LIFTING, RESTRICTING AND SIFTING INTEGRAL POINTS ON AFFINE HOMOGENEOUS VARIETIES

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Abstract. In [GN2] an effective solution of the lattice point counting problem in general domains in semisimple S-algebraic groups and affine symmetric varieties was established. The method relies on the mean ergodic theorem for the action of G on G/Γ, and implies uniformity in counting over families of lattice subgroups admitting a uniform spectral gap. In the present paper we extend some methods developed in [NS] and use them to establish several useful consequences of this property, including

(1) Effective upper bounds on lifting for solutions of congruences in affine homogeneous varieties,
(2) Effective upper bounds on the number of integral points on general subvarieties of semisimple group varieties,
(3) Effective lower bounds on the number of almost prime points on symmetric varieties,
(4) Effective upper bounds on almost prime solutions of Linnik-type congruence problems in homogeneous varieties.

1. Introduction and Statement of results

Throughout the paper, F denotes a number field, and V_F denotes the set of absolute values of F extending the standard normalised absolute values of the rational numbers. F_v, $v \in V_F$, will denote the corresponding local fields.

We introduce local and global heights. For Archimedean $v \in V_F$, and for $x = (x_1, \dots, x_d) \in F_v^d$, we set

$$\mathrm{H}_v(x) = \left(|x_1|_v^2 + \cdots + |x_d|_v^2\right)^{1/2},$$

and for non-Archimedean v,

$$\mathrm{H}_v(x) = \max\{|x_1|_v, \dots, |x_d|_v\}.$$

For $x = (x_1, \dots, x_d) \in F^d$, we set

$$\mathrm{H}(x) = \prod_{v \in V_F} \mathrm{H}_v(x).$$

1.1. **Effective lifting of solutions of congruences.** Let S be a finite subset of V_F containing all Archimedean absolute values, and

$$O_S = \{x \in F : |x|_v \le 1 \text{ for } v \notin S\}$$

is the ring of S-integers in F. We consider a system X of polynomial equations with coefficients in O_S. Given an ideal $\mathfrak{a}$ of O_S, we denote by $\mathsf{X}^{(\mathfrak{a})}$ the system of

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polynomial equations over the factor-ring $O_S/\mathfrak{a}$ obtained by reducing X modulo $\mathfrak{a}$. There is a natural reduction map

$$\pi_{\mathfrak{a}} : \mathsf{X}(O_S) \to \mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a}).$$

The question whether a solution in $\mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ can be lifted to an integral solution in $\mathsf{X}(O_S)$ is of fundamental importance in number theory. It is closely related to the strong approximation property for algebraic varieties (see [PR, §7.1]). For instance, if G is a connected F-simple simply connected algebraic group which is isotropic over S, then G satisfies the strong approximation property (see [PR, §7.4]) and, in particular, the map $\pi_{\mathfrak{a}}$ is surjective in this case. For more general homogeneous varieties, the map $\pi_{\mathfrak{a}}$ need not be surjective, but the image $\pi_{\mathfrak{a}}(\mathsf{X}(O_S))$ can be described using the Brauer–Manin obstructions (see [Hr, CTX, BD]).

In this paper, we consider the problem whether a solution in $\mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ can be lifted to an integral solution in $\mathsf{X}(O_S)$ *effectively* : given $\bar{x} \in \mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a})$, can one find $x \in \mathsf{X}(O_S)$, with $\mathrm{H}(x)$ bounded in terms of $|O_S/\mathfrak{a}|$, such that $\pi_{\mathfrak{a}}(x) = \bar{x}$?

We give a positive answer to this question for affine homogeneous varieties of semisimple groups. Let $\mathsf{X} \subset F^n$ be an affine variety defined over F and equipped with a transitive action (defined over F) of a connected simply connected F-simple algebraic group $\mathsf{G} \subset \mathrm{GL}_m$.

Let S be a finite subset of V_F, containing all Archemedean absolute values, such that action of G on X is defined over O_S, and $\mathrm{Lie}(\mathsf{G}) \cap \mathrm{M}_m(O_S)$ has a basis over O_S as an O_S-module. We note that every sufficiently large subset S of V_F satisfies the above assumptions. Moreover, the second assumption on S is satisfied when O_S is a principal ideal domain. In particular, the second assumption always holds when the field F has class number one.

Theorem 1.1. *There exist $q_0, \sigma > 0$ such that for every ideal $\mathfrak{a}$ of O_S satisfying $|O_S/\mathfrak{a}| \geq q_0$ and every $\bar{x} \in \pi_{\mathfrak{a}}(\mathsf{X}(O_S))$, there exists $x \in \mathsf{X}(O_S)$ such that*

$$\pi_{\mathfrak{a}}(x) = \bar{x} \quad \textit{and} \quad \mathrm{H}(x) \leq |O_S/\mathfrak{a}|^{\sigma}. \tag{1.1}$$

The parameter σ in (1.1) can be explicitly computed. For instance, for group varieties, an explicit value of σ is given in Theorem 2.1 below. The parameter q_0 is computable too (see Remark 2.3 below).

Remark 1.2. The finiteness of the exponent σ for the case of S-integral points in the group variety follows from the fact that the Cayley graphs $\mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ have logarithmic diameter. The bound provided by this approach depends on a choice of generating set of $\mathsf{G}(O_S)$, and when measured in terms of the height H is of lesser quality than the estimate on σ which is developed below explicitly in terms of geometric and representation-theoretic data of G. We thank Peter Sarnak for this remark.

Let us now consider the case of a connected F-simple simply connected algebraic group $\mathsf{G} \subset \mathrm{GL}_m$ which is isotropic over S. Then it is known to satisfy the strong approximation property, and our method gives an asymptotic formula for the number of solutions of (1.1):

Theorem 1.3. *For every $\sigma > \sigma_0$ (as in (2.1) below), every ideal $\mathfrak{a}$ in O_S and every $\bar{x} \in \mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})$,*

$$\begin{aligned}&|\{x \in \mathsf{G}(O_S);\ \pi_{\mathfrak{a}}(x) = \bar{x},\, \mathrm{H}(x) \le |O_S/\mathfrak{a}|^\sigma\}|\\ =&\frac{1}{|\mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})|} \cdot |\{x \in \mathsf{G}(O_S);\ \mathrm{H}(x) \le |O_S/\mathfrak{a}|^\sigma\}| \left(1 + O_\epsilon\left(|O_S/\mathfrak{a}|^{\dim(\mathsf{G})(1-\sigma_0^{-1}\sigma)+\epsilon}\right)\right)\end{aligned}$$

for every $\epsilon > 0$.

This result indicates that the properties $\pi_{\mathfrak{a}}(x) = \bar{x}$ and $\mathrm{H}(x) \le |O_S/\mathfrak{a}|^\sigma$ are asymptotically independent.

We illustrate our results on a classical example — the problem of representation a quadratic form by another quadratic form (see, for instance, [Om]).

Example 1.4. Let A be an integral nondegenerate symmetric $(n\times n)$-matrix and B be an integral nondegenerate symmetric $(m \times m)$-matrix with $n \le m$. The variety

$$\mathsf{X} = \{x \in \mathrm{M}_{m\times n}(\mathbb{C}) :\ {}^t x B x = A\} \tag{1.2}$$

parametrises all possible representation of the quadratic form corresponding to A by the quadratic form corresponding to B. For simplicity, we assume that $m-n \ge 3$ and A is isotropic over $\mathbb{R}$. Then if the equation ${}^t xBx = A$ has a solution over $\mathbb{R}$ and over $\mathbb{Z}_p$ for every p, then it has an integral solution, and the reduction map $\mathsf{X}(\mathbb{Z}) \to \mathsf{X}(\mathbb{Z}/q)$ is surjective for every $q \ge 1$ (see [Om, Ch. X]). Our results implies that under the same assumptions, for every $q \ge 1$ and $\bar{x} \in \mathrm{M}_{m\times n}(\mathbb{Z}/q)$ satisfying

$$ {}^t\bar{x}B\bar{x} = A \bmod q,$$

there exists $x \in \mathrm{M}_{m\times n}(\mathbb{Z})$ such that

$${}^t xBx = A, \qquad x = \bar{x} \bmod q, \qquad \mathrm{H}_\infty(x) \ll q^\sigma, \tag{1.3}$$

where $\sigma > 0$ is a computable constant. For instance, when B has the signature $(\lfloor m/2\rfloor, m - \lfloor m/2\rfloor)$, this estimate holds for

$$\sigma > \sigma_m = \begin{cases} \frac{4m(m^2-m+1)n_e}{m-1}, & \text{when } m \text{ is odd,}\\ \frac{4(m-1)(m^2-m+1)n_e}{m+2}, & \text{when } m \text{ is even,}\end{cases} \tag{1.4}$$

where n_e denotes the least even integer $\ge \lfloor m/2\rfloor$. We will provide details of this computation in Section 2.

The following example demonstrates that the polynomial bound established in Theorem 1.1 does not hold for other homogeneous varieties.

Example 1.5. Let F be a number field of degree d with an infinite group of units and $\{\xi_1, \dots, \xi_d\}$ be a basis of the ring O of integers of F. We consider the integral polynomial

$$f(x_1, \dots, x_d) = N_{K/\mathbb{Q}}(x_1\xi_1 + \cdots + x_d\xi_d)$$

and the variety $\mathsf{X} = \{f = 1\}$. Note that the set of integral points on X is exactly the group U of units in the number field F. We note that

$$|\{x \in \mathsf{X}(\mathbb{Z}) : \mathrm{H}_\infty(x) \le T\}| \ll (\log T)^{r+s-1} \tag{1.5}$$

where r and s denote the number of real and complex absolute values of F respectively. This claim can be checked by representing the group of units as a lattice in $\mathbb{R}^{r+s-1}$, similarly to the proof of Dirichlet's theorem.

We also note that there are infinitely many primes $\mathfrak{p}$ in the ring of integers O of F such that

$$(O/\mathfrak{p})^\times = U \bmod \mathfrak{p}. \tag{1.6}$$

It was proved in [CW] that the set of such primes has positive density if one assumes the Generalized Riemann Hyposesis, and in [N] that in most cases (for instance, when $[F:\mathbb{Q}]>3$), there are infinitely many such primes unconditionally. Now it follows from (1.6) that

$$|\pi_p(\mathsf{X}(\mathbb{Z}))| \ge p-1 \tag{1.7}$$

for infinitely many prime numbers p. Comparing (1.5) and (1.7), we conclude that the polynomial bound as in Corollary 1.1 is impossible in this case.

