

# DIOPHANTINE APPROXIMATION AND AUTOMORPHIC SPECTRUM

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ABSTRACT. The present paper establishes quantitative estimates on the rate of diophantine approximation in homogeneous varieties of semisimple algebraic groups. The estimates established generalize and improve previous ones, and in a number of important cases are shown to be best possible. In general, it is shown that the rate of diophantine approximation is determined by the spectrum of the automorphic representation, and thus subject to the generalised Ramanujan conjectures.

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## 1. INTRODUCTION, DEFINITIONS, AND STATEMENT OF MAIN RESULTS

**1.1. Introduction.** Diophantine approximation can be viewed as an attempt to quantify the density of the set  $\mathbb{Q}$  of rational numbers in the reals  $\mathbb{R}$  and, more generally, the density of a number field  $K$  in its completion  $K_v$ . In this paper we will be interested in the problem of Diophantine approximation on more general algebraic varieties. Let  $X$  be an algebraic variety defined over a number field  $K$ . Given a height function  $H : X(K) \rightarrow \mathbb{R}^+$  and a metric  $\text{dist}_v$  on  $X(K_v)$ , we introduce a function  $\omega_v(x, \epsilon)$  which measures the density of  $X(K)$  in  $X(K_v)$ , and thus the Diophantine properties of points  $x$  in  $X(K_v)$  with respect to  $X(K)$ . We define :

$$\omega_v(x, \epsilon) := \min\{H(z) : z \in X(K), \text{dist}_v(x, z) \leq \epsilon\} \quad (1.1)$$

(if no such  $z$  exists, we set  $\omega_v(x, \epsilon) = \infty$ ). This function is a natural generalization of the uniform irrationality exponent  $\hat{\omega}(\xi)$  of a real number  $\xi$  (see, for instance, [2]). Note that  $\omega_v(x, \epsilon)$  is a non-increasing function which is bounded as  $\epsilon \rightarrow 0^+$  if and only if  $x \in X(K)$  and is finite if and only if  $x \in \overline{X(K)}$ . For  $x \in \overline{X(K)} \setminus X(K)$ , it is natural to consider the growth rate of  $\omega_v(x, \epsilon)$  as  $\epsilon \rightarrow 0^+$ , which provides a quantitative measure of irrationality of  $x$  with respect to  $K$ .

Our paper is motivated by the work [44] of M. Waldschmidt who considered this problem in the case when  $X$  is an Abelian variety defined over  $\mathbb{Q}$  equipped with the Néron–Tate height. In terms of our notation, he proved upper estimates on the function  $\omega_\infty(x, \epsilon)$  and conjectured that for every  $\delta > 0$ ,  $\epsilon \in (0, \epsilon_0(\delta))$ , and  $x \in \overline{X(K)} \subset X(\mathbb{R})$ ,

$$\omega_\infty(x, \epsilon) \leq \epsilon^{-\frac{2 \dim(X)}{\text{rank}(X(\mathbb{Q}))} - \delta}.$$

This conjecture is remarkably strong as one can show that the exponent in this estimate is the best possible. We also mention that there is a similar conjecture in the case of algebraic tori (see [45, Conjecture 4.21]).

More generally, let  $X \subset \mathbb{A}^n$  be a quasi-affine variety defined over a number field  $K$ . We denote by  $V_K$  the set of normalised absolute values  $|\cdot|_v$  of  $K$ , by  $K_v$  the  $v$ -completion of  $K$ , by  $k_v$  the residue field and by  $q_v$  its cardinality. We define the height function on  $X(K)$ :

$$H(x) := \prod_{v \in V_K} \max_{1 \leq i \leq n} (1, |x_i|_v), \quad (1.2)$$

and the metric on  $X(K_v)$ :

$$\|x - y\|_v := \max_{1 \leq i \leq n} |x_i - y_i|_v. \quad (1.3)$$

In this paper we derive upper estimates on the functions  $\omega_v(x, \epsilon)$  for quasi-affine varieties which are homogeneous spaces of semisimple algebraic groups. Our upper bounds on the functions  $\omega_v(x, \epsilon)$  depend on information about the spectrum of the associated automorphic representations, and the *best possible* upper bounds will be established in several cases. To illustrate the relevance of the Ramanujan-Petersson conjectures to our analysis let us consider first the case of diophantine approximation on hyperboloids. Further examples, including the case of spheres of dimensions 2 and 3 where best possible upper bounds are obtained, will be discussed in §2.

