

Diophantine approximation exponents on homogeneous varieties

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ABSTRACT. Recent years have seen very important developments at the interface of Diophantine approximation and homogeneous dynamics. In the first part of the paper we give a brief exposition of a dictionary developed by Dani and Kleinbock-Margulis which relates Diophantine properties of vectors to distribution of orbits of flows on the space of unimodular lattices. In the second part of the paper we briefly describe an extension of this dictionary recently developed by the authors, which establishes an analogous dynamical correspondence for general lattice orbits on homogeneous spaces. We concentrate specifically on the problem of estimating exponents of Diophantine approximation by arithmetic lattices acting on algebraic varieties. In the third part of the paper, we exemplify our results by establishing explicit bounds for the Diophantine exponent of dense lattice orbits in a number of basic cases. These include the linear and affine actions on affine spaces, and the action on the variety of matrices of fixed determinant. In some cases, these exponents are shown to be best possible.

To S.G. Dani on the occasion of his 65th birthday.

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1. Introduction

The theory of Diophantine approximation has many deep and fruitful connections with dynamical properties of flows on homogeneous spaces. This connection

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has provided many fundamental new insights enriching both fields. For instance, E. Artin [1] used continued fractions to construct dense geodesics on the modular surface, and it was observed by C. Series [45] that classical continued fraction expansions can be constructed as cutting sequences for orbits of the geodesic flow on the modular surface.

This remarkable connection between Diophantine approximation and dynamics also exists in higher dimensions. It was realized by S.G. Dani [11] that Diophantine properties of vectors in Euclidean space can be encoded by orbits of a suitable one-parameter flow on the space of unimodular lattices. In particular, he showed that badly approximable vectors correspond to bounded orbits, and singular vectors correspond to divergent orbits. This work has inspired many subsequent investigations exploring properties of flows on homogeneous spaces, and the techniques developed gave rise to the solution of several longstanding open problems in number theory. For instance one can mention such notable advances as the solution of Sprindzhuk's conjecture in the theory of Diophantine approximation with dependent quantities by D. Kleinbock and G. Margulis [23] and the computation of the Hausdorff dimension of the set of singular vectors in \mathbb{R}^2 by Y. Cheung [7].

In this paper, we first discuss the correspondence between Diophantine properties of vectors and recurrence properties of flows developed by Dani, Kleinbock and Margulis. We then put these results in the context of recent works of the authors on Diophantine exponents on homogeneous varieties of semisimple algebraic groups. Finally, we present some new estimates for Diophantine exponents on specific homogeneous varieties, complementing the results of [19]. We thank the anonymous referees for very helpful suggestions.

2. Diophantine approximation and the shrinking target property

2.1. Classical Diophantine approximation. It is a well-known theorem of Dirichlet that given a vector $x \in \mathbb{R}^d$, for every $R > 1$ one can find $m \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ such that

$$(2.1) \quad \left\| x - \frac{m}{n} \right\|_{\infty} \leq n^{-1} R^{-1/d} \quad \text{and} \quad n \leq R.$$

Here $\| \cdot \|_{\infty}$ denotes the maximum norm. Dirichlet introduced his famous pigeonhole or box principle to prove the above theorem. A proof can also be provided using Minkowski's convex body theorem. We refer the reader to [43] for details. In particular, it follows from (2.1) that the inequality

$$(2.2) \quad \left\| x - \frac{m}{n} \right\|_{\infty} \leq n^{-1-1/d}$$

always has a solution with $m \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. Diophantine properties of vectors can also be studied in the context of linear forms, i.e. given $x \in \mathbb{R}^d$ one can study small values of the linear form

$$|m_1 x_1 + \cdots + m_d x_d + n|$$

for $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ and $n \in \mathbb{Z}$. This is referred to as the *linear setting*. These two settings are related by Khintchine's transference principle [43].

Dirichlet's theorem provides us with the first examples of *Diophantine exponents*. Let $x, x_0 \in \mathbb{R}^d$. Following Bugeaud and Laurent [4, 5, 6], we denote by $\omega(x, x_0)$

the supremum of real numbers ω such that there are arbitrarily large real numbers R for which the inequalities

$$(2.3) \quad \|nx - x_0 - m\|_\infty \leq R^{-\omega} \quad \text{and} \quad n \leq R$$

have a nonzero solution $n \in \mathbb{Z}, m \in \mathbb{Z}^d$. We define $\hat{\omega}(x, x_0)$ to be the supremum of ω such that (2.3) has a nonzero integer solution for *all* sufficiently large positive real numbers R . Taking $x_0 = (0, \dots, 0)$ gives the uniform homogeneous Diophantine exponents $\omega(x)$ and $\hat{\omega}(x)$. Then Dirichlet's theorem implies that

$$\omega(x) \geq \hat{\omega}(x) \geq 1/d.$$

The exponents ω and $\hat{\omega}$ are related by various transference principles which are still not completely understood in higher dimensions. We refer the reader to the works of Buegaud and Laurent cited above for details. In this paper, we will study analogues of the exponent $\hat{\omega}$ in the more general context of dense lattice orbits on homogeneous varieties. We begin with an example which is closely related to the above discussion. Let Γ denote a lattice subgroup of the group $\text{Aff}(\mathbb{R}^d)$ of affine transformations of \mathbb{R}^d . We equip $\text{Aff}(\mathbb{R}^d)$ with a norm which is a natural way to measure complexity of elements of Γ . Studying effective density of Γ -orbits amounts to estimating Diophantine exponents which are defined as follows.

DEFINITION 2.1. Assume that for $x, x_0 \in \overline{\Gamma x}$, there exist constants $c = c(x, x_0)$ and $\epsilon_0 = \epsilon_0(x, x_0)$ such that for all $\epsilon < \epsilon_0$, the system of inequalities

$$\|\gamma^{-1}x - x_0\|_\infty \leq \epsilon \quad \text{and} \quad \|\gamma\| \leq c\epsilon^{-\kappa}.$$

have a solution $\gamma \in \Gamma$. Define the Diophantine approximation exponent $\kappa_\Gamma(x, x_0)$ as the infimum of $\kappa > 0$ such that the foregoing inequalities have a solution.

Note that κ_Γ is closely related to the exponent $\hat{\omega}$ above and should be viewed as a generalization of $\hat{\omega}$ to the context of lattice actions on homogeneous varieties. Taking $\Gamma = \mathbb{Z}^\times \ltimes \mathbb{Z}^d$ acting on \mathbb{R}^d , we see that $\kappa_\Gamma(x, x_0)$ is the infimum of $\kappa > 0$ such that the inequalities

$$\|nx - x_0 - m\|_\infty < cR^{-1/\kappa} \quad \text{and} \quad n \leq R$$

has a solution for all sufficiently large R .

We will provide new examples of exponents κ_Γ in §3 based on our work [19]. For now we return to a discussion of dynamical approaches to the study of ω and $\hat{\omega}$, specifically questions regarding vectors in \mathbb{R}^d for which the general estimates (2.1) and (2.2) can - or cannot - be improved; in particular, one wishes to study the size of sets of such vectors. A vector x is called *badly approximable* if there exists $c > 0$ such that the inequality

$$(2.4) \quad \left\| x - \frac{m}{n} \right\|_\infty \leq cn^{-1-1/d}$$

has no solutions $m \in \mathbb{Z}^d$ and $n \in \mathbb{N}$.

At the other extreme, the vector x is called *singular* if for every $c > 0$ and $R \geq R(c)$, the system of inequalities

$$(2.5) \quad \left\| x - \frac{m}{n} \right\|_\infty \leq cn^{-1}R^{-1/d} \quad \text{and} \quad n \leq R$$

has a solution $m \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. In other words, Dirichlet's theorem can be infinitely improved for such vectors. One can show that a vector is badly approximable

(resp. singular) if and only if it also has the analogous property in the sense of the linear setting.