1.2. **Integral points on subvarieties.** We now turn to consider the problem of bounding the number of integral points on algebraic varieties. This has been an active field of research in recent years, and we refer the reader to the survey [HB3] and the book [B] for overviews of results and conjectures concerning upper estimates on the number of integral points. We will concentrate on homogeneous varieties and our methods and results are motivated by those developed in [NS, §4.3] .

Given an affine variety X defined over a number field F, we set

$$N_T(\mathsf{X}(O_S)) = |\{x\in \mathsf{X}(O_S):\ \mathrm{H}(x)\le T\}|$$

where O_S is a ring of S-integers in F. The problem we will focus on is establishing an upper estimate on $N_T(\mathsf{Y}(O_S))$ for arbitrary proper affine subvarieties Y of X. We will prove a *non-concentration phenomenon* for the collection of proper subvarieties of a semisimple group variety G, namely that the number of S-integral points on a subvariety Y has strictly lower rate of growth than the group variety G. We remark that this important property does not hold for general irreducible varieties X. Indeed a bound of the form

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{X},\deg(\mathsf{Y})} N_T(\mathsf{X}(O_S))^{1-\sigma_{\mathsf{Y}}}$$

with $\sigma_{\mathsf{Y}}>0$, where we write $\deg(\mathsf{Y})$ for the degree of the projective closure of Y, is false in general. This can be demonstrated by the variety $x_1^3+x_2^3+x_3^3+x_4^3=0$, where most of rational point lie on lines (see [HB1]). But in the case of S-algebraic group varieties we have the following:

Theorem 1.6. *Let G be a connected F-simple simply connected algebraic group defined over a number field F. Let $S\subset V_F$ be a finite subset containing all Archimedean absolute values such that G is isotropic over S. Then there exists $\sigma=\sigma(\mathsf{G},S,\dim(\mathsf{Y}))\in(0,1)$ such that for every absolutely irreducible proper affine subvariety Y of G defined over F, we have*

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{G},\deg(\mathsf{Y})} N_T(\mathsf{G}(O_S))^{1-\sigma}\ .$$

An explicit formula for the exponent σ is given in Theorem 3.1 below, demonstrating that σ depends only on $\dim\mathsf{Y}$, and increases monotonically with the codimension of Y.

To demonstrate Theorem 3.1 let us consider the case of integral points on subvarieties of the special linear group SL_n, $n\ge 2$.

Example 1.7. For every absolutely irreducible proper affine subvariety Y of SL_n defined over F, we have

$$N_T(\mathsf{Y}(\mathbb{Z})) \ll_{n,\deg(\mathsf{Y}),\epsilon} T^{n^2-n-\frac{n^2-1-\dim(\mathsf{X})}{(n^2+n)2n_e}+\epsilon}, \quad \epsilon > 0, \tag{1.8}$$

as $T \to \infty$, where n_e is the least even integer $\geq n-1$. This improves the trivial estimate $N_T(\mathsf{Y}(\mathbb{Z})) \ll T^{n^2-n}$. Details of this computation will be given in Section 3.

We note that the assumption of absolute irreducibility is not crucial for the conclusion of Theorem 1.6. Another version of Theorem 1.6 which can be proved using the argument of [NS, Lem. 4.2] is as follows.

Theorem 1.8. *With notation as in Theorem 1.6, for every proper affine subvariety Y of G defined over F, we have*

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{G},\mathsf{Y}} N_T(\mathsf{G}(O_S))^{1-\sigma} .$$

Let now Y_i, $1 \leq i \leq k$ be a collection of k hypersurfaces in G. Since the number of lattice points in each hypresurface has lower rate of growth than the number of lattice points in G, the same holds for their union. Thus, the rate of growth of the number of lattice points in the complement of these hypersurfaces is the same as the rate of growth of all lattice points. This observation gives rise to host of results asserting that the set of integral points which are generic (i.e. avoid the union of the hypersurfaces) has maximal possible rate of growth. Let us illustrate this principle concretely by the following example.

Example 1.9. Denote by N_T the number of unimodular integral $(n \times n)$-matrices $(n \geq 3)$ of norm bounded by T, and by N'_T the number of such matrices satisfying

- all the matrix entries are non-zero,
- all the principal minors do not vanish,
- all the eigenvalues are distinct,
- all the singular values (eigenvalues of A^tA) are distinct.

Then

$$N'_T = N_T \cdot \left(1 + O_\epsilon\left(T^{-\frac{1}{2n_e n(n+1)}+\epsilon}\right)\right), \quad \epsilon > 0,$$

where n_e is the least even integer $\geq n-1$.

1.3. **Almost prime points on varieties and orbits.** We now turn to the question of how often a polynomial map $f : \mathbb{Z}^n \to \mathbb{Z}$ admits prime (or, more realistically, almost prime) values. This problem has long been studied using sieve methods (see, for instance, [HR]). Recently substantial progress has been achieved in the papers [BGS, LS, NS], establishing results on the abundance of almost prime values for polynomials defined on homogeneous varieties and orbits of linear groups. The goal of this section is to generalise one of the main results of [NS] to the setting of symmetric varieties.

Let G be a connected $\mathbb{Q}$-simple simply connected algebraic group isotropic over $\mathbb{Q}$ and $\mathsf{G} \to \mathrm{GL}_n$ a representation of G which is also defined over $\mathbb{Q}$. Fix $v \in \mathbb{Z}^n$. We assume that $\mathsf{X} = \mathsf{G}v$ is Zariski closed, and $\mathsf{L} = \mathrm{Stab}_{\mathsf{G}}(v)$ is connected and has no nontrivial characters. Then the coordinate ring $\mathbb{C}[\mathsf{X}]$ is a unique factorisation domain (see Lemma 4.3 below). Let f be a regular function on X defined over $\mathbb{Q}$ such that it has a decomposition into irreducible factors $f = f_1 \cdots f_t$ where all f_i's are distinct and defined over $\mathbb{Q}$. Let $\mathcal{O} = \Gamma v$ be the orbit of $\Gamma = \mathsf{G}(\mathbb{Z})$.

We assume that f takes integral values on $\mathcal{O}$ and is weakly primitive (that is, $\gcd(f(x) : x \in \mathcal{O}) = 1$). The saturation number $r_0(\mathcal{O}, f)$ of the pair $(\mathcal{O}, f)$ is the least r such that the set of $x \in \mathcal{O}$ for which $f(x)$ has at most r prime factors is Zariski dense in X, which is the Zariski closure of $\mathcal{O}$ by the Borel density theorem. It is natural to ask whether the saturation number $r_0(\mathcal{O}, f)$ is finite and establish quantitative estimates on the set $\{x \in \mathcal{O} : f(x) \text{ has at most } r \text{ prime factors}\}$.

We fix a norm on $\mathbb{R}^n$ and set $\mathcal{O}(T) = \{w \in \mathcal{O} : \|w\| \le T\}$. It was shown in [NS] that when $\mathsf{X} \simeq \mathsf{G}$ is a group variety, the saturation number is finite and there exists explicit $r \ge 1$ such that

$$|\{x \in \mathcal{O}(T) : f(x) \text{ has at most } r \text{ prime factors}\}| \gg \frac{|\mathcal{O}(T)|}{(\log T)^{t(f)}} \tag{1.9}$$

as $T \to \infty$. As remarked in [BGS, NS], the assumption that $\mathsf{X} \simeq \mathsf{G}$ is not crucial if only finiteness of the saturation number is concerned, and $r_0(\mathcal{O}, f)$ is finite for general orbits. However, the effective lower estimate (1.9) is much more demanding, and so far it has only been established for 2-dimensional quadratic surfaces [LS] and for group varieties [NS]. Our goal here is to prove (1.9) for general symmetric varieties.

Theorem 1.10. *Let $\mathcal{O}$ and f be as above and assume in addition that $\mathsf{L} = \mathrm{Stab}_{\mathsf{G}}(v)$ is symmetric (that is, L is the set of fixed points of an involution of G). Then there exists $r \ge 1$ such that*

$$|\{x \in \mathcal{O}(T) : f(x) \text{ has at most } r \text{ prime factors}\}| \gg \frac{|\mathcal{O}(T)|}{(\log T)^{t(f)}}$$

as $T \to \infty$.

An explicit value of the number r is given in Theorem 4.2 below.

We illustrate Theorem 1.10 by three examples.

Example 1.11. Let Q be a nondegenerate integral quadratic form in n variables, which is indefinite over $\mathbb{R}$. Let $v \in \mathbb{Z}^n$, $\Gamma = \mathrm{Spin}(Q)(\mathbb{Z})$, and $\mathcal{O} = \Gamma v$. If we assume that $Q(v) \ne 0$, then the stabiliser of v in $\mathrm{Spin}(Q)$ is a symmetric subgroup of $\mathrm{Spin}(Q)$. Moreover, we assume that $n \ge 4$, which implies that this stabilizer is connected and has no nontrivial characters. Then Theorem 1.10 applies and (1.9) holds. An explicit estimate for the number r of prime factors is as follows. If Q has signature $(1, n-1)$ over $\mathbb{R}$, then (1.9) holds with r the least integer satisfying

$$r > \frac{9(n^2 - n + 2)(3n^2 - 3n + 2)}{2n - 4} \cdot n_e \cdot t(f) \deg(f),$$

where n_e is the least even integer $\ge 9(n-1)/7$. On the other hand, if Q has signature $(\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$ over $\mathbb{R}$, then (1.9) holds with r as above where n_e is the least even integer $\ge \lfloor n/2 \rfloor$. We will explain this computations in Section 4.

Example 1.12. Let A be a nondegenerate integral symmetric matrix of dimension n. We say that another matrix B is integrally equivalent of A if there exists $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ such that $B = {}^t\gamma A \gamma$, and write $B \sim_{\mathbb{Z}} A$. Let

$$\mathcal{O} = \{B \in \mathrm{M}_n(\mathbb{Z}) : B \sim_{\mathbb{Z}} A\}.$$

If $n \ge 3$, then Theorem 1.10 implies estimate (1.9) with

$$r > \frac{36n(3n^2 - 2)}{n - 1} \cdot n_e \cdot t(f) \deg(f),$$

where n_e is the least even integer $\geq n-1$. This will be explained in detail in Section 4.

