal{D}F\|_{L^\infty} \leq 2^{2^{k-1}-1} + o(1).$$

Proof.

$$|\mathcal{D}F(x)| \leq \mathbb{E}\left(\prod_{\omega \in \{0,1\}^k - \{0\}} (\nu + 1)(x + \omega h) | h \in \mathbb{Z}_N^{k-1} \right).$$

Multiplying out, the estimate follows from the linear form condition. □

Properties of dual functions

Proposition (uniform distribution for antiuniform functions)

Given function $F_1, \ldots, F_n : \mathbb{Z}_N \to \mathbb{R}$ *such that* $|F_i| \le \nu$,
a continuous function $\Phi : [-2^{2^k}, 2^{2^k}]^n \to \mathbb{R}$, *we define*

$$\phi(x) = \Phi(\mathcal{D}F_1(x), \ldots, \mathcal{D}F_n(x)).$$

Then

$$\langle \nu - 1, \phi \rangle = o_{n,\Phi}(1).$$

Proof

First, we show that $\|\nu - 1\|_{U^k} = o(1)$. We need to estimate

$$\mathbb{E}\left(\prod_{\omega \in \{0,1\}^{k-1}} (\nu(x + \omega \cdot h) - 1) | x \in \mathbb{Z}_N, h \in \mathbb{Z}_N^{k-1} \right)$$

$$= \sum_{A \subset \{0,1\}^{d-1}} (-1)^{|A|} \mathbb{E}\left(\prod_{\omega \in A} \nu(x + \omega \cdot h) | x \in \mathbb{Z}_N, h \in \mathbb{Z}_N^{k-1} \right)$$

By linear form condition, each expected value is $= 1 + o(1)$.
Hence, the claim follows from binomial theorem.
To finish the proof, we need to show that for any f with $\|f\|_{U^{k-1}} \le 1$,

$$\langle f, \phi \rangle = O_{n,\Phi}(1).$$

Approximating Φ by polynomials, to treat the case $\Phi =$ monomial.
We give a proof for $\Phi(x) = x$. We need to show

$$\left\langle f, \mathcal{D}F \right\rangle = O(1).$$

Proof

By the Gowers-Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}\left(f(x)\prod_{\omega\in\{0,1\}^{k-1}:\omega\neq 0^{k-1}} F(x+\omega\cdot h)\mid h\in\mathbb{Z}_N^{k-1}, x\in\mathbb{Z}_N\right)\\
\leq&\mathbb{E}\left(\|f\|_{U^{k-1}}\prod_{\omega\in\{0,1\}^{k-1}:\omega\neq 0^{k-1}} \|F(\cdot+\omega\cdot h)\|_{U^{k-1}}\mid h\in\mathbb{Z}_N^{k-1}\right)\\
\ll&\prod_{\omega\in\{0,1\}^{k-1}:\omega\neq 0^{k-1}}\mathbb{E}\left(\|F(\cdot+\omega\cdot h)\|_{U^{k-1}}^{2^{k-1}}\mid h\in\mathbb{Z}_N^{k-1}\right)^{1/2^{k-1}}.
\end{aligned}$$

Proof

We have to estimate

$$\begin{aligned}
&\mathbb{E}\left(\|F(\cdot+\omega\cdot h)\|_{U^{k-1}}^{2^{k-1}}\mid h\in\mathbb{Z}_N^{k-1}\right)\\
=&\mathbb{E}\left(\|F(\cdot+u)\|_{U^{k-1}}^{2^{k-1}}\mid u\in\mathbb{Z}_N\right)\\
=&\mathbb{E}\left(\prod_{\omega\in\{0,1\}^{k-1}} F(x+u+h\cdot\omega)\mid x\in\mathbb{Z}_N, h\in\mathbb{Z}_N^{k-1}, u\in\mathbb{Z}_N\right)\\
=&\mathbb{E}\left(\mathbb{E}\left(\prod_{\omega\in\{0,1\}^{k-1}} F(x+h\cdot\omega)\mid x\in\mathbb{Z}_N\right)\mid h\in\mathbb{Z}_N^{k-1}\right).
\end{aligned}$$