**Example 1.1.** Let  $Q$  be a non-degenerate quadratic form in three variables defined over a number field  $K \subset \mathbb{R}$ ,  $a \in K$ , and

$$X = \{Q(x) = a\}.$$

For a finite set of non-Archimedean places of  $K$ , we denote by  $O_S$  the ring of  $S$ -integers. We suppose that  $Q$  is isotropic over  $S$  and  $X(O_S) \neq \emptyset$ . Then assuming the Ramanujan-Petersson conjecture for  $\mathrm{PGL}_2$  over  $K$ , our main results imply that (w.r.t. the maximum norm  $\|\cdot\|_\infty$  on  $\mathbb{R}^3$ , the completion at  $v = \infty$ )

- (i) for almost every  $x \in X(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(x, \delta))$ , there exists  $z \in X(O_S)$  such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-2-\delta},$$

where the exponent 2 is the best possible (cf. (1.4) below).

- (ii) for every  $x \in X(\mathbb{R})$  with  $\|x\| \leq r$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(r, \delta))$ , there exists  $z \in X(O_S)$  such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-4-\delta}.$$

Using the best currently known estimates towards the Ramanujan-Petersson conjecture (see [27]), our method gives unconditional solutions to (i) and (ii) with

$$H(z) \leq \epsilon^{-\frac{18}{7}-\delta} \quad \text{and} \quad H(z) \leq \epsilon^{-\frac{36}{7}-\delta}$$

respectively. Moreover, when  $K = \mathbb{Q}$ , (i) and (ii) give unconditional solutions to the problem of diophantine approximation on the hyperboloid  $X(\mathbb{R})$  (when  $Q$  is isotropic over  $\mathbb{R}$ ), with

$$H(z) \leq \epsilon^{-\frac{64}{25}-\delta} \quad \text{and} \quad H(z) \leq \epsilon^{-\frac{128}{25}-\delta}$$

respectively, using [26, Appendix 2].

Finally, we note that for forms anisotropic over  $\mathbb{R}$  we will indeed establish the best possible bound unconditionally - see §2.

**1.2. Definitions and notation.** Before we state our results in full generality, we introduce some definitions and notation. First, let us observe that there is an (obvious) lower bound for the function  $\omega_v$ . For a subset  $Y$  of  $X(K_v)$ , we set

$$\omega_v(Y, \epsilon) := \sup_{y \in Y} \omega_v(y, \epsilon).$$

Assuming that  $Y$  is not contained in  $X(K)$ , one can give a lower estimate on  $\omega_v(Y, \epsilon)$  that depends on two properties of the set  $Y$ . These are the Minkowski dimension  $d(Y)$  of  $Y$  which measures the growth of the number of balls needed to cover  $Y$ , and the exponent  $a(Y)$  of  $Y$  which measures the growth of the set of relevant approximating rational points. The exact definition are as follows.

The Minkowski *dimension* of a subset  $Y$  of  $X(K_v)$  is defined by

$$d(Y) := \liminf_{\epsilon \rightarrow 0^+} \frac{\log D(Y, \epsilon)}{\log(1/\epsilon)},$$

where  $D(Y, \epsilon)$  denotes the least number of balls of radius  $\epsilon$  (w.r.t. the distance  $\text{dist}_v$ ) needed to cover  $Y$ . We recall that the relation of the Minkowsky dimension to the algebraic dimension is as follows. The set of nonsingular points in  $X(K_v)$  has a structure of analytic manifold over  $K_v$ . In particular, it is equipped with a canonical measure class. If a subset  $Y$  of  $X(K_v)$  has positive measure, then

$$d(Y) = r_v \dim(X)$$

where  $r_v = 2$  if  $K_v \simeq \mathbb{C}$  and  $r_v = 1$  otherwise.