1.4. **Linnik-type congruence problems on varieties and orbits.** Our next aim is to discuss an analogue of Linnik theorem [L1, L2] on the least prime in an arithmetic progression, which states that there exists $c, \sigma > 0$ such that for every coprime $b, q \in \mathbb{N}$ one can find a prime number p such that

$$p = b \bmod q \quad \text{and} \quad p \leq c\, q^{\sigma}.$$

It is a very challenging goal to establish such a result in the setting of the previous section, so to keep things more realistic, we settle for the existence of solutions of polynomially bounded size which are almost primes.

Let $\mathsf{G} \subset \mathrm{GL}_n$ be a connected $\mathbb{Q}$-simple simply connected algebraic group defined over $\mathbb{Q}$. We fix $v \in \mathbb{Z}^n$ and consider the orbit $\mathcal{O} = \Gamma v$ of $\Gamma = \mathsf{G}(\mathbb{Z})$. Let $f : \mathcal{O} \to \mathbb{Z}$ be a polynomial map. We assume that f is weakly primitive, and the regular function $\tilde{f} : \mathsf{G} \to \mathbb{C}$ defined by $\tilde{f}(g) = f(gv)$ decomposes as a product of t irreducible factors which are distinct and defined over $\mathbb{Q}$.

Theorem 1.13. *There exist $q_0, r, \sigma > 0$ (as in Theorem 5.1 below) such that for every coprime $b, q \in \mathbb{N}$ satisfying $q \geq q_0$ and $b \in f(\mathcal{O}) \bmod q$, one can find $x \in \mathcal{O}$ satisfying*

(i) *$f(x)$ is a product of at most r prime factors,*
(ii) *$f(x) = b \bmod q$ and $\|x\| \leq q^{\sigma}$.*

The explicit values of r and σ are given in Theorem 5.1 below, and q_0 could be computed, in principle, as well.

Coming back to Example 1.11, we conclude that for a polynomial function f on $\mathrm{M}_{m\times n}(\mathbb{C})$ satisfying above conditions, the system of equations

$${}^t xBx = A, \qquad f(x) = b \bmod q, \qquad \mathrm{H}_\infty(x) \ll q^{\sigma},$$

has a solution $x \in \mathrm{M}_{m\times n}(\mathbb{Z})$ such that $f(x)$ is a product of at most r prime factors, provided that

$${}^t xBx = A, \qquad f(x) = b$$

has a solution modulo q, and q is sufficiently large. For instance, when B has the signature $(\lfloor m/2 \rfloor, m - \lfloor m/2 \rfloor)$, this holds for $\sigma > \sigma_m$ as in (1.4), and

$$r > \frac{9\alpha_m\, \sigma}{\alpha_m\, \sigma - m(m-1)/2} \cdot \sigma_m \cdot t(f) \deg(f),$$

where $\alpha_m = (m-1)^2/4$ for odd m and $\alpha_m = m(m+2)/4$ for even m.

We remark that in Theorem 1.13 we do not assume that the stabiliser of v in G is symmetric. Under this assumption, our method implies a result on the number of solutions:

Theorem 1.14. *Under the additional assumption that* $\mathrm{Stab}_{\mathsf{G}}(v)$ *is symmetric, for every $\sigma > \sigma_0$ (as in (4.7)), r (as in (5.12)), and coprime $b, q \in \mathbb{N}$ satisfying $b \in f(\mathcal{O}) \bmod q$,*

$$\left| \left\{ x \in \mathcal{O}(q^{\sigma}) : \begin{array}{l} f(x) \text{ has at most } r \text{ prime factors} \\ f(x) = b \bmod q \end{array} \right\} \right| \gg_{\sigma} \frac{1}{|f(\mathcal{O}) \bmod q|} \cdot \frac{|\mathcal{O}(q^{\sigma})|}{(\log q)^{t(f)}}$$

for sufficiently large q.

1.5. **The fundamental lattice point counting result.** We now state the uniform solution given in [GN2, Th. 5.1] to the lattice point counting problem, which underlies the results in the present paper. Let G be a connected F-simple simply connected algebraic group defined over a number field F, S a finite subset of V_F, and O_S the ring of S-integers. Let $G = \prod_{v\in S} \mathsf{G}(F_v)$, $\Gamma = \mathsf{G}(O_S)$, and

$$\Gamma(\mathfrak{a}) = \{\gamma \in \Gamma : \gamma = I \bmod \mathfrak{a}\} \quad \text{for an ideal } \mathfrak{a} \text{ of } O_S .$$

We shall use the following notation throughout the paper :

$$\begin{aligned}
p_S &= \text{the least number such that all } L_0^2(G/\Gamma(\mathfrak{a})) \text{ are } L^{p_S^+}\text{-integrable ([GN1, Def. 5.2])},\\
n_e(p) &= \text{the least even integer} \geq p/2, \text{ if } p > 2, \text{ and } 1, \text{ if } p = 2,\\
d_S &= \sum_{v\in V_\infty} \dim \mathsf{G}(F_v),\\
B_T &= \{g \in G : \mathrm{H}(g) \leq T\},\\
a_S &= \text{the Hölder exponent of the family of sets } B_{e^t} \text{ (see [GN1, Def. 3.12])}.
\end{aligned}$$

We note that the finiteness of p_S is a manifestation of property (τ), established in full generality by Clozel [C]. Hölder-admissibility of the sets B_{e^t} was established in [GN1, Th. 7.19] and [BO]. In many cases, one can take $a_S = 1$ (see [GN1, Ch. 7]). For instance, this is the case when $F = \mathbb{Q}$ and $S = \{\infty\}$.

We can now state :

Theorem 1.15 ([GN2])**.** *For every $\gamma_0 \in \Gamma$ and all ideals $\mathfrak{a}$ of O_S,*

$$|\gamma_0\Gamma(\mathfrak{a}) \cap B_T| = \frac{\mathrm{vol}(B_T)}{[\Gamma : \Gamma(\mathfrak{a})]} + O_\epsilon\left(\mathrm{vol}(B_T)^{1-(2n_e(p_S))^{-1}a_S/(a_S+d_S)+\epsilon}\right), \quad \epsilon > 0,$$

where the Haar measure on G is normalised so that $\mathrm{vol}(G/\Gamma) = 1$.

We set

$$\alpha_S(G) = \limsup_{T\to\infty} \frac{\log|\{\gamma \in \Gamma : \mathrm{H}(\gamma) \leq T\}|}{\log T} = \limsup_{T\to\infty} \frac{\log \mathrm{vol}(B_T)}{\log T}. \tag{1.10}$$

We note that $\alpha_S(G) > 0$ provided that G is isotropic over S (see [GW, Sec. 7], [M], [GOS, Sec. 6]).

We will also have occasion below to consider the volume growth in a homogeneous space G/H, in which case we will denote the exponent by $\alpha(G/H)$.

Although the asymptotics of $|\{\gamma \in \Gamma : \mathrm{H}(\gamma) \leq T\}|$ is also known, this will not be needed in our argument.

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2. Effective lifting of solutions of congruences

We first establish a version of Theorem 1.1 in the case of group varieties, and Theorem 1.1 will be deduced from the following result.

Theorem 2.1. *Let $\mathsf{G} \subset \mathrm{GL}_m$ be a connected simply connected F-simple algebraic group, and let S be a finite subset of V_F, containing all Archimedean absolute values,* *such that* G *is isotropic over* S *and* $\mathrm{Lie}(\mathsf{G}) \cap \mathrm{M}_m(O_S)$ *has a basis over* O_S *as an* O_S*-module. Let*

$$\sigma > \sigma_0 := \alpha_S(G)^{-1} \dim(\mathsf{G}) \frac{a_S + d_S}{a_S} 2n_e(p_S). \tag{2.1}$$

Then there exists $q_0 > 0$ *such that for every ideal* $\mathfrak{a}$ *of* O_S *satisfying* $|O_S/\mathfrak{a}| \geq q_0$ *and every* $\bar{x} \in \mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})$*, there exists* $x \in \mathsf{G}(O_S)$ *such that*

$$\pi_{\mathfrak{a}}(x) = \bar{x} \quad \textit{and} \quad \mathrm{H}(x) \leq |O_S/\mathfrak{a}|^{\sigma}. \tag{2.2}$$

We note that the exponent σ can be further improved, for instance, by considering a smooth density on the sets $\{\mathrm{H} \leq T\}$, and when $p_S = 2$, this leads to essentially optimal bound on σ. However, we do not pursue this direction in the present papers and rely only on the counting estimate of Theorem 1.15.

Proof. Since G is isotropic over S, it satisfies the strong approximation property with respect to S (see [PR, §7.4]). Then it follows that the map $\pi_{\mathfrak{a}}$ is surjective, and there exists $\gamma_0 \in \Gamma$ such that $\bar{x} = \pi_{\mathfrak{a}}(\gamma_0)$ for some $\gamma_0 \in \Gamma$. Moreover, we have $\bar{x} = \pi_{\mathfrak{a}}(\gamma_0 \Gamma(\mathfrak{a}))$.

By Theorem 1.15, for every $\delta < \delta_0 = (2n_e(p_S))^{-1} a_S/(a_S + d_S)$ and $c_\delta > 0$, we have

$$\left| |\gamma_0 \Gamma(\mathfrak{a}) \cap B_T| - \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|} \right| \leq c_\delta \; \mathrm{vol}(B_T)^{1-\delta}. \tag{2.3}$$

It is important to emphasize here that this estimate is uniform over all $\gamma_0 \in \Gamma$ and all ideals $\mathfrak{a}$ of O_S. It follows from (2.3) that for T satisfying

$$\mathrm{vol}(B_T) > (c_\delta \, |\Gamma : \Gamma(\mathfrak{a})|)^{1/\delta}, \tag{2.4}$$

there exists $x \in \gamma_0 \Gamma(\mathfrak{a}) \cap B_T$. Then we have $\pi_{\mathfrak{a}}(x) = \bar{x}$ and $\mathrm{H}(x) \leq T$.