Proof

By the correlation condition,

$$\mathbb{E}\left(\prod_{\omega\in\{0,1\}^{k-1}} \nu(x+h\cdot\omega) \mid x\in\mathbb{Z}_N\right) \leq \sum_{\omega,\omega'\in\{0,1\}^{k-1}:\omega\neq\omega'} \tau(h\cdot(\omega-\omega')).$$

Finally,

$$\mathbb{E}(\tau(h\cdot(\omega-\omega')) \mid h\in\mathbb{Z}_N^{k-1}) = \mathbb{E}(\tau) = O(1).$$

σ-algebras generated by Bohr sets

For $\varepsilon,\eta>0$, $G:\mathbb{Z}_N\to[-2^{2^k},2^{2^k}]$,
define a σ-algebra $\mathcal{B}_{\varepsilon,\eta}(G)$ on $\mathbb{Z}_N$ satisfying:

- For any σ-algebra $\mathcal{B}$ on $\mathbb{Z}_N$,

$$\|G-\mathbb{E}(G|\mathcal{B}\vee\mathcal{B}_{\varepsilon,\eta}(G))\|_{L^\infty}\leq\varepsilon.$$

- The σ-algebra $\mathcal{B}_{\varepsilon,\eta}(G)$ is generated by at most $O(1/\varepsilon)$ atoms.
- If A is any atom of $\mathcal{B}_{\varepsilon,\eta}(G)$, then there exists a continuous function $\Psi_A:[-2^{2^k},2^{2^k}]\to[0,1]$ such that

$$\|(1_A-\Psi_A(G))(\nu+1)\|_{L^1}=O(\eta).$$

Construction $\mathcal{B}_{\varepsilon,\eta}(G)$

$\mathcal{B}_{\varepsilon,\eta}(G)$ consists of atoms

$$G^{-1}([\varepsilon(n+\alpha), \varepsilon(n+1+\alpha))), \quad n \in \mathbb{Z}.$$

Then diameter of $G(\{\text{atom of } \mathcal{B}\})$ is $\leq \varepsilon$.
The number of atoms is $O(1/\varepsilon)$ because G is bounded.
By Fubini theorem,

$$\int_0^1 \sum_{n\in\mathbb{Z}} \mathbb{E}(1_{G(x)\in[\varepsilon(n-\eta+\alpha),\varepsilon(n+\eta+\alpha))})(\nu(x)+1)|x \in \mathbb{Z}_N)d\alpha = 2\eta\mathbb{E}(\nu+1).$$

Hence, for some α,

$$\sum_{n\in\mathbb{Z}} \mathbb{E}(1_{G^{-1}([\varepsilon(n-\eta+\alpha),\varepsilon(n+\eta+\alpha)))}(\nu+1)) = O(\eta).$$

Take

$\Psi_A =$ cont. approximation of char. function of $[\varepsilon(n+\alpha), \varepsilon(n+1+\alpha))$.

Generalized Koopman–von Neumann decomposition

Theorem

Let ν be a k-pseudorandom measure,
$f : \mathbb{Z}_N \to \mathbb{R}$ such that $0 \leq f \leq \nu$.
Let $\varepsilon > 0$ be a small parameter and $N > N_0(\varepsilon)$ sufficiently large.
Then there exists σ-algebra $\mathcal{B}$ and exceptional set $\Omega \in \mathcal{B}$ such that

$$\mathbb{E}(1_\Omega \nu) = o_\varepsilon(1),$$
$$\|(1-1_\Omega)\mathbb{E}(\nu - 1|\mathcal{B})\|_{L^\infty} = o_\varepsilon(1),$$
$$\|(1-1_\Omega)(f - \mathbb{E}(f|\mathcal{B}))\|_{U^{k-1}} \leq \varepsilon^{1/2^k} \quad \textit{(uniformity).}$$