The *exponent* of a subset  $Y$  of  $X(K_v)$  is defined by

$$\mathbf{a}_v(Y) := \inf_{\mathcal{O} \supset Y} \limsup_{h \rightarrow \infty} \frac{\log A_v(\mathcal{O}, h)}{\log h},$$

where  $\mathcal{O}$  runs over open neighborhoods of  $Y$  in  $X(K_v)$ , and

$$A_v(\mathcal{O}, h) := |\{z \in X(K) : H(z) \leq h, z \in \mathcal{O}\}|.$$

Note that since  $Y \not\subseteq X(K)$ , we have  $\omega_v(Y, \epsilon) \rightarrow \infty$  as  $\epsilon \rightarrow 0^+$ . Hence, for a sufficiently small neighbourhood  $\mathcal{O}$  of  $Y$ , every  $\delta_1, \delta_2 > 0$  and  $0 < \epsilon < \epsilon_0(\mathcal{O}, \delta_1, \delta_2)$ , we have

$$\epsilon^{-d(Y)+\delta_1} \leq D(Y, \epsilon) \leq A_v(\mathcal{O}, \omega_v(Y, \epsilon)) \leq \omega_v(Y, \epsilon)^{\mathbf{a}_v(Y)+\delta_2}.$$

This implies that for every  $\delta > 0$  and sufficiently small  $\epsilon > 0$  depending on  $\delta$ ,

$$\omega_v(Y, \epsilon) \geq \epsilon^{-\frac{d(Y)}{\mathbf{a}_v(Y)}+\delta}.$$

In particular, when  $Y$  has positive measure, we always have the lower bound

$$\omega_v(Y, \epsilon) \geq \epsilon^{-r_v \frac{\dim(X)}{\mathbf{a}_v(Y)}+\delta} \tag{1.4}$$

for every  $\delta > 0$  and  $\epsilon \in (0, \epsilon_0(Y, \delta))$ .

More generally, we consider the problem of diophantine approximation for points  $x = (x_v)_{v \in S}$  with  $S \subset V_K$  and  $x_v \in X(K_v)$ . Let

$$X_S := \{(x_v)_{v \in S} : x_v \in X(K_v); x_v \in X(O_v) \text{ for almost all } v\}, \quad (1.5)$$

where  $O_v = \{x \in K_v : |x|_v \leq 1\}$  is the ring of integers in  $K_v$  for non-Archimedean  $v$ . The set  $X_S$ , equipped with the topology of the restricted direct product, is a locally compact second countable space. One of the fundamental questions in arithmetic geometry is to understand the closure  $\overline{X(K)}$  in  $X_S$  where  $X(K)$  is embedded in  $X_S$  diagonally. We say that the *approximation property* with respect to  $S$  holds if  $\overline{X(K)} = X_S$ . Alternatively, denote the ring of  $S$ -integers of  $K$  by

$$O_S = \{x \in K : |x|_v \leq 1 \text{ for non-Archimedean } v \notin S\}^1.$$

Then the approximation property with respect to  $S$  can be reformulated as follows: for every  $x \in X_S$  and every  $\epsilon = (\epsilon_v)_{v \in S'}$ , where  $\epsilon_v \in (0, 1)$  and  $S'$  is a finite subset of  $S$ , there exists  $z \in X(O_{(V_K \setminus S) \cup S'})$  such that

$$\|x_v - z\|_v \leq \epsilon_v \text{ for all } v \in S'.$$

Our aim is to establish a quantitative version of this property. Given  $x$  and  $(\epsilon_v)_{v \in S'}$  as above, we consider

$$\omega_S(x, (\epsilon_v)_{v \in S'}) := \min \{H(z) : z \in X(O_{(V_K \setminus S) \cup S'}), \|x_v - z\|_v \leq \epsilon_v, v \in S'\}. \quad (1.6)$$