We note that a number $x \in \mathbb{R}$ is badly approximable if and only if its continued fraction expansion has bounded digits, and a number $x \in \mathbb{R}$ is singular if and only if it is rational. These properties are more difficult to characterise in higher dimensions, but it turns out they have a very convenient interpretation based on dynamics of certain flows on the space of unimodular lattices. Denote by \mathcal{L}_{d+1} the space of unimodular lattices in \mathbb{R}^{d+1} , which can be identified with the homogeneous space $\mathrm{SL}_{d+1}(\mathbb{Z}) \backslash \mathrm{SL}_{d+1}(\mathbb{R})$. It has an invariant probability measure as well as a metric, inherited from a left invariant metric on $\mathrm{SL}_{d+1}(\mathbb{R})$. The quotient is non-compact and its compact subsets are described by Mahler's compactness criterion. For $x \in \mathbb{R}^d$ we define the lattice

$$\Lambda_x := \{(n, m - nx) : m \in \mathbb{Z}^d, n \in \mathbb{Z}\} \in \mathcal{L}_{d+1}.$$

We consider the action on \mathcal{L}_{d+1} by the one-parameter subgroups

$$g_t := \mathrm{diag}(e^{-t}, e^{t/d}, \dots, e^{t/d}).$$

The following results follow from the work of Dani:

PROPOSITION 2.2 (Dani [11]). *With notation as above,*

- (i) $x \in \mathbb{R}^d$ is badly approximable if and only if the semiorbit $\Lambda_x g_t$, $t > 0$, is bounded in \mathcal{L}_{d+1} .
- (ii) $x \in \mathbb{R}^d$ is singular if and only if the semiorbit $\Lambda_x g_t$, $t > 0$, is divergent, i.e. leaves every compact set in \mathcal{L}_{d+1} .

The idea of the proof of Proposition 2.2 is based on the observation that if the lattice $\Lambda_x g_t$ contains a small non-zero vector, this gives a solution of the relevant Diophantine inequalities. Indeed, let

$$(2.6) \quad \Omega(\delta) := \{\Lambda \in \mathcal{L}_{d+1} : \exists z \in \Lambda - \{0\} : \|z\|_\infty < \delta\}.$$

Then if for some $t > 0$, we have $\Lambda_x g_t \in \Omega(\delta)$ with $\delta \in (0, 1)$, the system of inequalities

$$(2.7) \quad \left\| x - \frac{m}{n} \right\|_\infty < \delta e^{-t/d} \quad \text{and} \quad n < \delta e^t$$

has a solution $m \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. By Mahler's compactness criterion, the family of sets $\Omega(\delta)$ form a basis of neighborhoods of infinity of \mathcal{L}_{d+1} . Thus, Diophantine properties of the vector x are determined by visits of the semiorbit $\Lambda_x g_t$ to neighborhoods of infinity.

2.2. Schmidt's game and bounded orbits. The points whose g_t semiorbits are bounded (resp. divergent) form a set of measure zero. Nevertheless, the former are quite abundant. In [10, 11] Dani used the above correspondence (i.e. Proposition 2.2 (i)), along with Schmidt's results on his game to show that bounded orbits for certain partially hyperbolic flows on homogeneous spaces have full Hausdorff dimension. Schmidt's game was introduced in [42] and is played in a complete metric space X . Two players, Bob and Alice start with a subset $W \subseteq X$, and two parameters $0 < \alpha, \beta < 1$. The game consists of choosing a sequence of nested closed balls. Bob begins by choosing a ball B_0 and Alice continues by choosing A_0 and so on.

$$B_0 \supset A_0 \supset B_1 \supset A_1 \supset \dots$$

The radii of the balls are related by the parameters α and β :

$$r(A_i) = \alpha r(B_i) \quad \text{and} \quad r(B_{i+1}) = \beta r(A_i).$$

Alice wins this game if $\bigcap_n B_n$ intersects W . The set W is called (α, β) -winning if Alice can find a winning strategy, α -winning if it is (α, β) -winning for all $0 < \beta < 1$ and winning if it is α -winning for some $\alpha > 0$.

Schmidt games have many nice properties. Most prominently, a winning subset of \mathbb{R}^d is *thick*, i.e. the intersection of a winning set with every open set in \mathbb{R}^d has Hausdorff dimension d . Schmidt showed that badly approximable vectors form an α -winning set for $0 < \alpha \leq 1/2$. Subsequent to Dani's introduction of Schmidt's game in homogeneous dynamics, a general conjecture on abundance of bounded orbits was formulated by Margulis [37] in his Kyoto ICM address, generalizing Dani's results. The conjecture was proved in stages by Kleinbock and Margulis [25] and Kleinbock and Weiss [26]. There has been intense activity in this subject, and Schmidt games and their variations have been used to prove a wide variety of results. Recently in [13], Dani and H. Shah have introduced a new topological variant of Schmidt's game.

2.3. Dani's correspondence and Khintchine's theorem. Proposition 2.2 and Proposition 2.4 below are examples of what Kleinbock and Margulis have termed the *Dani correspondence*. In their paper [24], they further developed this correspondence to handle inequalities of the form

$$(2.8) \quad \left\| x - \frac{m}{n} \right\|_{\infty} \leq n^{-1} \psi(n),$$

where ψ is a general nonincreasing function, and to obtain a new proof of Khintchine's theorem¹ using homogeneous dynamics. Recall that Khintchine's theorem states that the set of $x \in \mathbb{R}^d$ for which inequality (2.8) has infinitely many solutions $m \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ has zero (resp. full) measure depending on the convergence (resp. divergence) of the sum

$$\sum_{n=1}^{\infty} \psi(n)^d.$$

As before, the original question in Diophantine approximation is transferred to a problem about visits of the semiorbit $\Lambda_x g_t$ to a family of shrinking neighbourhoods of infinity. The rate at which these neighbourhoods shrink is determined by the decay rate of the function ψ . Kleinbock and Margulis then use the exponential mixing property of the flow g_t in conjunction with a very general and quantitative form of the Borel–Cantelli lemma due to Sprindzhuk, to establish a zero-one law and deduce the Khintchine-Groshev theorem as a corollary. As another corollary they also obtain logarithm laws for geodesic excursions to shrinking neighbourhoods of cusps of locally symmetric spaces, thereby generalising Sullivan's logarithm law. Further, they established versions of zero-one laws for multi-parameter actions, thereby confirming, in stronger form, a conjecture of Skriganov in the geometry of numbers.

¹In fact, their paper deals with systems of linear forms, i.e. the Khintchine-Groshev theorem.