Now it remains to analyse for which values of T inequality (2.4) holds. By Lemma 2.2 below,

$$|\Gamma : \Gamma(\mathfrak{a})| \ll |O_S/\mathfrak{a}|^{\dim(\mathsf{G})}. \tag{2.5}$$

By (1.10), for every $\alpha < \alpha_S$ and $T > T(\alpha)$,

$$\mathrm{vol}(B_T) \geq T^{\alpha}. \tag{2.6}$$

Therefore, we conclude that (2.4) holds for $T = |O_S/\mathfrak{a}|^{\sigma}$ with $\sigma > \dim(\mathsf{G})/(\alpha\delta)$ and sufficiently large $|O_S/\mathfrak{a}|$. Since this is the case for every $\alpha < \alpha_S(G)$ and $\delta < \delta_0$, this concludes the proof. □

To complete the proof of Theorem 2.1, we therefore only have to establish the following

Lemma 2.2. *Let* $\mathsf{G} \subset \mathrm{GL}_n$ *be a connected algebraic group, and let* S *be a finite subset of* V_F*, containing all Archemedean absolute values, such that* $\mathrm{Lie}(\mathsf{G}) \cap \mathrm{M}_m(O_S)$ *has a basis over* O_S*. Then*

$$|\Gamma : \Gamma(\mathfrak{a})| \asymp |O_S/\mathfrak{a}|^{\dim(\mathsf{G})}$$

uniformly over ideals $\mathfrak{a}$ *of* O_S*.*

Proof. Let O_v denote the local ring with the prime ideal $\mathfrak{p}_v$ corresponding to non-Archemedean $v \in V_F$. It follows from the formulas for local Tamagawa measures (see, for instance, [V, 14.2-14.3]) and the Lang–Weil estimates [LW] that for all valuations v outside S and all $n \geq 0$,

$$|\mathsf{G}^{(\mathfrak{p}_v^n)}(O_v/\mathfrak{p}_v^n)| \asymp |O_v/\mathfrak{p}_v|^{n\dim(\mathsf{G})} = |O_v/\mathfrak{p}_v^n|^{\dim(\mathsf{G})},$$

and it follows the Chinese reminder theorem that for every ideal $\mathfrak{a}$ of O_S,

$$|\mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})| \asymp |O_S/\mathfrak{a}|^{\dim(\mathsf{G})}.$$

Since the kernel of the reduction map $\pi_{\mathfrak{a}} : \Gamma \to \mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ is equal to $\Gamma(\mathfrak{a})$, this implies the claim of the lemma. □

Remark 2.3. The constant q_0 in our results can be computed in principle. It depends on the implicit constant in Theorem 1.15, which is given explicitly in [GN2], and on $T(\alpha)$ in (2.6). An explicit value of $T(\alpha)$ can derived from the asymptotic formula for $\mathrm{vol}(B_T)$ (see [GN1, Ch. 7]).

Proof of Theorem 1.1. By the Borel–Harish-Chandra theorem [BHC], the set $\mathsf{X}(O_S)$ is a union of finitely many orbits of $\Gamma = \mathsf{G}(O_S)$. Hence, it suffices to prove the claim for every $\bar{x} \in \mathsf{X}^{(\mathfrak{a})}(O_S/\mathfrak{a})$ that lifts to a point $x \in \mathsf{X}(O_S)$ contained in a Γ-orbit Γx_0 for some fixed $x_0 \in \mathsf{X}(O_S)$. If G is anisotropic over every $v \in S$, then Γ is finite, and the claim is trivial. Hence, we may assume that G is isotropic for some $v \in S$. Then Theorem 2.1 applies. We have

$$\bar{x} = \pi_{\mathfrak{a}}(\gamma \cdot x_0) = \pi_{\mathfrak{a}}(\gamma) \cdot \pi_{\mathfrak{a}}(x_0)$$

for some $\gamma \in \Gamma$. By Theorem 2.1, there exists $\gamma' \in \Gamma$ such that

$$\pi_{\mathfrak{a}}(\gamma') = \pi_{\mathfrak{a}}(\gamma) \quad \text{and} \quad \mathrm{H}(\gamma') \leq |O_S/\mathfrak{a}|^{\sigma}$$

where σ is as in Theorem 2.1. Since $\bar{x} = \pi_{\mathfrak{a}}(\gamma') \cdot \pi_{\mathfrak{a}}(x_0) = \pi_{\mathfrak{a}}(\gamma' \cdot x_0)$, it remains to observe that

$$\mathrm{H}(\gamma' \cdot x_0) \ll \mathrm{H}(\gamma')^N \tag{2.7}$$

for some uniform $N > 0$ determined by the action. □

Proof of Theorem 1.3. Let $\gamma_0 \in \Gamma$ be such that $\pi_{\mathfrak{a}}(\gamma_0) = \bar{x}$. By Theorem 1.15,

$$|\gamma_0\Gamma(\mathfrak{a}) \cap B_T| = \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|}\left(1 + O_\delta\left(\frac{|\Gamma : \Gamma(\mathfrak{a})|}{\mathrm{vol}(B_T)^\delta}\right)\right)$$

for every $\delta < \delta_0 = (2n_e(p_S))^{-1}a_S/(a_S + d_S)$. Hence, it follows from (2.5), and (2.6) that for every $\alpha < \alpha_S(G)$ and $T > T(\alpha)$, we have

$$|\gamma_0\Gamma(\mathfrak{a}) \cap B_T| = \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|}\left(1 + O_\delta\left(|O_S/\mathfrak{a}|^{\dim(\mathsf{G})}T^{-\alpha\delta}\right)\right).$$

Hence, if we pick $T = |O_S/\mathfrak{a}|^\sigma$ with $\sigma > \sigma_0$ as in (2.1) and sufficiently large $|O_S/\mathfrak{a}|$, then

$$\begin{aligned}
|\gamma_0\Gamma(\mathfrak{a}) \cap B_T| &= \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|}\left(1 + O_{\alpha,\delta}\left(|O_S/\mathfrak{a}|^{\dim(\mathsf{G}) - \sigma\alpha\delta}\right)\right) \\
&= \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma(\mathfrak{a})|}\left(1 + O_\epsilon\left(|O_S/\mathfrak{a}|^{\dim(\mathsf{G}) - \dim(\mathsf{G})\sigma_0^{-1}\sigma + \epsilon}\right)\right)
\end{aligned}$$

for every $\epsilon > 0$, where we have used that $\sigma_0 = \dim(\mathsf{G})/(\alpha_S\delta_0)$. Finally, to complete the proof, we note that

$$|\Gamma : \Gamma(\mathfrak{a})| = |\mathsf{G}^{(\mathfrak{a})}(O_S/\mathfrak{a})|$$

and by Theorem 1.15,

$$\mathrm{vol}(B_T) = |\{x \in \mathsf{G}(O_S);\ \mathrm{H}(x) \le |O_S/\mathfrak{a}|^\sigma\}| \left(1 + O_{\alpha,\delta}\left(|O_S/\mathfrak{a}|^{-\sigma\alpha\delta}\right)\right).$$

□

Coming back to Example 1.4, we note that the variety X defined in (1.2) is a homogeneous space of the spinor group $\mathsf{G} = \mathrm{Spin}(B)$. We have $\dim(\mathsf{G}) = m(m-1)/2$. The Hölder exponent is $a_S = 1$ by [GN1, Prop. 7.3]. Now we assume that G has maximal $\mathbb{R}$-rank (i.e., the signature of B is $(\lfloor m/2\rfloor, m - \lfloor m/2\rfloor)$). Then the integrability exponent is $p_S = m-1$ for odd m and $p_S = m$ for even m by [Li, Oh]. By [DRS, EM], the growth rate $\alpha_S(G)$ of integral points in $\mathsf{G}(\mathbb{Z})$ can be estimated in terms of volume growth of the norm balls which is computable in terms of the root data of G (see [M, GOS]). This gives $\alpha_S(G) = (m-1)^2/4$ for odd m and $\alpha_S(G) = m(m+2)/4$ for even m. Hence, Theorem 2.1 holds with

$$\sigma > \begin{cases} \frac{2m(m^2-m+1)n_e}{m-1}, & \text{when } m \text{ is odd,} \\ \frac{2(m-1)(m^2-m+1)n_e}{m+2}, & \text{when } m \text{ is even,} \end{cases}$$

where n_e denotes the least even integer $\ge \lfloor m/2\rfloor$. We note that the action of $\mathrm{Spin}(B)$ on X can be given by the standard Clifford algebra construction (see [D, Ch. II, §7])) which implies that (2.7) holds with $N = 2$. This explains (1.3).

3. Integral points on subvarieties

The following result is a precise version of Theorem 1.6 from Introduction. In the statement we use notation introduced in Theorem 1.15.

Theorem 3.1. *Let G be a connected F-simple simply connected algebraic group defined over a number field F. Let $S \subset V_F$ be a finite subset containing all Archemedean absolute values such that G is isotropic over S. Then for every absolutely irreducible proper affine subvariety Y of G defined over F, we have*

$$N_T(\mathsf{Y}(O_S)) \ll_{\mathsf{G},\deg(\mathsf{X}),\epsilon} N_T(\mathsf{G}(O_S))^{1-\frac{a_S(\dim(\mathsf{G})-\dim(\mathsf{Y}))}{\dim(\mathsf{G})(a_S+d_S)2n_e(p_S)}+\epsilon}, \quad \epsilon > 0,$$

as $T \to \infty$.

Coming back to Example 1.7, we note that in this case, $\dim(\mathrm{SL}_n(\mathbb{R})) = n^2-1$, and so $N_T(\mathrm{SL}_n(\mathbb{Z})) \sim c_n T^{n^2-n}$ with $c_n > 0$. Furthermore $p_S = 2(n-1)$ (see [DRS]), and $a_S = 1$ (see [GN1, Prop. 7.3]). Hence, estimate (1.8) is a special case of Theorem 3.1.

Proof of Theorem 3.1. For non-Archemedean $v \in V_F$, we denote by f_v the corresponding residue field and by $\mathfrak{p}_v$ the corresponding prime ideal.

We consider the reduction $\mathsf{Y}^{(v)}$ of the variety Y modulo a valuation v. Then by Noether's theorem, $\mathsf{Y}^{(v)}$ is absolutely irreducible for almost all v. Moreover, $\dim(\mathsf{Y}^{(v)}) = \dim(\mathsf{Y})$ and $\deg(\mathsf{Y}^{(v)}) = \deg(\mathsf{Y})$ for almost all v (see [Od, Sec. 1]). Therefore, by [GL, Prop. 12.1], we have the following estimate

$$|\mathsf{Y}^{(v)}(f_v)| \ll_{\deg(\mathsf{Y})} |f_v|^{\dim(\mathsf{Y})}, \tag{3.1}$$

valid for almost all v. We observe that each fiber of the reduction map $\mathsf{Y}(O_S) \to \mathsf{Y}^{(v)}(f_v)$ is contained in a coset of the subgroup $\Gamma_v = \{\gamma \in \Gamma : \gamma = I \bmod \mathfrak{p}_v\}$ of $\Gamma = \mathsf{G}(O_S)$. Hence, it follows that $\mathsf{Y}(O_S)$ is contained in a union of at most $O_{\deg(\mathsf{Y})}\left(|f_v|^{\dim(\mathsf{Y})}\right)$ cosets $\gamma\Gamma_v$ with $\gamma \in \Gamma$.