**Remark 1.2.** To clarify our notation somewhat, consider the group variety  $G \subset GL_n$ . Fixing the finite set  $S' \subset V_K$ , we want to establish a rate for simultaneous diophantine approximation of all (or almost all) elements in the group  $\prod_{v \in S'} G_v$ . The elements in the group  $G(K)$  of  $K$ -rational points which are allowed in the approximation process are determined by the choice of a set  $S$  containing  $S'$ , which may be finite or infinite. One choice is  $S = S'$ , in which case the approximation property calls for using elements from  $O_{V_K} = G(K)$ , namely there are no restrictions at all on the  $K$ -rational matrices allowed. Thus for example when  $S = \{v\} = S'$ , the approximation rate  $\omega_S$  defined above is given by the function  $\omega_v(x, \epsilon)$  as defined in (1.1). Another choice is  $S = V_K \setminus v_0$ , where  $v_0$  is a valuation not in  $S'$ . In this case the approximation property calls for using only  $K$ -rational matrices whose elements are in  $O_v$  (namely  $v$ -integral) for every  $v \in V_K$ , with the exception of  $v \in S'$  and  $v = v_0$ . We will of course assume that  $G$  is isotropic over  $K_{v_0}$  in this case. We also admit any other intermediate choice of  $S$  containing  $S'$ , namely we allow imposing *arbitrary* integrality conditions on the set of approximating  $K$ -rational matrices. The integrality conditions are that the matrices should be  $v$ -integral for  $v \in S \setminus S'$ , and we assume that  $G$  is isotropic over  $V_K \setminus S$ .

<sup>1</sup>Note that  $O_{\{v\}} \neq O_v$ , the ring of integers defined above !

Given a variety  $X$ , we define the *exponent* of a subset  $Y$  of  $X_S$  as

$$\mathfrak{a}_S(Y) := \inf_{\mathcal{O} \supset Y} \limsup_{h \rightarrow \infty} \frac{\log A_S(\mathcal{O}, h)}{\log h}, \quad (1.7)$$

where  $\mathcal{O}$  runs over open neighborhoods of  $Y$  in  $X_S$ , and

$$A_S(\mathcal{O}, h) := |\{z \in X(K) : H(z) \leq h, z \in \mathcal{O}\}|.$$

More appropriately, the notation should be  $\mathfrak{a}_S(Y, X)$  but we will suppress the dependence on  $X$  in the notation. We also define the *exponent*  $\mathfrak{a}_S(X)$  of the variety  $X$  as the supremum of  $\mathfrak{a}_S(Y)$  as  $Y$  runs over bounded subsets of  $X_S$ . Since our variety  $X$  will be fixed and we will not consider subvarieties of it, this notation should cause no conflict.

As in (1.4), one can show that given  $Y \subset X_S$  and finite  $S' \subset S$  such that the projection of  $Y$  to  $X_{S'}$  has positive measure, we have

$$\sup_{y \in Y} \omega_S(y, (\epsilon_v)_{v \in S'}) \geq \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(X)}{\mathfrak{a}_S(Y)} + \delta} \quad (1.8)$$

for every  $\delta > 0$  and  $\epsilon_v \in (0, \epsilon_0(Y, S', \delta))$ . In particular, it follows that we have the following universal lower bound

$$\sup_{y \in Y} \omega_S(y, (\epsilon_v)_{v \in S'}) \geq \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(X)}{\mathfrak{a}_S(X)} + \delta}, \quad (1.9)$$

provided that the projection of  $Y$  to  $X_{S'}$  has positive measure.

We shall derive upper estimates on the function  $\omega_S$  of comparable quality in the case when  $X$  is a homogeneous quasi-affine variety of a semisimple group. We shall start by considering the case of the group variety itself.

Let  $G$  be a connected almost simple algebraic  $K$ -group. Our main results depend on properties of unitary representations  $\pi_v$  of the groups  $G_v := G(K_v)$ , which we now introduce. We fix a suitable maximal compact subgroup  $U_v$  of  $G_v$ , whose choice is discussed in Section 3. The group  $G(K)$  embeds in the restricted direct product group  $G_{V_K}$  diagonally as a discrete subgroup and  $\text{vol}(G_{V_K}/G(K)) < \infty$  (see [37, §5.3]). We consider the Hilbert space  $L^2(G_{V_K}/G(K))$  consisting of square-integrable functions on  $G_{V_K}/G(K)$ . A unitary continuous character  $\chi$  of  $G_{V_K}$  is called automorphic if  $\chi(G(K)) = 1$ . Then  $\chi$  can be considered as an element of  $L^2(G_{V_K}/G(K))$ . We denote by  $L_{00}^2(G_{V_K}/G(K))$  the subspace of  $L^2(G_{V_K}/G(K))$  orthogonal to all automorphic characters. The translation action of the group  $G_v$  on  $G_{V_K}/G(K)$  defines the unitary representation  $\pi_v$  of  $G_v$  on  $L_{00}^2(G_{V_K}/G(K))$ . We define the spherical integrability exponent of  $\pi_v$  w.r.t.  $U_v$  as follows

$$\mathfrak{q}_v(G) := \inf \left\{ q > 0 : \forall U_v\text{-inv. } w \in L_{00}^2(G_{V_K}/G(K)) \right. \\ \left. \langle \pi_v(g)w, w \rangle \in L^q(G_v) \right\}. \quad (1.10)$$