2.4. Diophantine approximation on manifolds. One says that the vector $x \in \mathbb{R}^d$ is *very well approximable* if there exists $\epsilon > 0$ such that the inequality

$$\left\| x - \frac{m}{n} \right\|_{\infty} \leq m^{-1-1/d-\epsilon}$$

has infinitely many solutions $m \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. The exponent in (2.2) is the best possible, and it follows from the Borel-Cantelli lemma that the set of $x \in \mathbb{R}^d$ which are very well approximable has zero Lebesgue measure. Note that the above definition and the results discussed in this section pertain to the exponent ω . It is easy to see that $x \in \mathbb{R}^d$ is very well approximable if and only if $\omega(x) = d$. The subject of metric Diophantine approximation on manifolds seeks to study the extent to which generic Diophantine properties are inherited by proper submanifolds of \mathbb{R}^d , in particular inheritance of values of Diophantine exponents as above. In 1932, Mahler's investigations into the classification of numbers according to their Diophantine properties, led him to conjecture that almost every point on the curve

$$(x, x^2, \dots, x^d)$$

is not very well approximable. Mahler's conjecture was resolved by Sprindzhuk who in turn conjectured a more general form of his theorem. Let M be a k -dimensional submanifold of \mathbb{R}^d parametrised by a C^l -map $f : U \rightarrow M$. We say that M is nondegenerate if for almost every $x \in U$, the space spanned by the partial derivatives of f up to order l coincides with \mathbb{R}^d . The following was a long standing conjecture of Sprindzhuk (in the case of analytic manifolds) and proved by Kleinbock and Margulis²:

THEOREM 2.3 (Kleinbock–Margulis [23]). *Almost every point on a nondegenerate submanifold $M \subset \mathbb{R}^d$ is not very well approximable.*

The proof once more follows a dynamical route. Namely, it was observed by Kleinbock and Margulis that the property of being very well approximable also has a convenient interpretation in terms of dynamics on the space of unimodular lattices:

PROPOSITION 2.4 (Kleinbock–Margulis [23]). *A vector $x \in \mathbb{R}^d$ is very well approximable if and only if there exists $\alpha > 0$ such that $\Lambda_x g_{t_i} \in \Omega(e^{-\alpha t_i})$ for a sequence $t_i \rightarrow \infty$.*

Hence, in order to improve the exponent of Diophantine approximation for a vector $x \in \mathbb{R}^d$, one needs to establish that the semiorbit $\Lambda_x g_t$ visits the sequence of exponentially shrinking sets $\Omega(e^{-\alpha t})$ infinitely often. This is a common feature of many chaotic dynamical systems, and is usually called the *shrinking target property*. With the help of Proposition 2.4, the proof of Theorem 2.3 reduces to an analysis of visits of translated submanifolds $\Lambda_{f(U)} g_t$ to the neighbourhoods $\Omega(e^{-\alpha t})$. The crucial and the most difficult part of the argument is an explicit estimate on the measure of the set of $x \in U$ such that $\Lambda_{f(x)} g_t \in \Omega(e^{-\alpha t})$. This estimate generalises the non-divergence properties of unipotent flows discovered by Margulis in [36] and developed further in quantitative forms by Dani in [10, 12].

The relevant result is stated as follows. Let C, α be positive numbers, B an open subset of \mathbb{R}^k and λ denote Lebesgue measure of appropriate dimension. Say

²In fact, they proved more general *multiplicative* versions of the conjecture.

that a function $f : B \rightarrow \mathbb{R}$ is (C, α) -good on B if for any open ball $J \subset B$ and any $\epsilon > 0$,

$$\lambda(\{x \in J : |f(x)| < \epsilon\}) \leq \left(\frac{\epsilon}{\sup_{x \in J} |f(x)|} \right)^\alpha \lambda(B).$$

The main property of unipotent flows which allows for nondivergence is precisely the (C, α) -good property. Kleinbock and Margulis showed that more generally, smooth nondegenerate maps also have this property. Let $\mathcal{L}(\mathbb{Z}^d)$ be the poset of primitive subgroups of \mathbb{Z}^d . For discrete subgroups Λ we define $\|\Lambda\|$ as the norm of the corresponding vector in a suitable wedge product. Now we state the main estimate in [23], which plays crucial role in the proof of Theorem 2.3:

THEOREM 2.5. *Let an open ball $B(x_0, r_0) \subset \mathbb{R}^k$, $C, \alpha > 0$, $0 < \rho < 1/d$ and a continuous map $h : B(x_0, 3^d r_0) \rightarrow \mathrm{SL}_d(\mathbb{R})$ be given. We assume that for every $\Delta \in \mathcal{L}(\mathbb{Z}^d)$,*

- (1) *the function $x \rightarrow \|h(x)\Delta\|$ is (C, α) -good on $B(x_0, 3^d r_0)$,*
- (2) *$\sup_{x \in B(x_0, r_0)} \|h(x)\Delta\| \geq \rho$.*

Then for every $\epsilon \in (0, \rho]$,

$$\lambda(\{x \in J : \mathbb{Z}^d h(x) \notin \Omega(\epsilon)\}) \leq D(d, k) C \left(\frac{\epsilon}{\rho} \right)^\alpha \lambda(B),$$

where $D(d, k)$ is a constant depending on d and k only.

In subsequent work, [28, 29, 30], D. Kleinbock has further studied the Diophantine exponents ω in the context of affine subspaces and manifolds. In particular, in [30] it is proved that if an analytic manifold contains a point x_0 such that $\omega(x) \geq c$ then $\omega(x) \geq c$ for almost all x in the manifold. We refer the reader to [31] for an overview of further developments.

We note that in the theory of metric Diophantine approximation on manifolds, one is concerned with approximating points on manifolds by *all rational points in the ambient Euclidean space*. We now turn in the next section to discuss a completely different, but equally natural, question. Namely we will consider Diophantine approximation of a general point on a variety intrinsically, by *rational points lying on the variety itself*.

2.5. Intrinsic Diophantine approximation on algebraic varieties. The question of Diophantine approximation on algebraic varieties by rational points on the variety itself was raised already half a century ago by Serge Lang but the results in this direction are still very scarce. On page 14 of his paper [33], Lang asks if homogeneous and inhomogeneous versions of metric Diophantine approximation can be developed for group varieties and homogeneous varieties over number fields. Specifically, he asks for analogues of Khintchine's theorem in this context. In some cases one can deduce that the set $X(\mathbb{Q})$ of rational points on an algebraic variety X is dense in the set $X(\mathbb{R})$ of real points using a rational parametrisation of X (for example, the stereographic projection for the quadratic surfaces), but this approach usually provides poor bounds on Diophantine exponents that depend on the degree of the parametrisation map. More generally, one can consider the problem of Diophantine approximation by the set $X(\mathbb{Z}[1/p])$ of $\mathbb{Z}[1/p]$ -points in X . Here even establishing density is a nontrivial task.

One of the most natural examples of algebraic varieties with rich structure of rational points is given by algebraic groups and their homogeneous spaces. Here

several results regarding quantitative density of rational points have been proved. This includes elliptic curves and abelian varieties [46], general homogeneous spaces of semisimple algebraic groups [17, 18], and quadratic surfaces [14, 44, 32, 16]. In the latter two cases, one can also use dynamical correspondences which relates Diophantine properties of points to shrinking target properties of orbits for the corresponding dynamical systems. Let us now turn to a brief description of these correspondences.

Let X be an algebraic variety in \mathbb{C}^d defined over \mathbb{Q} equipped with an action of a connected almost simple algebraic group $G \subset \mathrm{GL}_d(\mathbb{C})$ defined over \mathbb{Q} . For simplicity of exposition, let us consider here the problem of Diophantine approximation in $X(\mathbb{R})$ by rational points in $X(\mathbb{Z}[1/p])$ where p is prime. A basic observation is that since rational points on X can be parametrized using orbits of the group $G(\mathbb{Z}[1/p])$, they can be studied using techniques from the theory of dynamical systems. The relevant dynamical system here is the space

$$Y = G(\mathbb{Z}[1/p]) \backslash (G(\mathbb{R}) \times G(\mathbb{Q}_p))$$

with the action of the group $G(\mathbb{Q}_p)$ by right multiplication. The following proposition can be viewed as an analogue the classical Dani correspondence described in the previous section and is a special case of results in §5 of [17]. However, we note that the latter is an if and only if statement while the statement below is only in one direction. Moreover, Dani's correspondence is for cuspidal excursions of one parameter groups while the statement below involves the shrinking target problem for larger groups where the target is a point in the space. For a prime p (possibly infinite), we denote by \mathbb{Q}_p the corresponding completion of \mathbb{Q} and define metrics on $X(\mathbb{Q}_p)$ (resp $G(\mathbb{Q}_p)$) as follows:

$$\|x - y\|_p := \max_{1 \leq i \leq d} |x_i - y_i|_p,$$

and denote $\|x - e\|_p$ by $\|x\|_p$.