The crucial ingredient of the proof is Theorem 1.15, which gives an estimate of the number points in the cosets $\gamma\Gamma_v$ uniformly over $\gamma \in \Gamma$. More precisely, by Theorem 1.15, for all all v,

$$|\gamma\Gamma_v \cap B_T| = \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma_v|} + O_\epsilon\left(\mathrm{vol}(B_T)^{1-\frac{a_S}{(a_S+d_S)2n_e(p_S)}+\epsilon}\right), \quad \epsilon > 0. \tag{3.2}$$

For almost all v, the reduction $\mathsf{G}^{(v)}$ is smooth geometrically irreducible variety of dimension $\dim \mathsf{G}$. Therefore, we have the Lang–Weil estimate (see [LW])

$$|\mathsf{G}^{(v)}(f_v)| = |f_v|^{\dim(\mathsf{G})} + O_{\mathsf{G}}\left(|f_v|^{\dim(\mathsf{G})-1/2}\right).$$

Since G is simply connected F-simple and isotropic over S, it follows from the strong approximation property (see [PR, Theorem 7.12]) that the reduction map $\Gamma \to \mathsf{G}^{(v)}(f_v)$ is surjective for all $v \notin S$. This implies the estimate

$$|\Gamma : \Gamma_v| = |\mathsf{G}^{(v)}(f_v)| \gg |f_v|^{\dim(\mathsf{G})}$$

for almost all v.

Finally, we conclude from (3.1) and (3.2) that for all v,

$$|\mathsf{Y}(O_S) \cap B_T| \ll_{\mathsf{G},\deg(\mathsf{Y}),\epsilon} |f_v|^{\dim(\mathsf{Y})}\left(\frac{\mathrm{vol}(B_T)}{|f_v|^{\dim(\mathsf{G})}} + \mathrm{vol}(B_T)^{1-\frac{a_S}{(a_S+d_S)2n_e(p_S)}+\epsilon}\right), \quad \epsilon > 0.$$

To optimise this estimate, we take v such that

$$\mathrm{vol}(B_T)^{\frac{a_S}{(a_S+d_S)2n_e(p_S)}} \le |f_v|^{\dim(\mathsf{G})} \le 2\ \mathrm{vol}(B_T)^{\frac{a_S}{(a_S+d_S)2n_e(p_S)}}.$$

For sufficiently large T, such v exists by the prime number theorem for the ring of integers O in F. This gives the estimate

$$N_T(\mathsf{Y}(O_S)) = |\mathsf{Y}(O_S) \cap B_T| \ll_{\mathsf{G},\deg(\mathsf{Y}),\epsilon} \mathrm{vol}(B_T)^{1-\frac{a_S(\dim(\mathsf{G})-\dim(\mathsf{Y}))}{\dim(\mathsf{G})(a_S+d_S)2n_e(p_S)}+\epsilon}, \quad \epsilon > 0,$$

as $T \to \infty$. Since $N_T(\mathsf{G}(O_S)) \sim \mathrm{vol}(B_T)$ by Theorem 1.15, this completes the proof. $\square$

4. Almost prime points on varieties and orbits

We now turn to the problem of establishing the existence of almost prime points on symmetric varieties. We shall use the notation from Section 1.3. In particular, G is a connected $\mathbb{Q}$-simple simply connected algebraic group defined over $\mathbb{Q}$ and L is a symmetric $\mathbb{Q}$-subgroup. Let $G = \mathsf{G}(\mathbb{R})$ and $L = \mathsf{L}(\mathbb{R})$. Then G is a connected semisimple Lie group with finite center and L is a closed symmetric subgroup of G. We shall use the structure theory of affine symmetric spaces (see [HS, Part II]). Fix a maximal compact subgroup K of G compatible with L and a Cartan subgroup A for the pair (K, L). Then the Cartan decomposition

$$G = KA^+L$$

holds where A^+ denotes a closed positive Weyl chamber in A. Let M denote the centraliser of A in $K \cap L$. We fix a bounded subset Ψ of $M\backslash L$ with nonempty interior which we assume to be Lipschitz well-rounded (in the sense of [GN2, Sec. 7]). We also denote by $\dot{A}^+$ the interior of the Weyl chamber A^+ and set

$$S_T = \{g \in K\dot{A}^+\Psi : \|gv\| \le T\}.$$

We note that it was shown in [GN2, Prop. 8.4] that the sets S_{e^t} are Hölder well-rounded with exponent 1/3.

Our main tool is the following result on counting of lattice points in S_T for the congruence subgroups $\Gamma(q) = \{\gamma \in \Gamma : \gamma = I \bmod q\}$ of $\Gamma = \mathsf{G}(\mathbb{Z})$. We note that representations $L^2_0(G/\Gamma(q))$ are all L^{p+} with uniform $p > 0$ by [C], so that the following theorem is a special case of [GN2, Th. 8.1].

Theorem 4.1 ([GN2]). *For every $\gamma_0 \in \Gamma$ and $q \ge 1$,*

$$|\gamma_0\Gamma(q) \cap S_T| = \frac{\mathrm{vol}(S_T)}{[\Gamma : \Gamma(q)]} + O_\epsilon\left(\mathrm{vol}(S_T)^{1-(2n_e(p))^{-1}(1+3\dim G)^{-1}+\epsilon}\right), \quad \epsilon > 0,$$

where the Haar measure is normalised so that $\mathrm{vol}(G/\Gamma) = 1$.

We note that by [DRS, EM]

$$|\mathcal{O}(T)| \sim \frac{\mathrm{vol}(L/(L\cap\Gamma))}{\mathrm{vol}(G/\Gamma)} \cdot \mathrm{vol}(S_T v) \quad \text{as } T \to \infty, \tag{4.1}$$

where vol denote G-invariant measures on the corresponding spaces. It was shown in [GOS, Sec. 6] that

$$\mathrm{vol}(S_T v) \sim v_0 T^{\alpha(G/H)} (\log T)^\beta \quad \text{as } T \to \infty, \tag{4.2}$$

for some $v_0 > 0$, $\alpha(G/H) \in \mathbb{Q}^+$, and $\beta \in \mathbb{Z}^+$. Also, it is clear that

$$\mathrm{vol}(S_T) = \mathrm{vol}(S_T v) \cdot \mathrm{vol}(\Psi). \tag{4.3}$$

Now we prove the following theorem, which is a more explicit version of Theorem 1.10 stated in §1.3 (we refer there for the notation used below).

Theorem 4.2. *With the notation above, let r be the least integer satisfying*

$$r > 9\alpha(G/H)^{-1}\,(1+\dim(G))(1+3\dim(G))2n_e(p)\cdot t(f)\deg(f).$$

Then

$$|\{x \in \mathcal{O}(T) : f(x) \text{ has at most } r \text{ prime factors}\}| \gg \frac{|\mathcal{O}(T)|}{(\log T)^{t(f)}}$$

as $T \to \infty$.

In the case of Example 1.11, we have $\dim(\mathrm{Spin}(Q)) = n(n-1)/2$ and $\alpha(G/H) = n-2$. When Q has signature $(n,1)$, one can take $p = 9(n-1)/7$ (see [BS]). For other signatures, the group $\mathrm{Spin}(Q)(\mathbb{R})$ has $\mathbb{R}$-rank at least 2 and we can utilise the estimates on integrability exponents obtained in [Li, Oh]. In particular, when Q has signature $(\lfloor n/2\rfloor, n-\lfloor n/2\rfloor)$ (i.e., when $\mathrm{Spin}(Q)$ is split over $\mathbb{R}$), we have $p = n-1$ for odd n and $p = n$ for even n.

In the case of Example 1.12, we have $\dim(\mathrm{SL}_n) = n^2-1$, $\alpha = (n^2-n)/2$ (see [GOS, Sec. 2.3]), and $p = 2(n-1)$ (see [DRS]).

Before we start the proof of Theorem 4.2, we show that the decomposition $f = f_1 \cdots f_t$ into irreducible factors is well-defined.

Lemma 4.3. *Let G be a connected semisimple simply connected algebraic group and L a closed connected subgroup with no nontrivial characters. Then the coordinate ring $\mathbb{C}[\mathsf{G}/\mathsf{L}]$ is a unique factorisation domain.*

Proof. We refer to [FI, KKV, P] for computation of Picard groups of homogeneous spaces. There is an exact sequence

$$\mathcal{X}(\mathsf{G}) \to \mathcal{X}(\mathsf{L}) \to \mathrm{Pic}(\mathsf{G}/\mathsf{L}) \to \mathrm{Pic}(\mathsf{G})$$

where $\mathcal{X}(\mathsf{G})$ and $\mathcal{X}(\mathsf{L})$ denote the character groups. Since G is simply connected, $\mathrm{Pic}(\mathsf{G}) = 1$. Hence, it follows from the exact sequence that $\mathrm{Pic}(\mathsf{G}/\mathsf{L}) = 1$, and $\mathbb{C}[\mathsf{G}/\mathsf{L}]$ is a unique factorisation domain by [Ht, Prop. 6.2]. □

Proof of Theorem 4.2. Using the dominant map $\mathsf{G} \to \mathsf{X}$, every element $f \in \mathbb{C}[\mathsf{X}]$ lifts to an element $\tilde{f} \in \mathbb{C}[\mathsf{G}]$. Since G is simply connected, the ring $\mathbb{C}[\mathsf{G}]$ is a unique factorisation domain. We claim that the decomposition of $\tilde{f}$ into irreducible factors in $\mathbb{C}[\mathsf{G}]$ is of the form $\tilde{f} = \tilde{f}_1 \cdots \tilde{f}_t$, where $f = f_1 \cdots f_t$ is the decomposition in $\mathbb{C}[\mathsf{X}]$. Indeed, suppose that $\tilde{f}_i = g_1 \cdots g_s$ for $g_1, \ldots, g_s \in \mathbb{C}[\mathsf{G}]$ is the decomposition into irreducibles. We consider the right action of L on $\mathbb{C}[\mathsf{G}]$. Since $\tilde{f}_i$ is L-invariant and L is connected, it follows from uniqueness of the decomposition that each g_i is also L-invariant and descends to a function on $\mathbb{C}[\mathsf{X}]$, which implies that this decomposition must be trivial. Hence, $\tilde{f}_i$'s are irreducible.