As we shall see in §3.3, Corollary 3.6, an equivalent definition is that  $\mathfrak{q}_v(G)$  is the infimum of  $q > 0$  having the property that every irreducible

spherical unitary representation  $\tau_v$  of  $G_v$  which is weakly contained in  $\pi_v$  is an  $L^q$ -representation, namely has a dense subspace giving rise to matrix coefficients in  $L^q(G_v)$ .

If  $\mathfrak{q}_v(\mathbf{G}) = 2$ , then we say that the representation  $\pi_v$  is *tempered*<sup>2</sup>.

It is one of the fundamental results in the theory of automorphic representations that the integrability exponent  $\mathfrak{q}_v(\mathbf{G})$  is finite. (see [7, Theorem 3.1]). Moreover,  $\mathfrak{q}_v(\mathbf{G})$  is uniformly bounded over  $v \in V_K$  (see [15]). The precise value of  $\mathfrak{q}_v(\mathbf{G})$  is related to generalised Ramanujan conjecture and Langlands functoriality conjectures (see [38]). For instance, the generalised Ramanujan conjecture for  $\mathrm{SL}_2$  is equivalent to  $\mathfrak{q}_v(\mathrm{SL}_2) = 2$  for all  $v \in V_K$ , and the best currently known estimate established in [27, 26] gives  $\mathfrak{q}_v(\mathrm{SL}_2) \leq \frac{18}{7}$  for general number fields  $K$  and  $\mathfrak{q}_v(\mathrm{SL}_2) \leq \frac{64}{25}$  for  $K = \mathbb{Q}$ .

We define the *exponent* of a subset  $S$  of  $V_K$  by

$$\sigma_S := \limsup_{N \rightarrow \infty} \frac{1}{\log N} |\{v \in S : q_v \leq N\}|, \quad (1.11)$$

where  $q_v$  denotes the cardinality of the residue field for non-Archimedean  $v$ . Let

$$\mathfrak{q}_S(\mathbf{G}) = (1 + \sigma_S) \sup_{v \in S} \mathfrak{q}_v(\mathbf{G}). \quad (1.12)$$

This parameter will appear below as a bound on the integrability exponent of the automorphic representation restricted to the group  $G_S$ .

**1.3. Statement of main results.** We now turn to state our main results. First let us note that the approximation property for algebraic varieties we defined above, namely the density of the closure of  $X(K)$  in  $X_S$ , has been studied extensively in the setting of algebraic groups (see [37, Ch. 7]). It is known that when  $\mathbf{G}$  is connected simply connected almost simple algebraic group defined over  $K$ , then the approximation property holds with respect to  $S$  provided that  $\mathbf{G}$  is isotropic over  $V_K \setminus S$ . Our first result can be viewed as a quantitative version of this fact.

It is convenient to set  $I_v = \{q_v^{-n}\}_{n \geq 1}$  for non-Archimedean  $v \in V_K$  and  $I_v = (0, 1)$  for Archimedean  $v \in V_K$ .

**Theorem 1.3.** *Let  $G$  be a connected simply connected almost simple algebraic  $K$ -group and  $S$  a (possibly infinite) subset of  $V_K$  such that  $G$  is isotropic over  $V_K \setminus S$ . Then*

- (i) *There exists a subset  $Y$  of full measure in  $G_S$  such that for every  $\delta > 0$ , finite  $S' \subset S$ ,  $x \in Y$ , and  $(\epsilon_v)_{v \in S'}$  with  $\epsilon_v \in$*

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<sup>2</sup>This is equivalent to the standard notion of a tempered representation defined in terms of weak containment, by [10, Theorem 1].

$I_v \cap (0, \epsilon_0(x, S', \delta))$ , we have

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \leq \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G)}{\mathfrak{a}_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2}.$$

In particular, when the representations  $\pi_v$ ,  $v \in V_K \setminus S$ , are tempered and  $\sigma_S = 0$ , then the above exponent is the best possible (cf. (1.8))!