PROPOSITION 2.6 (Ghosh–Gorodnik–Nevo [17]). *Given $x \in X(\mathbb{R})$, there exists $y_x \in Y$ and a sequence of neighborhoods \mathcal{O}_ϵ of the identity coset in Y such that if*

$$y_x \cdot b \in \mathcal{O}_\epsilon \quad \text{for some } b \in G(\mathbb{Q}_p) \text{ with } \|b\|_p \leq R,$$

then the system of inequalities

$$\left\| x - \frac{m}{n} \right\|_\infty \leq \epsilon \quad \text{and} \quad n \leq c(x)R$$

has a solution $\frac{m}{n} \in X(\mathbb{Z}[1/p])$, with the constant $c(x)$ uniform over x in compact sets.

Proposition 2.6 shows that the problem of Diophantine approximation reduces to the shrinking target problem for the orbit $y_x G(\mathbb{Q}_p)$ with respect to the sequence of neighbourhoods \mathcal{O}_ϵ .

Let us assume that the set $X(\mathbb{Z}[1/p])$ is not empty and the group G is isotropic over \mathbb{Q}_p . Then the closure $\overline{X(\mathbb{Z}[1/p])}$ in $X(\mathbb{R})$ is open and closed in $X(\mathbb{R})$ (see [17]). In particular, when $X(\mathbb{R})$ is connected, $X(\mathbb{Z}[1/p])$ is dense. To measure the quality of Diophantine approximation, we use Diophantine approximation exponents for homogeneous varieties.

DEFINITION 2.7. Assume that for $x \in \overline{X(\mathbb{Z}[1/p])}$, there exist constants $c = c(x)$ and $\epsilon_0 = \epsilon_0(x)$, such that for all $\epsilon < \epsilon_0$, the system of inequalities

$$(2.9) \quad \left\| x - \frac{m}{n} \right\|_\infty \leq \epsilon \quad \text{and} \quad n \leq c\epsilon^{-\kappa}$$

has a solution $\frac{m}{n} \in X(\mathbb{Z}[1/p])$. Define the *Diophantine approximation exponent* $\kappa_p(x)$ as the infimum of $\kappa > 0$ such that the foregoing inequalities have a solution.

We note that the exponent above is related to the exponent κ_Γ from the introduction. A unified notion of exponent for lattices in locally compact groups encompassing both, can be found in [19].

A lower bound on the exponents $\kappa_p(x)$ can be deduced from the following pigeon-hole argument. Let us introduce the growth exponent of the number of rational points

$$a_p(X) = \sup_{\text{compact } K \subset X(\mathbb{R})} \limsup_{R \rightarrow \infty} \frac{\log N_R(K, X(\mathbb{Z}[1/p]))}{\log R}$$

where $N_R(K, X(\mathbb{Z}[1/p]))$ denotes the number of points $\frac{m}{n} \in K \cap X(\mathbb{Z}[1/p])$ such that $n \leq R$. One can show that if the constant $c = c(x)$ is uniform over compact sets in $X(\mathbb{R})$, then

$$(2.10) \quad \kappa_p(X) \geq \frac{\dim(X)}{a_p(X)}.$$

The question about upper bounds for $\kappa_p(X)$ is much deeper. Indeed, any such upper bound would quantify density of $X(\mathbb{Z}[1/p])$ in $\overline{X(\mathbb{Z}[1/p])}$. In view of Proposition 2.6, this question can be answered by studying the problem of establishing quantitative equidistribution for orbits of $G(\mathbb{Q}_p)$ in Y . Let us therefore introduce the family of averaging operators $A_R : L^2(Y) \rightarrow L^2(Y)$ given by

$$A_R \phi(y) := \frac{1}{|B_R|} \int_{B_R} \phi(yb) db$$

where $B_R = \{b \in G(\mathbb{Q}_p) : \|b\|_p \leq R\}$. It was shown in [17] that there exists $\theta > 0$ such that for every $\eta > 0$, there exists $C := C(\eta)$ such that

$$(2.11) \quad \|A_R(\phi) - P(\phi)\| \leq C |B_R|^{-\theta+\eta} \|\phi\|_2,$$

for every $R \geq R_\eta$ where P is an explicit projection operator on $L^2(Y)$. Let θ_p denote the supremum over θ 's for which the estimate (2.11) holds. This parameter provides the crucial input to deduce the following upper bound on the Diophantine exponent:

THEOREM 2.8 (Ghosh, Gorodnik, Nevo [17]). *With notation as above,*

(i) *For almost every $x \in \overline{X(\mathbb{Z}[1/p])}$,*

$$\kappa_p(x) \leq (2\theta_p)^{-1} \frac{\dim(X)}{a_p(G)}.$$

(ii) *For every $x \in \overline{X(\mathbb{Z}[1/p])}$,*

$$\kappa_p(x) \leq \theta_p^{-1} \frac{\dim(X)}{a_p(G)}.$$

Moreover, the constant $c = c(x)$ in (2.9) is uniform over x in compact sets.

Typically, $a_p(G) \geq a_p(X)$, and in that case, it follows that $\theta_p = 1/2$, the bound in Theorem 2.8(i) matches with the lower bound (2.10), so that it is best possible. For instance, Theorem 2.8 gives a sharp bound for Diophantine approximation by $\mathbb{Z}[1/p]$ -points on the two-dimensional sphere. We note that the exponent θ_p is closely related to the integrability exponents of automorphic representations, which has been extensively studied in relation to the generalised Ramanujan conjectures, and explicit estimates on θ_p are available in a number of cases (see [41, 2] for a detailed account).

A fruitful approach to the problem of Diophantine approximation by the set of all rational points on nonsingular quadratic surfaces has been developed in [14, 32, 16]. In particular, it turns out that the problem of Diophantine approximation on the sphere is related to a shrinking target problem for a suitable one-parameter flow g_t on the space $Y = \Gamma \backslash G$ where $G = \mathrm{SO}(d, 1)$ and Γ is a subgroup of integral matrices in G . We note that Y can be naturally embedded in the space of unimodular lattices \mathcal{L}_{d+1} and one can set $\Omega_Y(\delta) = Y \cap \Omega(\delta)$ where $\Omega(\delta)$ is defined in (2.6). Given a vector x on the d dimensional unit sphere S^d , one can associate a point $y_x \in Y$ such that the following dynamical correspondence holds:

PROPOSITION 2.9 (Kleinbock–Merrill [32]). *Suppose that there exists $t > 0$ such that $y_x g_t \in \Omega_Y(\delta)$. Then the system of inequalities*

$$\left\| x - \frac{m}{n} \right\|_{\infty} < \frac{2\delta^{1/2} e^{-t/2}}{n^{1/2}} \quad \text{and} \quad n < \delta e^t$$

has a solution with $\frac{m}{n} \in S^d$.

For instance, Proposition 2.9 is used to prove the following analogue of the classical Dirichlet theorem (cf. (2.1) in the case of the unit spheres:

THEOREM 2.10 (Kleinbock–Merrill [32]). *There exists $c > 0$ such that for every $x \in S^d$ and $R > 1$, the system of inequalities*

$$\left\| x - \frac{m}{n} \right\|_{\infty} \leq \frac{c}{n^{1/2} R^{1/2}} \quad \text{and} \quad n \leq R$$

has a solution with $\frac{m}{n} \in S^d$.