Now we apply the argument of [NS] to the polynomial function $\tilde{f} : \Gamma \to \mathbb{Z}$ and the sets $\Gamma \cap S_T$ (instead of sets $\{\gamma \in \Gamma : \|\gamma\| < T\}$). It follows from Theorem 4.1 that for every $q \geq 1$ and $\gamma_0 \in \Gamma$,

$$\frac{|\gamma_0\Gamma(q) \cap S_T|}{\mathrm{vol}(S_T)} = \frac{1}{[\Gamma : \Gamma(q)]} + O_\epsilon\left(\mathrm{vol}(S_T)^{-(2n_e(p))^{-1}(1+3\dim G)^{-1}+\epsilon}\right), \quad \epsilon > 0. \tag{4.4}$$

Therefore, by (4.2)–(4.3),

$$\frac{|\gamma_0\Gamma(q) \cap S_T|}{\mathrm{vol}(S_T)} = \frac{1}{[\Gamma : \Gamma(q)]} + O_\epsilon\left(T^{-\frac{\theta}{1+3\dim(G)}+\epsilon}\right), \quad \epsilon > 0,$$

where $\theta = \frac{\alpha(G/H)}{2n_e(p)}$. This estimate is a substitute for [NS, Th. 3.2]. Given the estimate above for a family of sets S_T, the argument in [NS] for norm balls can be carried out without change, and we conclude that for sufficiently large T,

$$\sum_{\gamma \in \Gamma \cap S_T : \gcd(\tilde{f}(\gamma), P_z) = 1} 1 \gg \frac{|\Gamma \cap S_T|}{(\log |\Gamma \cap S_T|)^{t(f)}}, \tag{4.5}$$

where

$$P_z = \prod_{p \leq z} p, \quad z = |\Gamma \cap S_T|^\kappa, \quad \kappa = (9t(f)(1+\dim(G))(1+3\dim(G))2n_e(p))^{-1}.$$

For every $\gamma \in \Gamma \cap S_T$, we have

$$|\tilde{f}(\gamma)| = |f(\gamma v)| \ll T^{\deg(f)}.$$

On the other hand, if $\gcd(\tilde{f}(\gamma), P_z) = 1$, then every prime factor of $\tilde{f}(\gamma)$ is at least z, and $z \gg T^{\alpha(G/H)\kappa}$ by (4.4) and (4.2)–(4.3). Therefore, for every term in the sum (4.5), the number of prime factors of $\tilde{f}(\gamma)$ is bounded above by

$$\frac{\deg(f)}{\alpha(G/H)\kappa} = 9\alpha(G/H)^{-1}(1+\dim(G))(1+3\dim(G))2n_e(p)t(f)\deg(f)$$

provided T is sufficiently large. We conclude that

$$|\{\gamma \in \Gamma \cap S_T : f(\gamma v) \text{ has at most } r \text{ prime factors}\}| \gg \frac{|\Gamma \cap S_T|}{(\log |\Gamma \cap S_T|)^{t(f)}}. \tag{4.6}$$

To finish the proof, we consider the projection map

$$\pi : \Gamma \cap S_T \to \mathcal{O}(T) : \gamma \mapsto \gamma v.$$

It follows from the uniqueness properties of the Cartan decomposition (see [HS, p. 108]) that if $\gamma_0, \gamma \in \Gamma \cap S_T$ satisfy $\gamma_0 v = \gamma v$, then their KA^+-components are equal modulo M, and $\gamma_0^{-1}\gamma \in \Psi^{-1}\Psi$. Hence,

$$\pi^{-1}(\gamma_0 v) \subset \gamma_0 \Psi^{-1}\Psi \cap \Gamma,$$

and the cardinality of every fiber of π is bounded by $|\Psi^{-1}\Psi \cap \Gamma|$. It follows from (4.6) that

$$|\{w \in \mathcal{O}(T) :\ f(w) \text{ has at most } r \text{ prime factors}\}| \gg \frac{|\Gamma \cap S_T|}{(\log |\Gamma \cap S_T|)^{t(f)}}$$

as $T \to \infty$. Since $|\Gamma \cap S_T| \asymp \mathrm{vol}(S_T)$, the claim of the theorem now follows from (4.1)–(4.3). □

We also establish a quantitative version of Theorem 1.1 for lifting solutions of congruences in $\mathcal{O}$, which will be used to prove Theorem 1.14 in Section 5.

Theorem 4.4. *For every*

$$\sigma > \sigma_0 := \alpha(G/H)^{-1} \dim(G)(1 + 3\dim(G))2n_e(p), \tag{4.7}$$

sufficiently large q, and $b \in \mathcal{O} \bmod q$,

$$|\{x \in \mathcal{O}(q^\sigma);\ \ x = b \bmod q\}| \gg_\sigma |\mathcal{O}(q^\sigma)| \cdot \frac{1}{|\mathcal{O} \bmod q|}.$$

Proof. Using Theorem 4.1 and arguing exactly as in the proof of Theorem 1.3, we get the estimate

$$|\gamma\Gamma(q) \cap S_T| \gg_\sigma \frac{1}{|\Gamma : \Gamma(q)|} \cdot |\Gamma \cap S_T|.$$

for $T = q^\sigma$ with sufficiently large q and every $\gamma \in \Gamma$. This implies that

$$|\{\gamma \in \Gamma \cap S_T;\ \ \gamma v = b \bmod q\}| = \sum_{\gamma \in \Gamma/\Gamma(q):\, \gamma v = b \bmod q} |\gamma\Gamma(q) \cap S_T|$$

$$\gg_\sigma \frac{|\mathrm{Stab}_\Gamma(b \bmod q) : \Gamma(q)|}{|\Gamma : \Gamma(q)|} \cdot |\Gamma \cap S_T| = \frac{1}{|\mathcal{O} \bmod q|} \cdot |\Gamma \cap S_T|.$$

Recall from the previous proof that the cardinality of the fibers of the map

$$\pi : \Gamma \cap S_T \to \mathcal{O}(T) : \gamma \mapsto \gamma v.$$

is uniformly bounded. Therefore,

$$|\{x \in \mathcal{O}(T);\ \ x = b \bmod q\}| \gg |\{\gamma \in \Gamma \cap S_T;\ \ \gamma v = b \bmod q\}|.$$

Since $|\Gamma \cap S_T| \asymp |\mathcal{O}(T)|$ by (4.1), this completes the proof. □

5. Linnik-type congruence problems on varieties and orbits

We start by proving Theorem 1.13 for group varieties. Let $\mathsf{G} \subset \mathrm{GL}_n$ be a connected $\mathbb{Q}$-simple simply connected algebraic group defined over $\mathbb{Q}$. We assume that G is isotropic over $\mathbb{R}$, and denote by $\alpha = \alpha(G) > 0$ the volume growth exponent of $\mathsf{G}(\mathbb{R})$, defined as in (1.10). Let f be a regular function on G defined over $\mathbb{Q}$ that decomposes into product of $t = t(f)$ absolutely irreducible factors defined over $\mathbb{Q}$. We assume that $f(\mathsf{G}(\mathbb{Z})) \subset \mathbb{Z}$ and f is weakly primitive.

Theorem 5.1. *Let*

$$\sigma > \sigma_0 := \alpha^{-1} \dim(\mathsf{G})(1 + \dim(\mathsf{G}))2n_e(p),$$
$$r > \frac{9\alpha\sigma}{\alpha\sigma - \dim(\mathsf{G})} \cdot \sigma_0 \cdot t(f) \deg(f).$$

Then there exists $q_0 > 0$ such that for every coprime $b, q \in \mathbb{N}$ satisfying $q \geq q_0$ and $b \in f(\mathsf{G}(\mathbb{Z})) \bmod q$, one can find $x \in \mathsf{G}(\mathbb{Z})$ such that

(i) *$f(x)$ is a product of at most r prime factors,*
(ii) *$f(x) = b \bmod q$ and $\|x\| \leq q^\sigma$.*

Proof. We write $f(x) = \frac{1}{N} g(x)$ where $g(x)$ is a polynomial with integral coefficients and $N \in \mathbb{N}$. Since f is weakly primitive,

$$\gcd(g(\gamma) : \gamma \in \Gamma) = N. \tag{5.1}$$

Let $N = N_1 N_2$ where N_1 is the product of all prime factors coprime to q. Then the condition $f(\gamma) = b \bmod q$ is equivalent to $g(\gamma) = bN \bmod qN$. Moreover, because of (5.1), it is equivalent to $g(\gamma) = bN \bmod qN_2$.

According to our assumptions, there exists $\gamma_0 \in \mathsf{G}(\mathbb{Z})$ such that $f(\gamma_0) = b \bmod q$. We set

$$\Gamma = \mathsf{G}(\mathbb{Z}),$$
$$\Gamma_q = \Gamma(qN_2) = \{\gamma \in \Gamma : \gamma = id \mod qN_2\},$$
$$\mathcal{O}_q(T) = \{\gamma \in \gamma_0\Gamma_q : \|\gamma\| \leq T\}.$$

Note that every $\gamma \in \gamma_0\Gamma_q$ satisfies $f(\gamma) = b \bmod q$.

Let $\mathcal{P}_{q,z}$ be the set of prime numbers which are coprime to q and bounded by z. Our aim is to estimate from below the cardinality of points $\gamma \in \mathcal{O}_q(T)$ such that $f(\gamma)$ is coprime to $\mathcal{P}_{q,z}$, which we denote by $S(T, q, z)$. This will achieved by applying the combinatorial sieve as in [HR, Thm 7.4] and [NS, Sec. 2]. Let

$$a_k = |\{\gamma \in \mathcal{O}_q(T) : f(\gamma) = k\}| \quad \text{and} \quad X = |\mathcal{O}_q(T)| = \sum_{k \geq 0} a_k.$$

In order to apply the combinatorial sieve, we need to verify the following conditions:

(A_0) For every square-free d in $\mathcal{P}_{q,z}$,

$$\sum_{k=0 \bmod d} a_k = \frac{\rho(d)}{d} X + R_d, \tag{5.2}$$

where $\rho(d)$ is a nonnegative multiplicative function such that for primes $p \in \mathcal{P}_{q,z}$, we have

$$\frac{\rho(p)}{p} \leq c_1 \tag{5.3}$$

for some $c_1 < 1$.