The Koopman–von Neumann decomposition is obtained by setting

$$f_U = (1-1_\Omega)(f - \mathbb{E}(f|\mathcal{B})), \; f_{U^\perp} = (1-1_\Omega)\mathbb{E}(f|\mathcal{B}), \; E = 1_\Omega f$$

Sketch of the proof

In the proof, we use a parameter $\eta \to 0^+$.
First, we set $\mathcal{B}_0 = \{\emptyset, \mathbb{Z}_N\}$ and $\Omega_0 = \emptyset$.
Then all properties hold except possibly uniformity.
If uniformity fails, we set

$$\begin{aligned} F_1 &:= (1 - 1_{\Omega_0})(f - \mathbb{E}(f|\mathcal{B}_0)), \\ \mathcal{B}_1 &:= \mathcal{B}_0 \vee \mathcal{B}_{\varepsilon,\eta}(\mathcal{D}F_1), \end{aligned}$$

and define the exceptional set Ω_1 to be the union of the atoms $A \in \mathcal{B}_1$ such that $\mathbb{E}(1_A(\nu+1)) \leq \eta^{1/2}$.Then

$$\mathbb{E}(1_{\Omega_1}(\nu+1)) = O_\varepsilon(\eta^{1/2}),$$

Sketch of the proof

We claim that

$$\|(1 - 1_{\Omega_1})\mathbb{E}(\nu - 1|\mathcal{B}_1)\|_{L^\infty} = O_\varepsilon(\eta^{1/2}).$$

Let A be an atom of $\mathcal{B}_1$ with $A \cap \Omega_1 = \emptyset$.
Using equidistribution for antiuniform functions, we get

$$\mathbb{E}((\nu - 1)1_A) = O_\varepsilon(\eta).$$

Also,

$$\mathbb{E}(1_A) = \frac{1}{2}(\mathbb{E}((\nu+1)1_A) - \mathbb{E}((\nu-1)1_A)) \geq O_\varepsilon(\eta^{1/2}).$$

Hence,

$$\frac{\mathbb{E}((\nu-1)1_A)}{\mathbb{E}(1_A)} = O_\varepsilon(\eta^{1/2}).$$

This implies the claim.
Continue...

Sketch of the proof

Set

$$F_n := (1 - 1_{\Omega_{n-1}})(f - \mathbb{E}(f|\mathcal{B}_{n-1})).$$

Since $f \leq \nu$,

$$\begin{aligned}\|(1 - 1_{\Omega_{n-1}})\mathbb{E}(f|\mathcal{B}_{n-1})\|_{L^\infty} &\leq 1 + O_\varepsilon(\eta^{1/2}),\\ \|F_n\|_{L^\infty} &\leq \|(1 - 1_{\Omega_{n-1}})(f - \mathbb{E}(f|\mathcal{B}_{n-1}))\|_{L^\infty}\\ &\leq (1 + O_\varepsilon(\eta^{1/2}))(\nu + 1).\end{aligned}$$

Hence,

$$\|\mathcal{D}F_n\|_{L^\infty} \leq 2^{2^{k-1}-1} + O_{n,\varepsilon}(\eta^{1/2}),$$

and we define σ-algebras

$$\mathcal{B}_n := \mathcal{B}_{n-1} \vee \mathcal{B}_{\varepsilon,\eta}(\mathcal{D}F_n),$$

and exceptional sets $\Omega_n \in \mathcal{B}_n$ as above so that

$$\begin{aligned}&\mathbb{E}(1_{\Omega_n}\nu) = O_{\varepsilon,n}(\eta^{1/2}),\\ &\|(1 - 1_{\Omega_n})\mathbb{E}(\nu - 1|\mathcal{B}_n)\|_{L^\infty} = O_{\varepsilon,n}(\eta^{1/2}),\end{aligned}$$