We also mention that this approach also allows to prove an analogue of the Khinchin–Groshev theorem [32] and to study the set of badly approximable vectors [32] and well approximable vectors [14] in the context of the intrinsic Diophantine approximation.

3. Diophantine exponents for group actions

The problem of establishing Diophantine exponents discussed in the previous sections is an instance of a much more general problem, namely establishing a rate of distribution for dense lattice orbits on homogeneous spaces of the ambient group. In [19] we developed a general approach to establishing quantitative density of orbits which is based on a *duality principle* combined with a *quantitative mean ergodic theorem*. Then we establish a dynamical correspondence similar to the correspondences discussed in the previous sections. More explicitly, the duality principle implies that quantitative density of lattice orbits Γx in the homogeneous space Gx is equivalent to the quantitative density of the corresponding orbits of H_x in the space $Y = \Gamma \backslash G$, where H_x denotes the stabilizer of x in G . On a larger

note, it would be interesting to study analogues of the various results discussed in §2 in the context of dense lattice orbits, for instance notions of badly approximable points, analogues of Schmidt's game etc. We do not discuss these themes. Indeed, this theory is in its infancy and we are not aware of any results in this direction, other than those mentioned in §2.

To motivate the discussion by a concrete example, consider a case where the approach mentioned above works especially well, namely when Γ is a lattice in an algebraic subgroup G of the group of affine transformations. In classical inhomogeneous Diophantine approximation, one is interested in minimizing the quantity $|nx - m + x_0|$ for given $x, x_0 \in \mathbb{R}^d$, as n varies over \mathbb{Z} and m varies over \mathbb{Z}^d . This problem can be interpreted as establishing quantitative density for orbits of the semigroup $\Gamma = \mathbb{Z}^\times \times \mathbb{Z}^d$ acting on \mathbb{R}^d . There has been considerable recent interest in the study of density of orbits for more general groups of affine transformations. One example is $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acting on the punctured plane, where the Γ -orbit of a point with irrational slope is dense. In [34, 35], Laurent and Nogueira have studied effective versions of this density. In [39], Maucourant and Weiss have used effective equidistribution results for horocycle flows to obtain effective results for dense Γ -orbits on the plane, where Γ is an arbitrary lattice in $\mathrm{SL}_2(\mathbb{R})$.

In our discussion below Γ will denote for instance a discrete subgroup of the group $\mathrm{Aff}(\mathbb{R}^d)$ of affine transformations of \mathbb{R}^d . We recall the definition of Diophantine exponent for Γ from the introduction.

DEFINITION 3.1. Assume that for $x, x_0 \in \overline{\Gamma x}$, there exist constants $c = c(x, x_0)$ and $\epsilon_0 = \epsilon_0(x, x_0)$ such that for all $\epsilon < \epsilon_0$, the system of inequalities

$$\|\gamma^{-1}x - x_0\|_\infty \leq \epsilon \quad \text{and} \quad \|\gamma\| \leq c\epsilon^{-\kappa}.$$

has a solution $\gamma \in \Gamma$. Define the Diophantine approximation exponent $\kappa_\Gamma(x, x_0)$ as the infimum of $\kappa > 0$ such that the foregoing inequalities have a solution.

We note that the exponent defined above generalizes the Diophantine exponent for *uniform* approximation by $\mathrm{SL}_2(\mathbb{Z})$ -orbits in \mathbb{R}^2 as considered by Laurent and Nogueira [34]. In fact, the definition above makes sense for general homogeneous varieties of semisimple groups X equipped with a distance dist as in §2. In the next section, we present a wide variety of new estimates for Diophantine exponents.

The exponent κ_Γ above is naturally related to the exponent κ_p discussed in the previous section (which is associated with dense orbits of the group $\Gamma = G(\mathbb{Z}[\frac{1}{p}])$).

A basic geometric argument leads to a lower bound on the Diophantine exponent for dense lattice orbits on homogeneous varieties. Let us define the growth exponent of Γ -orbits by

$$a_\Gamma(x) = \sup_{\text{compact } K \subset X(\mathbb{R})} \limsup_{R \rightarrow \infty} \frac{\log N_R(K, x)}{\log R}$$

where $N_R(K, x)$ denotes the number of elements $\gamma \in \Gamma$ such that $\|\gamma\| \leq R$ and $\gamma^{-1}x$ belongs to K . It is not hard to show that for almost every $x_0 \in X$,

$$\kappa_\Gamma(x, x_0) \geq \frac{\dim(X)}{a_\Gamma(x)}.$$

The quantity $a_\Gamma(x)$ could be difficult to estimate in general, but if the variety X is homogeneous, it can be estimated in terms of volume growth of a suitable subgroup.

X can then be identified with the homogeneous space G/H where H is a closed subgroup of G . We set

$$a(X) = \limsup_{t \rightarrow \infty} \frac{\log m_H(H_t)}{t},$$

where $H_t = \{h \in H; \log \|h\| \leq t\}$ and m_H is a right-invariant Haar measure on H . Then one can show using discreteness of Γ that for every $x \in X$,

$$a_\Gamma(x) \leq a(X).$$

In particular, for almost every $x_0 \in X$,

$$(3.1) \quad \kappa_\Gamma(x, x_0) \geq \frac{\dim(X)}{a(X)}.$$

Thus the fundamental question that arises is to determine when this lower bound is in fact sharp (for almost all x), and in general to give an estimate for the upper bound.

We show that in the above homogeneous setting one can reduce the original problem of quantitative density of Γ -orbits in X to the problem of quantitative density of the corresponding H orbits in the space $Y := \Gamma \backslash G$. More precisely, we have the following dynamical correspondence:

PROPOSITION 3.2 (Ghosh–Gorodnik–Nevo [19]). *Let $x = gH \in X$, $x_0 = g_0H \in X$ and $y = \Gamma g \in Y$, $y_0 = \Gamma g_0 \in Y$. There exists a sequence of neighbourhoods $\mathcal{O}_\epsilon(y_0)$ of y_0 such that if there exists $h \in H$ such that*

$$y \cdot h \in \mathcal{O}_\epsilon(y_0) \quad \text{and} \quad \|h\| \leq R,$$

then there exists $\gamma \in \Gamma$ such that

$$\|\gamma^{-1}x - x_0\|_\infty \leq \epsilon \quad \text{and} \quad \|\gamma\| \leq c(x, x_0)R,$$

where $c(x, x_0)$ is uniform over x, x_0 in compact sets.

Hence, the problem of establishing upper bounds on $\kappa_\Gamma(x, x_0)$ is closely related to the shrinking target problem for the orbit yH in Y with respect to the family of neighbourhoods $\mathcal{O}_\epsilon(y_0)$. The latter can be approached using a quantitative mean ergodic theorem for the action of H on Y .

Now we assume that the discrete group Γ has finite covolume in G and consider a family of averaging operators $\pi_Y(\beta_t) : L^2(Y) \rightarrow L^2(Y)$ defined by

$$\phi(y) := \frac{1}{m_H(H_t)} \int_{H_t} \phi(yh) dm_H(h).$$

Let us suppose that there exist $C, \theta > 0$ such that for all sufficiently large t ,

$$(3.2) \quad \left\| \pi_Y(\beta_t)(\phi) - \int_Y \phi dm_Y \right\| \leq C m_H(H_t)^{-\theta} \|\phi\|_2,$$

where m_Y denotes the normalised Haar measure on Y . Let $\theta_\Gamma(X)$ denote the supremum over θ 's for which the estimate (2.11) holds.