(A_1) Summing over square-free d in $\mathcal{P}_{q,z}$,

$$\sum_{d\le X^\tau}{}' |R_d| \le c_2 X^{1-\zeta}$$

for some $c_2, \tau, \zeta > 0$.

(A_2) For every $2 \le w \le z$,

$$-l \le \sum_{p\in\mathcal{P}_{q,z}: w\le p<z} \frac{\rho(p)\log p}{p} - t\log\frac{z}{w} \le c_3 \tag{5.4}$$

for some $c_3, l, t > 0$.

Assuming that (A_0), (A_1), and (A_2), [HR, Th. 7.4] implies that for $z = X^{\tau/s}$ with $s > 9t$, the following estimate holds:

$$S(T,q,z) \ge XW(z)\left(C_1 - C_2 l\frac{(\log\log 3X)^{3t+2}}{\log X}\right), \tag{5.5}$$

where

$$W(z) = \prod_{p\in\mathcal{P}_{q,z}: p\le z}\left(1-\frac{\rho(p)}{p}\right),$$

and the constants $C_1, C_2 > 0$ are determined by c_1, c_2, c_3, τ, ζ, t.

We deduce (A_0) and (A_1) from the estimates on the cardinality of lattice points given by Theorem 1.15. Let $\pi_{dN_1} : \Gamma \to \Gamma(dN_1)$ denotes the factor map. It follows from the strong approximation property that $\gamma_0\Gamma_q$ surjects onto $\Gamma \to \Gamma(dN_1)$ under π_{dN_1}. We set $B_T = \{h \in \mathsf{G}(\mathbb{R}) : \|h\| \le T\}$. By Theorem 1.15, for every d coprime to q and $\delta \in \Gamma/\Gamma_q(dN_1)$, we have $\Gamma_q(dN_1) = \Gamma(qdN)$ and

$$\begin{aligned}
|\delta\Gamma_q(dN_1)\cap B_T| &= \frac{\mathrm{vol}(B_T)}{[\Gamma:\Gamma(qdN)]} + O_\epsilon\left(\mathrm{vol}(B_T)^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}\right)\\
&= \frac{\mathrm{vol}(B_T)}{[\Gamma:\Gamma(dN_1)]\cdot[\Gamma:\Gamma_q]} + O_\epsilon\left(\mathrm{vol}(B_T)^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}\right)\\
&= \frac{|\gamma_0\Gamma_q\cap B_T|}{[\Gamma:\Gamma(dN_1)]} + O_\epsilon\left(\mathrm{vol}(B_T)^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}\right)\\
&= \frac{X}{|\mathsf{G}(\mathbb{Z}/(dN_1))|} + O_\epsilon\left(X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}\right)
\end{aligned}$$

for every $\epsilon > 0$. We note that for d coprime to q, we have $f(\gamma) = 0 \bmod d$ if and only if $g(\gamma) = 0 \bmod dN_1$. Restricting the sums below to d coprime to q, we have

$$\begin{aligned}
\sum_{k=0 \bmod d} a_k &= |\{\gamma\in\gamma_0\Gamma_q\cap B_T;\ f(\gamma)=0 \bmod d\}|\\
&= \sum_{\delta\in\pi_{dN_1}(\gamma_0\Gamma_q): g(\delta)=0 \bmod dN_1} |\delta\Gamma_q(dN_1)\cap B_T|\\
&= |\mathsf{G}(\mathbb{Z}/(dN_1))\cap\{g=0\}|\cdot\left(\frac{X}{|\mathsf{G}(\mathbb{Z}/(dN_1))|} + O_\epsilon\left(X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}\right)\right)\\
&= \frac{\rho(d)}{d}X + O_\epsilon\left(|\mathsf{G}(\mathbb{Z}/(dN_1))\cap\{g=0\}| X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon}\right),
\end{aligned}$$

where

$$\rho(d) = \frac{d|\mathsf{G}(\mathbb{Z}/(dN_1))\cap\{g=0\}|}{|\mathsf{G}(\mathbb{Z}/(dN_1))|}.$$

As in [NS, Sec. 4.1], we deduce that ρ is multiplicative function, (5.3) holds, and

$$\rho(p) = t(f) + O_f(p^{-1/2}). \tag{5.6}$$

Using that

$$|\mathsf{G}(\mathbb{Z}/(dN_1)) \cap \{g = 0\}| \ll d^{\dim(\mathsf{G})-1},$$

we obtain

$$\begin{aligned}\sideset{}{'}\sum_{d \le X^\tau} |R_d| &\ll_\epsilon \sum_{d\le X^\tau} d^{\dim(\mathsf{G})-1} X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon} \\ &\ll (X^\tau)^{\dim(\mathsf{G})} X^{1-(2n_e(p))^{-1}(1+\dim(\mathsf{G}))^{-1}+\epsilon} \ll X^{1-\zeta}\end{aligned}$$

for some $\zeta > 0$, provided that $\tau < \tau_0 = (2n_e(p))^{-1}\dim(\mathsf{G})^{-1}(1+\dim(\mathsf{G}))^{-1}$. This concludes the proof of (A_0) and (A_1).

To prove (A_2), we observe that it follows from (5.6) (see [MV, Th. 2.7(b)]) that

$$-c_3 \le \sum_{z\le p\le w} \frac{\rho(p)\log p}{p} - t(f)\log\frac{z}{w} \le c_3$$

for some $c_3 > 0$. This implies the upper estimate in (5.4). The lower estimate with $l = O(\log\log q)$ follows from Lemma 5.2 below.

Now it follows from (5.5) that

$$S(T, q, X^{\tau/s}) \gg \frac{X}{(\log X)^{t(f)}}\left(C_1 - C_2'(\log\log q)\frac{(\log\log X)^{3t(f)+2}}{\log X}\right). \tag{5.7}$$

Here we used that $W(z) \gg (\log z)^{-t(f)}$, which follows from (5.6).

We apply (5.7) with $T = q^\sigma$ with $\sigma > \sigma_0$ and sufficiently large q. Then by Theorem 1.3, Lemma 2.2, and (2.6),

$$X = |\gamma_0\Gamma_q \cap B_T| \gg_\sigma \frac{\mathrm{vol}(B_T)}{|\Gamma : \Gamma_q|} \gg_{\alpha'} q^{\alpha'\sigma-\dim(\mathsf{G})} \tag{5.8}$$

with $\alpha' < \alpha$. Hence, for sufficiently large q,

$$S(T, q, X^{\tau/s}) \gg_{\sigma,\alpha'} \frac{X}{(\log X)^{t(f)}} \tag{5.9}$$

We note that every point γ which is counted in $S(T, q, X^{\tau/s})$ satisfies conclusion (ii) of the theorem, and

$$|f(\gamma)| \ll T^{\deg(f)} = q^{\sigma\deg(f)},$$

and every prime p which is coprime to q and divides $f(\gamma)$ must satisfy

$$p > X^{\tau/s} \gg_{\sigma,\alpha'} q^{(\alpha'\sigma-\dim(\mathsf{G}))\tau s^{-1}}.$$

Hence, the number of such prime factors is bounded from above by

$$\frac{\sigma\deg(f)}{(\alpha'\sigma-\dim(\mathsf{G}))\tau s^{-1}}$$

provided that q is sufficiently large. Moreover, since b and q are coprime, $f(\gamma)$ is not divisible by any prime which divides q. Hence, the number of prime factors of $f(\gamma)$ (with multiplicities) is bounded by

$$r > \frac{\sigma\deg(f)}{(\alpha\sigma-\dim(\mathsf{G}))\tau_0(9t(f))^{-1}} = \frac{9\alpha\sigma}{\alpha\sigma-\dim(\mathsf{G})}\cdot\sigma_0\cdot t(f)\deg(f).$$

Hence, every γ counted in $S(T,q,X^{\tau/s})$ satisfies conclusion (i) of the theorem as well.

We have shown that every γ counted in $S(T,q,X^{\tau/s})$ satisfies (i) and (ii). Since it follows from (5.8) and (5.9) that $S(T,q,X^{\tau/s})\ge 1$ for sufficiently large q, this completes the proof. □

In order to complete the proof of the theorem, we need to show the following

Lemma 5.2. *$\sum_{p|q}\frac{\log p}{p}=O(\log\log q)$.*

Proof. It is sufficient to prove the claim for square-free q. Moreover, since the function $p\mapsto(\log p)/p-c_1\log(p+c_2)$, $c_1,c_2>0$, is decreasing for $p\ge 3$, it remains to verify the estimate when q is product of all consecutive primes less than z. In this case, by [MV, Th. 2.7(b)],

$$\sum_{p|q}\frac{\log p}{p}=O(\log z),$$

and by the Prime Number Theorem,

$$\log q=\sum_{p\le z}\log p\sim z,$$

which implies the claim. □

We note the proof of Theorem 5.1 not only implies existence of solutions for congruences, but also gives the following quantitative estimate.

Theorem 5.3. *Under the notation of Theorem 5.1,*

$$\left|\left\{x\in\mathsf{G}(\mathbb{Z}):\ \begin{array}{l} f(x)\ \textit{has at most } r \textit{ prime factors}\\ f(x)=b \bmod q\ \textit{and } \|x\|\le q^{\sigma}\end{array}\right\}\right|\gg_\sigma \frac{1}{|f(\mathsf{G}(\mathbb{Z}))\bmod q|}\cdot\frac{N_{q^\sigma}(\mathsf{G}(\mathbb{Z}))}{(\log q)^{t(f)}}$$

for sufficiently large q.