Sketch of the proof

It remains to show that after finitely many steps, we get

$$\|F_n\|_{U^{k-1}} \leq \varepsilon^{1/2^k}.$$

This follows from

Claim (energy increment property). *If* $\|F_n\|_{U^{k-1}} > \varepsilon^{1/2^k}$, *then*

$$\|(1 - 1_{\Omega_n})\mathbb{E}(f|\mathcal{B}_n)\|_{L^2}^2 > \|(1 - 1_{\Omega_{n-1}})\mathbb{E}(f|\mathcal{B}_{n-1})\|_{L^2}^2 + 2^{-2^k+1}\varepsilon.$$

On the other hand,

$$\|(1 - 1_{\Omega_n})\mathbb{E}(f|\mathcal{B}_n)\|_{L^\infty} \leq 1 + O_{n,\varepsilon}(\eta^{1/2})$$

Proof of energy increment property, assuming $\Omega = \emptyset$

$$\langle F_n, \mathcal{D}F_n \rangle = \|F_n\|_{U^{k-1}}^{2^{k-1}} > \varepsilon^{1/2}.$$

By definition of $\mathcal{B}_n$,

$$|\langle F_n, \mathcal{D}F_n - \mathbb{E}(\mathcal{D}F_n|\mathcal{B}_n)\rangle| \le \mathbb{E}(|F_n|)\cdot\|\mathcal{D}F_n - \mathbb{E}(\mathcal{D}F_n|\mathcal{B}_n)\|_{L^\infty} = O(\varepsilon).$$

Hence,

$$\langle F_n, \mathbb{E}(\mathcal{D}F_n|\mathcal{B}_n)\rangle = \langle f - \mathbb{E}(f|\mathcal{B}_{n-1}), \mathbb{E}(\mathcal{D}F_n|\mathcal{B}_n)\rangle > \varepsilon^{1/2} + O(\varepsilon),$$

Since

$$\langle f - \mathbb{E}(f|\mathcal{B}_{n-1}), \mathbb{E}(\mathcal{D}F_n|\mathcal{B}_n)\rangle = \langle \mathbb{E}(f|\mathcal{B}_n) - \mathbb{E}(f|\mathcal{B}_{n-1}), \mathbb{E}(\mathcal{D}F_n|\mathcal{B}_n)\rangle,$$

by the Cauchy-Schwarz inequality,

$$\|\mathbb{E}(f|\mathcal{B}_n) - \mathbb{E}(f|\mathcal{B}_{n-1})\|_{L^2}\cdot(2^{2^{k-1}-1} + O_{n,\varepsilon}(\eta^{1/2})) > \varepsilon^{1/2} + O(\varepsilon).$$

By the Pythagoras theorem (recall that $\mathcal{B}_n \subset \mathcal{B}_{n+1}$),

$$\|\mathbb{E}(f|\mathcal{B}_n)\|_{L^2}^2 = \|\mathbb{E}(f|\mathcal{B}_n) - \mathbb{E}(f|\mathcal{B}_{n-1})\|_{L^2}^2 + \|\mathbb{E}(f|\mathcal{B}_{n-1})\|_{L^2}^2.$$

This implies the claim.

Outline of the proof

- We set

  $$f_U = (1 - 1_\Omega)(f - \mathbb{E}(f|\mathcal{B})),\ f_{U^\perp} = (1 - 1_\Omega)\mathbb{E}(f|\mathcal{B}),\ E = 1_\Omega f$$

  for a suitably constructed σ-algebra $\mathcal{B}$ of $\mathbb{Z}_N$ and $\Omega \in \mathcal{B}$.
- The σ-algebra $\mathcal{B}$ is constructed inductively.
- First, we start with $\mathcal{B} = \{\emptyset, \mathbb{Z}_N\}$. Then all properties hold except possibly "uniformity".
- Obstruction to uniformity is detected by dual function $\mathcal{D}(f_U)$.
- We refine $\mathcal{B}$ by adding "level sets" of $\mathcal{D}(f_U)$.
- In finitely many steps, we achieve uniformity because $\|f_{U^\perp}\|_{L^2}$ increases (energy increment property), but $\|f_{U^\perp}\|_\infty$ is bounded.