THEOREM 3.3 (Ghosh–Gorodnik–Nevo [19]). *For every $x_0 \in X$ and almost every $x \in X$,*

$$\kappa_\Gamma(x, x_0) \leq (2\theta_\Gamma(X))^{-1} \frac{\dim(X)}{a(X)}.$$

Moreover, the constants $c(x, x_0)$ and $\epsilon_0(x, x_0)$ in Definition 3.1 are uniform over x, x_0 in compact sets.

Looking at the case $\theta_\Gamma(X) = 1/2$ we have the following sample conclusion:

COROLLARY 3.4. *If G and the stability group H are semisimple and non-compact, and the representation of H on $L_0^2(\Gamma \backslash G)$ is a tempered representation of H , then the Diophantine exponent of Γ -action on $X = G/H$ is best possible, and is given by*

$$\kappa_\Gamma(x, x_0) = \frac{\dim(X)}{a(X)}$$

for every x_0 and almost every x in X .

We note that the results established in Theorem 3.3 and Corollary 3.4 hold not only for linear and affine action on varieties in \mathbb{R}^d , but also for linear and affine actions over other local fields and more generally for actions on general homogeneous spaces of locally compact groups satisfying some natural assumptions. We refer to [19] for general statements of these results. In this general setting one defines the dimension of X as

$$\dim(X) = \limsup_{\epsilon \rightarrow 0} \frac{\log m_X(B_\epsilon(x_0))}{\log \epsilon},$$

where $B_\epsilon(x_0)$ is the ϵ -ball around x_0 in X , and m_X is the measure on X induced by the chosen measures on G and H . We note that under the assumption imposed in [19] this limit is independent of x_0 .

The bound obtained in Theorem 3.3 for the approximation exponent depends on spectral information pertaining to automorphic representations $L^2(\Gamma \backslash G)$ as well as arithmetic data related to the group and the lattice. As such, it is a highly non-trivial task to compute these parameters in any given example. One of the main advantages in our approach is that this task is feasible in many interesting cases, and in fact often leads to best possible results. This necessarily involves a detailed study of the spectral theory of unitary representations of semisimple groups, and we now turn to demonstrating the method in some cases.

4. Examples of Diophantine exponents

In this section we give some concrete examples of explicit estimates of Diophantine exponents arising in Theorem 3.3, complementing some of the examples presented in [19]. For simplicity of exposition, let now F denote the fields \mathbb{R} , \mathbb{C} or \mathbb{Q}_p .³ Let G be a linear algebraic subgroup of the group $\mathrm{SL}_n(F) \ltimes F^n$ considered as a group of affine transformations of F^n . We fix a norm on \mathbb{R}^n and \mathbb{C}^n , and a (vector space) norm on $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$. In the local field case we take the standard valuation on the field, and the standard maximum norm on the linear space F^n , and on $M_n(F)$. We view the affine group $\mathrm{SL}_n(F) \ltimes F^n$, $n \geq 2$ as a subgroup of $\mathrm{SL}_{n+1}(F)$, specifically as the stability group of the standard basis vector e_{n+1} , and consider norms on it by restriction from $\mathrm{SL}_{n+1}(F) \subset M_{n+1}(F)$. Let $X \subset F^n$ be an affine subvariety which is invariant and homogeneous under the G -action, so that $X \simeq G/H$ where H is closed subgroup of G . We define the distance on X by restricting the norm defined on F^n . Let Γ be a lattice subgroup of G such that almost every Γ -orbit is dense in X .

³Although we do not treat this case, the methods in [19] are very general and also hold for local fields of positive characteristic. In this setting one also has the advantage of better spectral estimates in certain instances, for example arising from work of Drinfeld and Lafforgue.

We now proceed to describe several examples of classical Diophantine approximation problems in this context and estimate the exponents $\kappa_\Gamma(x, x_0)$ defined in Definition 3.1.

We remark that any choice of norm on \mathbb{R}^n or \mathbb{C}^n (and hence on X) and on $M_n(\cdot)$ (and hence on G and H) does not change the estimate of the exponent. We will thus choose the norm most convenient for us, namely one whose restriction to H has convenient properties.

For simplicity, we use notation $A \ll B$ if $A \leq cB$ for some constant c , and $A \asymp B$ if $c_1 B \leq A \leq c_2 B$ for some constants c_1, c_2 .

4.1. Homogeneous Diophantine approximation in linear space. Consider first the classical case of homogeneous Diophantine approximation in the linear action of a lattice subgroup of $\mathrm{SL}_n(F)$ on $F^n \setminus \{0\}$.

PROPOSITION 4.1. *For $n \geq 3$ the exponent of Diophantine approximation for an arbitrary lattice Γ in $G = \mathrm{SL}_n(F)$ acting on $X = F^n \setminus \{0\}$ is estimated by*

$$\frac{n}{(n-1)^2} \leq \kappa_\Gamma(x, x_0) \leq \frac{n}{n-1}$$

for almost every $x, x_0 \in F^n \setminus \{0\}$.

It is also possible to give an upper bound for the Diophantine exponent for lattices in $\mathrm{SL}_2(F)$ acting on the plane $F^2 \setminus \{0\}$. The bound depends on the lattice subgroup in question, and involves different and more elaborate considerations. In the case of $\mathrm{SL}_2(\mathbb{Z})$ the bound obtained is $\kappa_\Gamma(x, x_0) \leq 6$, and the same bound holds for any tempered lattice, namely any lattice Γ for which $L_0^2(\Gamma \setminus G)$ is a tempered representation of G . For further details and results about Diophantine approximation by lattice orbits in the real and complex plane we refer to [19].

- REMARK 4.2.**
- The only results we are aware of in the literature regarding estimates of the Diophantine exponent for lattice actions on homogeneous varieties are due to Laurent–Nogueira [34, 35] and to Maucourant–Weiss [39]. Laurent and Nogueira established that generically $\kappa_\Gamma(x, x_0) \leq 3$ for $\mathrm{SL}_2(\mathbb{Z})$ acting on the plane by explicitly constructing a sequence of approximants using a suitable continued fractions algorithm. Maucourant and Weiss have also established an (explicit, but not as sharp) upper bound for $\kappa_\Gamma(x, x_0)$ for arbitrary lattice subgroups of $\mathrm{SL}_2(\mathbb{R})$ using effective equidistribution of horocycle flows.
 - We note that for the linear action, determining the exact value of $\kappa_\Gamma(x, x_0)$ generically remains an open problem, for any lattice subgroup, over any field, in any dimension.

PROOF. *Part I: volume estimates.* $F^n \setminus \{0\}$ is a homogeneous space of $\mathrm{SL}_n(F)$, and the stability H group of the standard basis vector e_n is isomorphic to the group $\mathrm{SL}_{n-1}(F) \ltimes F^{n-1}$. We choose a vector space norm on $M_n(F)$ with the property that its restriction to $h \in H = \mathrm{SL}_{n-1}(F) \ltimes F^{n-1}$, written as $h = (h_1, v_1)$, is given by

$$\|(h_1, v_1)\| = \max\{\|h_1\|, \|v_1\|\},$$

where $\|\cdot\|$ denotes the Euclidean norm when $F = \mathbb{R}, \mathbb{C}$ and the maximal norm when $F = \mathbb{Q}_p$.