Proof. Since by (5.8) and Theorem 1.15, for every $\gamma_0\in\Gamma_q$ and sufficiently large q,

$$X=|\gamma_0\Gamma_q\cap B_{q^\sigma}|\gg_\sigma\frac{N_{q^\sigma}(\mathsf{G}(\mathbb{Z}))}{|\Gamma:\Gamma_q|},$$

the claim of the theorem follows from (5.9) by summing over $\gamma_0\in\Gamma/\Gamma_q$ such that $f(\gamma)=b\bmod q$. □

Proof of Theorem 1.13. If G is anisotropic over $\mathbb{R}$, then Γ is finite. Hence, we may assume that G is isotropic. We apply Theorem 5.1 with the function $\tilde f:\mathsf{G}\to\mathbb{C}$ given by $\tilde f(g)=f(gv)$. Since $\|\gamma v\|\ll\|\gamma\|$, the claim of Theorem 1.13 follows. □

Proof of Theorem 1.14. We apply the argument of the proof of Theorem 5.1 with the sets $S_T\subset\mathsf{G}(\mathbb{R})$ introduced in Section 4 (in place of the sets B_T) and the polynomial function $\tilde f$ on G defined by $\tilde f(x)=f(xv)$. Using the estimate on $|\delta\Gamma_q(dN_1)\cap S_T|$ provided by Theorem 4.1, this argument carries out with no changes. Let $T=q^\sigma$ with σ as in Theorem 4.4. We conclude from (4.2) that for sufficiently large q,

$$X=|\gamma_0\Gamma_q\cap S_T|\gg_\sigma\frac{\mathrm{vol}(S_T)}{|\Gamma:\Gamma_q|}\gg q^{\alpha\sigma-\dim(\mathsf{G})},\tag{5.10}$$

where α is as in (4.2), and

$$S(T,q,X^{\tau/s})\gg_\sigma\frac{X}{(\log X)^{t(f)}},\tag{5.11}$$

where $\tau < \tau_0 = (2n_e(p))^{-1}\dim(\mathsf{G})^{-1}(1+3\dim(\mathsf{G}))^{-1}$. As in the proof of Theorem 5.1, we conclude that every γ counted in $S(T,q,X^{\tau/s})$ is a product of at most r factors, where

$$
\begin{aligned}
r &> \frac{\sigma\deg(f)}{(\alpha\sigma-\dim(\mathsf{G}))\tau_0(9t(f))^{-1}} \\
&= \frac{\sigma}{\alpha\sigma-\dim(\mathsf{G})}9t(f)\deg(f)\dim(\mathsf{G})(1+3\dim(\mathsf{G}))2n_e(p).
\end{aligned} \tag{5.12}
$$

Now using (5.10) and (5.11), for every $\gamma_0\in\Gamma$ such that $f(\gamma_0 v) = b \bmod q$, we have the estimate

$$
\begin{aligned}
&|\{\gamma\in\gamma_0\Gamma_q\cap S_T:\ f(x)\text{ has at most } r \text{ prime factors}\}| \\
&\qquad\gg_\sigma \frac{X}{(\log X)^{t(f)}} \gg_\sigma \frac{1}{|\Gamma:\Gamma_q|}\cdot\frac{|\Gamma\cap S_T|}{(\log q)^{t(f)}}.
\end{aligned}
$$

Since every $\gamma\in\gamma_0\Gamma_q$ satisfies $f(\gamma v)=b\bmod q$, we conclude that

$$
\begin{aligned}
&\left|\left\{\gamma\in\Gamma\cap S_T:\ \begin{array}{l} f(\gamma v)\text{ has at most } r \text{ prime factors} \\ f(\gamma v)=b\bmod q\end{array}\right\}\right| \\
\gg_\sigma\ & \frac{|\{\gamma\in\Gamma/\Gamma_q:\ f(\gamma v)=b\bmod q\}|}{|\Gamma:\Gamma_q|}\cdot\frac{|\Gamma\cap S_T|}{(\log q)^{t(f)}} \\
=\ & \frac{1}{|f(\mathcal{O})\bmod q|}\cdot\frac{|\Gamma\cap S_T|}{(\log q)^{t(f)}}.
\end{aligned}
$$

Since the cardinality of the fibers of the map $\pi:\Gamma\cap S_T\to\mathcal{O}(T):\gamma\mapsto\gamma v$ is uniformly bounded, we conclude that

$$
\left|\left\{x\in\mathcal{O}(T):\ \begin{array}{l} f(x)\text{ has at most } r \text{ prime factors} \\ f(x)=b\bmod a\end{array}\right\}\right| \gg_\sigma \frac{1}{|f(\mathcal{O})\bmod q|}\cdot\frac{|\Gamma\cap S_T|}{(\log q)^{t(f)}},
$$

which implies the theorem because of (4.1) and (4.3). □

References

[BO] Y. Benoist and H. Oh, Effective equidistribution of S-integral points on symmetric varieties. ArXiv:0706.1621.

[BHC] A. Borel and Harish-Chandra, Arithmetic subgroups of algebraic groups. Ann. of Math. 75 (1962), 485–535.

[BD] M. Borovoi and C. Demarche, Manin obstruction to strong approximation for homogeneous spaces. ArXiv:0912.0408v1.

[BGS] J. Bourgain, A. Gamburd, and P. Sarnak, Affine linear sieve, expanders, and sum-product. Invent. Math. 179 (2010), no. 3, 559–644.

[B] T.Browning, Quantitative arithmetic of projective varieties. Progress in Mathematics, 277. Birkhäuser Verlag, Basel, 2009.

[BHBS] T. Browning, D. R. Heath-Brown, P. Salberger, Counting rational points on algebraic varieties. Duke Math. J. 132 (2006), no. 3, 545–578.

[BS] M. Burger and P. Sarnak, Ramanujan duals. II. Invent. Math. 106 (1991), no. 1, 1–11.

[CTX] J.-L. Colliot-Thélène and F. Xu, Brauer–Manin obstruction for integral points of homogeneous spaces and representation by integral quadratic forms. Compositio Math. 145 (2009), no. 2, 309–363.

[CW] G. Cooke and P. Weinberger, On the construction of division chains in algebraic number rings, with applications to SL_2. Comm. Algebra 3 (1975), 481–524.

[C] L. Clozel, Démonstration de la conjecture τ. Invent. Math. 151 (2003), no. 2, 297–328.

[D] J. Dieudonné, La géométrie des groupes classiques. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[DRS] W. Duke, Z. Rudnick, P. Sarnak, Density of integer points on affine homogeneous varieties. Duke Math. J. 71 (1993), no. 1, 143–179.

[EM] A. Eskin and C. McMullen, Mixing, counting, and equidistribution in Lie groups. Duke Math. J. 71 (1993), no. 1, 181–209.

[FI] R. Fossum and B. Iversen, On Picard groups of algebraic fibre spaces. J. Pure Appl. Algebra 3 (1973), 269–280.

[GL] S. Ghorpade and G. Lachaud, Étale cohomology, Lefschetz theorems and number of points of singular varieties over finite fields. Moscow Mathematical Journal 2, no. 3, 589–631, 2002.

[GN1] A. Gorodnik and A. Nevo, The ergodic theory of lattice subgroups. Annals of Mathematics Studies, 172. Princeton University Press, Princeton, NJ, 2010.

[GN2] A. Gorodnik and A. Nevo, Counting lattice points. To appear in J. Reine Angew. Math. ArXiv:0903.1515.

[GOS] A. Gorodnik, H. Oh and N. Shah, Integral points on symmetric varieties and Satake compactifications. Amer. J. Math. 131 (2009), no. 1, 1–57.

[GW] A. Gorodnik and B. Weiss, Distribution of lattice orbits on homogeneous varieties. Geom. Funct. Anal. 17 (2007), no. 1, 58–115.

[HR] H. Halberstam and H. Richert, Sieve methods. Academic Press, 1974.

[Hr] D. Harari, Le défaut d'approximation forte pour les groupes algébriques commutatifs. Algebra Number Theory 2 (2008), no. 5, 595–611.

[Ht] R. Hartshorne, Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[HB1] D. R. Heath-Brown, The density of rational points on cubic surfaces. Acta Arith. 79 (1997), no. 1, 17–30.

[HB2] D. R. Heath-Brown, The density of rational points on curves and surfaces. Ann. of Math. (2) 155 (2002), no. 2, 553–595.

[HB3] D. R. Heath-Brown, Counting rational points on algebraic varieties. Analytic number theory, 51–95, Lecture Notes in Math., 1891, Springer, Berlin, 2006.

[HS] G. Heckman and H. Schlichtkrull, Harmonic analysis and special functions on symmetric spaces, Perspectives in Mathematics, 16. Academic Press, Inc., 1994.

[KKV] F. Knop, H. Kraft, T. Vust, The Picard group of a G-variety. Algebraische Transformationsgruppen und Invariantentheorie, 77–87, DMV Sem., 13, Birkhäuser, Basel, 1989.

[LW] S. Lang and A. Weil, Number of points of varieties in finite fields. Amer. J. Math. 76, (1954) 819–827.

[Li] J.-S. Li, The minimal decay of matrix coefficients for classical groups. Harmonic analysis in China, 146–169, Math. Appl., 327, Kluwer Acad. Publ., Dordrecht, 1995.

[L1] Y. Linnik, On the least prime in an arithmetic progression. I. The basic theorem. Rec. Math. [Mat. Sbornik] N.S. 15(57) (1944), 139–178.

[L2] Y. Linnik, On the least prime in an arithmetic progression. II. The Deuring-Heilbronn phenomenon. Rec. Math. [Mat. Sbornik] N.S. 15(57) (1944), 347–368.

[LS] J. Liu and P. Sarnak, Almost primes on quadrics in 3 variables. Preprint.

[M] F. Maucourant, Homogeneous asymptotic limits of Haar measures of semisimple linear groups and their lattices. Duke Math. J. 136 (2007), no. 2, 357–399.

[MV] H. Montgomery and R. Vaughan, Multiplicative number theory. I. Classical theory. Cambridge Studies in Advanced Mathematics, 97. Cambridge University Press, Cambridge, 2007.

[N] W. Narkiewicz, Units in residue classes. Arch. Math. (Basel) 51 (1988), no. 3, 238–241.

[NS] A. Nevo and P. Sarnak, Prime and Almost Prime Integral Points on Principal Homogeneous Spaces. ArXiv:0902.0692v1.

[Od] R. W. K. Odoni, A proof by classical methods of a result of Ax on polynomial congruences modulo a prime. Bull. London Math. Soc. 11 (1979), no. 1, 55–58.

[Oh] H. Oh, Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants, Duke Math. J. 113 (2002), 133–192.

[Om] T. O'Meara, Introduction to quadratic forms. Classics in Mathematics. Springer-Verlag, Berlin, 2000.

[PR] V. Platonov and A. Rapinchuk, Algebraic groups and number theory. Pure and Applied Mathematics, 139. Academic Press, Inc., Boston, MA, 1994.

[P] V. L. Popov, Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector fiberings. Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 294–322.

[V] V. E. Voskresenski, Algebraic groups and their birational invariants. Translations of Mathematical Monographs, 179. American Mathematical Society, Providence, RI, 1998.

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