We can now evaluate the volume growth of $H_t = \{h \in H; \log \|h\| < t\}$. Due to our choice of the norm,

$$m_H(H_t) = \left(\int_{\|h_1\| \leq e^t} dh_1 \right) \cdot \left(\int_{\|v_1\| \leq e^t} dv_1 \right).$$

The first integral can be estimated using [15, Appendix 1] for $F = \mathbb{R}, \mathbb{C}$ or [20, Sec. 7] for general local fields. This gives

$$(4.1) \quad \int_{\|h_1\| \leq e^t} dh_1 \asymp \begin{cases} e^{t((n-1)^2 - (n-1))} & \text{for } F = \mathbb{R}, \mathbb{Q}_p, \\ e^{2t((n-1)^2 - (n-1))} & \text{for } F = \mathbb{C}. \end{cases}$$

Also clearly,

$$\int_{\|v_1\| \leq e^t} dv_1 \asymp \begin{cases} e^{t(n-1)} & \text{for } F = \mathbb{R}, \mathbb{Q}_p, \\ e^{2t(n-1)} & \text{for } F = \mathbb{C}. \end{cases}$$

Hence,

$$a(X) = \limsup_{t \rightarrow \infty} \frac{\log m_H(H_t)}{t} \asymp \begin{cases} (n-1)^2 & \text{for } F = \mathbb{R}, \mathbb{Q}_p, \\ 2(n-1)^2 & \text{for } F = \mathbb{C}. \end{cases}$$

Now it follows from (3.1) that for almost every $x_0 \in X$,

$$\kappa_\Gamma(x, x_0) \geq \frac{\dim(X)}{a(X)} = \frac{n}{(n-1)^2},$$

which proves the lower bound in the proposition.

Part II : spectral estimates. Let π denote the representation of H on $L_0^2(\Gamma \backslash G)$. We proceed to estimate the decay of the operator norm $\pi(\beta_t)$, where β_t are the Haar-uniform averages supported on the subsets H_t . The cases $n \geq 4$ and $n = 3$ require separate arguments.

When $n \geq 4$, we use that for the group $\mathrm{SL}_k(F)$, any unitary representation without invariant vectors is L^p -integrable for $p > 2(k-1)$, provided that $k \geq 3$ (see e.g. [21, 40]). Hence, using the spectral transfer principle [38], when $n \geq 4$ any unitary representation σ of $\mathrm{SL}_{n-1}(F)$ without invariant vectors has the property that $\sigma^{\otimes(n-2)}$ is weakly contained in the regular representation of $\mathrm{SL}_{n-1}(F)$. In particular, this statement holds for the representation of $\mathrm{SL}_{n-1}(F)$ of the form $\sigma = \pi|_{\mathrm{SL}_{n-1}(F)}$, for any lattice subgroup $\Gamma \subset \mathrm{SL}_n(F)$. Thus the restriction of the tensor power $\pi^{\otimes(n-2)}$ to the closed subgroup $\mathrm{SL}_{n-1}(F)$ is weakly contained in the regular representation of $\mathrm{SL}_{n-1}(F)$. By [8], it follows that the K -finite matrix coefficients of the representation $\pi|_{\mathrm{SL}_{n-1}(F)}$ are dominated along the split Cartan subgroup by a scalar multiple of $\Xi_{n-1}^{1/(n-2)}$, where Ξ_{n-1} denotes the Harish-Chandra function of $\mathrm{SL}_{n-1}(F)$.

We claim that the last estimate also holds when $n = 3$. To prove this, we consider the restriction of the representation π to $H = \mathrm{SL}_2(F) \ltimes F^2$. Since this representation has no F^2 -invariant vectors, it follows from the Kazhdan's original argument [22] that $\pi|_{\mathrm{SL}_2(F)}$ is weakly contained in the regular representation of $\mathrm{SL}_2(F)$, so that the same estimate as above applies.

Applying the general method of spectral estimates on groups with an Iwasawa decomposition (see [19] for details), and using the fact that the sets H_t are

bi-invariant by a maximal compact subgroup of $\mathrm{SL}_{n-1}(F)$, we can estimate the operator norm by integrating the bound $\Xi_{n-1}^{1/(n-2)}$ over H_t , where we view the function as independent of the second variable $v_1 \in F^{n-1}$. This gives

$$\begin{aligned} \|\pi(\beta_t)\| &\leq \frac{1}{m_H(H_t)} \int_{H_t} \Xi_{n-1}^{1/(n-2)}(h_1) dh_1 dv_1 \\ &= \frac{1}{m_H(H_t)} m_{F^{n-1}}(B_{e^t}) \int_{\|h_1\| \leq e^t} \Xi_{n-1}^{1/(n-2)}(h_1) dh_1 \end{aligned}$$

Since the function $\Xi_{n-1}^{1/(n-2)}$ is $L^{2(n-2)+\eta}$ -integrable for every $\eta > 0$, we can apply the following simple estimate, based on Hölder's inequality for the conjugate exponents $\frac{1}{p} = \frac{1}{2(n-2)+\eta}$ and $\frac{1}{q} = 1 - \frac{1}{2(n-2)+\eta}$:

$$\begin{aligned} \int_{\|h_1\| \leq e^t} \Xi_{n-1}^{1/(n-2)}(h_1) dh_1 &\leq \|\Xi_{n-1}^{1/(n-2)}\|_p \cdot \|\chi_{\{\|h_1\| \leq e^t\}}\|_q \\ &\ll m_{\mathrm{SL}_{n-1}(F)}(\{\|h_1\| \leq e^t\})^{1 - \frac{1}{2(n-2)+\eta}}. \end{aligned}$$

Hence, using the above volume estimates, we deduce that

$$\|\pi(\beta_t)\| \ll m_{\mathrm{SL}_{n-1}(F)}(\{\|h_1\| \leq e^t\})^{-\frac{1}{2(n-2)+\eta}} \ll m_H(H_t)^{-\theta+\eta'}$$

for every $\eta' > 0$ with $\theta = \frac{1}{2(n-1)}$. Now it follows from Theorem 3.3 that for almost every $x \in X$,

$$\kappa_\Gamma(x, x_0) \leq \frac{\dim(X)}{2\theta a(X)} = \frac{n}{n-1},$$

as claimed. \square

4.2. Simultaneous approximation on 3-dimensional space. We consider the standard action of $\mathrm{SL}_3(F)$ on the space $U = F^3 \setminus \{0\}$ and the action of $\mathrm{SL}_3(F)$ on the space $U \times U$ defined by $g(v, w) = (gv, (g^{-1})^t w)$. Given a lattice Γ in $\mathrm{SL}_3(F)$, we are interested in the problem of simultaneous Diophantine approximation in U , namely, we will seek to solve the inequalities

$$\|\gamma v - v'\| \leq \epsilon, \quad \|(\gamma^{-1})^t w - w'\| \leq \epsilon \quad \text{and} \quad \|\gamma\| \leq \epsilon^{-\zeta}.$$

We note that the action on $U \times U$ preserves the standard bilinear form $J(v, w) = \sum_{i=1}^3 v_i w_i$ and hence each of the subvarieties

$$W_\alpha = \{(v, w) \in U \times U(F); J(v, w) = \alpha\}.$$

We establish the following best possible result for Diophantine exponents, uniformly for all lattices.

PROPOSITION 4.3. *For an arbitrary lattice Γ in $\mathrm{SL}_3(F)$, the exponent of Diophantine approximation for Γ -action on W_α with $\alpha \neq 0$ is given by*

$$\kappa_\Gamma(x, x_0) = 5/2$$

for almost every $x, x_0 \in W_\alpha$.

PROOF. We first observe that the action of $\mathrm{SL}_3(F)$ on W_α with $\alpha \neq 0$ is transitive. Clearly W_α contains the orbit $\mathrm{SL}_3(F) \cdot (e_3, \alpha e_3)$ where $e_3 = (0, 0, 1)^t$. This is in fact an equality, namely if $J(v, w) = \alpha$, then for some $g \in \mathrm{SL}_3(F)$ we have $ge_3 = v$ and $(g^{-1})^t \alpha e_3 = w$, or equivalently $g^t w = \alpha e_3$. Indeed the first condition amounts to the third column of g being equal to v , and the second is solved by choosing the first two columns of g to be a basis of w^\perp (under the form

J), linearly independent of v , and adjusting one of the basis vectors so that g has determinant 1.

Since the action is transitive, $W_\alpha \simeq \mathrm{SL}_3(F)/H$ where H is the copy of $\mathrm{SL}_2(F)$ embedded in $\mathrm{SL}_3(F)$ in the upper left hand corner. It follows from Kazhdan's original argument [22] (proving property T) that the unitary representation of H on $L_0^2(\Gamma \backslash \mathrm{SL}_3(F))$ is tempered. Hence, by Corollary 3.4,

$$\kappa_\Gamma(x, x_0) = \frac{\dim(W_\alpha)}{a(W_\alpha)} = 5/2$$

for almost every $x, x_0 \in W_\alpha$. Here we used that $a(W_\alpha) = 2$ when $F = \mathbb{R}, \mathbb{Q}_p$ and $a(W_\alpha) = 4$ when $F = \mathbb{C}$, which follows from the volume estimates (4.1). \square

4.3. Inhomogeneous Diophantine approximation. We turn to consider the action of a lattice subgroup Γ of the affine group $\mathrm{SL}_n(F) \ltimes F^n$ acting on F^n .

PROPOSITION 4.4. *For $n \geq 3$ the exponent of Diophantine approximation for an arbitrary lattice Γ in $G = \mathrm{SL}_n(F) \ltimes F^n$ acting on $X = F^n \setminus \{0\}$ is estimated by*

$$(4.2) \quad \frac{1}{n-1} \leq \kappa_\Gamma(x, x_0) \leq 1$$

for almost every $x, x_0 \in X$.

We also remark that when $n = 2$ and the lattice satisfies a suitable spectral condition, the best possible exponent $\kappa_\Gamma(x, x_0) = \frac{1}{n-1}$ is achieved generically. This applies, for example, to the lattice $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ acting on \mathbb{R}^2 , and the lattice $\mathrm{SL}_2(\mathbb{Z}[i]) \ltimes \mathbb{Z}[i]^2$ acting on \mathbb{C}^2 . For full details on this matter we refer to [19].

PROOF. We note that $X \simeq G/H$ where $H = \mathrm{SL}_n(F)$ is the stabiliser of the origin. We introduce the norm on G as in the proof of Proposition 4.1. Then it follows from (4.1) that

$$(4.3) \quad a(X) = \limsup_{t \rightarrow \infty} \frac{\log m_H(H_t)}{t} \asymp \begin{cases} n^2 - n & \text{for } F = \mathbb{R}, \mathbb{Q}_p, \\ 2(n^2 - n) & \text{for } F = \mathbb{C}. \end{cases}$$

Now the lower bound in (4.5) follows from (3.1). To establish an upper bound, we need to estimate decay of the operator norm of $\pi(\beta_t)$ where π denotes the unitary representation of H on $L_0^2(\Gamma \backslash G)$, and β_t is the Haar uniform average supported on the set H_t . As in the proof of Proposition 4.1, the representation $\pi^{\otimes(n-1)}$ is weakly contained in the regular representation, and using bi-invariance of β_t under the maximal compact subgroup, the estimate of general matrix coefficients by a power of the Harish-Chandra function from [8], and Hölder's inequality, we deduce that

$$(4.4) \quad \|\pi(\beta_t)\| \leq \frac{1}{m_H(H_t)} \int_{H_t} \Xi_n^{1/(n-1)}(h) dm_H(h) \ll m_H(H_t)^{-\frac{1}{2(n-1)+\eta}}.$$

Hence, the upper bound in (4.5) now follows from Theorem 3.3. \square

4.4. The variety of matrices with a fixed determinant. Consider the variety of matrices with a determinant $k \neq 0$:

$$X = \{x \in \mathrm{M}_n(F); \det(x) = k\}.$$

The group $G = \mathrm{SL}_n(F) \times \mathrm{SL}_n(F)$ acts transitively on X , via $(g_1, g_2)x = g_1 x g_2^{-1}$. We introduce a norm on G by embedding it diagonally in $\mathrm{SL}_{2n}(F)$. Given a lattice Γ in G , we are interested in estimating the exponents of Diophantine approximation

for Γ -orbits in X , namely, to investigate existence of solutions $(\gamma_1, \gamma_2) \in \Gamma$ of the inequalities

$$\|\gamma_1 x \gamma_2^{-1} - x_0\| \leq \epsilon \quad \text{and} \quad \|(\gamma_1, \gamma_2)\| \leq \epsilon^{-\kappa}.$$

In this setting we establish the following bounds:

PROPOSITION 4.5. *For $n \geq 3$ the exponent of Diophantine approximation for an arbitrary irreducible lattice Γ in $G = \mathrm{SL}_n(F) \times \mathrm{SL}_n(F)$ acting on X is estimated by*

$$\frac{n+1}{n} \leq \kappa_\Gamma(x, x_0) \leq \frac{n^2-1}{2n}$$

for almost every $x, x_0 \in X$.

When $n = 3$ the lower and upper bounds match, so that the optimal exponent is given by $4/3$.

We note that when $n = 2$ it is also possible to obtain estimate of the Diophantine exponent, but these depend on the irreducible lattice chosen. We refer to [19] for these results.

PROOF. We observe that $X \simeq G/H$ where $H = \{(h, h); h \in \mathrm{SL}_n(F)\} \simeq \mathrm{SL}_n(F)$ is the stabiliser of the identity matrix. The growth rate of $m_H(H_t)$ can be estimated as in the previous sections (cf. (4.3)), and $\dim(X) = n^2 - 1$ when $F = \mathbb{R}, \mathbb{Q}_p$ and $\dim(X) = 2(n^2 - 1)$ when $F = \mathbb{C}$. The lower bound follows from (3.1). To prove the upper bound, we need to estimate $\|\pi(\beta_t)\|$ where π denotes the representation of H on $L_0^2(\Gamma \backslash G)$. The bound (4.4) is valid for all unitary representations of $\mathrm{SL}_n(F)$ without invariant vectors. Hence, it applies in our case as well, and it follows from Theorem 3.3 that with $\theta = \frac{1}{2(n-1)}$,

$$(4.5) \quad \kappa_\Gamma(x, x_0) \leq \frac{\dim(X)}{2\theta a(X)} = \frac{n^2-1}{n}$$

for almost every $x \in X$.

However, in the present situation it is possible to give a better estimate. Indeed, since the lattice is irreducible, the spectral decomposition of the representation of $G = \mathrm{SL}_n(F) \times \mathrm{SL}_n(F)$ in $L_0^2(\Gamma \backslash G)$ involves tensor products $\pi_1 \otimes \pi_2$ of irreducible infinite dimensional representations of $\mathrm{SL}_n(F)$. It follows that the restriction of the associated K -finite matrix coefficients to the diagonally embedded group $H \simeq \mathrm{SL}_n(F)$ is in fact not just in $L^{2(n-1)+\eta}(H)$ but in $L^{n-1+\eta}(H)$ for every $\eta > 0$. For a detailed account of this argument we refer to [19]. It follows that the spectral estimate obtained is

$$\|\pi(\beta_t)\| \ll m_H(H_t)^{-\frac{1}{n-1+\eta}}.$$

Hence, (4.5) holds with $\theta = \frac{1}{n-1}$, and this implies the upper bound in the proposition. \square

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