- (ii) For every  $\delta > 0$ , bounded  $\Omega \subset G_S$ , and  $(\epsilon_v)_{v \in S'}$  with finite  $S' \subset S$  and  $\epsilon_v \in I_v \cap (0, \epsilon_v^0(\Omega, \delta))$ , where  $\epsilon_v^0(\Omega, \delta) \in (0, 1]$  and  $\epsilon_v^0(\Omega, \delta) = 1$  for almost all  $v$ , we have

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \leq \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G)}{\mathfrak{a}_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)},$$

uniformly over  $x \in \Omega$  and finite  $S' \subset S$ .

**Remark 1.4.** Comparing the estimate in Theorem 1.3(i) and (1.8), we conclude that the integrability exponents always satisfy  $\mathfrak{q}_v(G) \geq 2$ . While this fact was previously known (see, for instance, [3]), it is curious that it follows from diophantine approximation considerations as well. This point is further addressed in Corollary 1.8 below.

We also prove a version of Theorem 1.3(i) which is uniform over finite  $S' \subset S$ .

**Theorem 1.5.** *With the notation as in Theorem 1.3, there exists a subset  $Y$  of full measure in  $G_S$  such that for every  $\delta > 0$ ,  $x \in Y$ , and  $(\epsilon_v)_{v \in S'}$  with finite  $S' \subset S$  and  $\epsilon_v \in I_v \cap (0, \epsilon_v^0(x, \delta))$ , where  $\epsilon_v^0(x, \delta) \in (0, 1]$  and  $\epsilon_v^0(x, \delta) = 1$  for almost all  $v$ , we have*

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \leq \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G) + \sigma_S}{\mathfrak{a}_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2}.$$

More generally, let  $X \subset \mathbb{A}^n$  be a quasi-affine algebraic variety defined over  $K$  equipped with a transitive action of a connected almost simple algebraic  $K$ -group  $G \subset \mathrm{GL}_n$ . Given a subset  $S$  of  $V_K$ , we consider the problem of Diophantine approximation in  $X_S$  by the rational points in  $X(K)$ , or equivalently the problem of Diophantine approximation in  $X_{S'}$ , where  $S'$  is a finite subset of  $S$ , by points in  $X(O_{(V_K \setminus S) \cup S'})$ . The closure  $\overline{X(O_{(V_K \setminus S) \cup S'})}$  in  $X_{S'}$  can be described explicitly (see Lemma 6.3 below). In particular, it follows that  $\overline{X(O_{(V_K \setminus S) \cup S'})}$  is open  $X_{S'}$  provided that  $G$  is isotropic over  $V_K \setminus S$ . Our next result gives a quantitative version of the density (in the closure) where the estimates are sharp in many cases (see Example 1.1 and Section 2).

**Theorem 1.6.** *Assume that  $G$  is isotropic over  $V_K \setminus S$ . Then*

- (i) For every finite  $S' \subset S$ , there exists a subset  $Y$  of full measure in  $\overline{X(O_{(V_K \setminus S) \cup S'})} \subset X_{S'}$  such that for every  $\delta > 0$ ,  $x \in Y$ , and  $(\epsilon_v)_{v \in S'}$  with  $\epsilon_v \in I_v \cap (0, \epsilon_0(x, S', \delta))$ , we have

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \leq \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(X)}{\mathfrak{a}_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2}.$$

- (ii) For every finite  $S' \subset S$ ,  $\delta > 0$ , bounded  $Y \subset \overline{X(O_{(V_K \setminus S) \cup S'})}$ , and  $(\epsilon_v)_{v \in S'}$  with  $\epsilon_v \in I_v \cap (0, \epsilon_0(Y, S', \delta))$ , we have

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \leq \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(X)}{\mathfrak{a}_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)},$$

uniformly over  $x \in Y$ .

**Remark 1.7.** We note that the estimates in Theorem 1.6 are stated in terms of  $\mathfrak{a}_S(G)$ , but not in terms of  $\mathfrak{a}_S(X)$ . While the latter quantity is difficult to compute in general, in many cases we have  $\mathfrak{a}_S(G) \geq \mathfrak{a}_S(X)$ . For instance, this is so when the rational points in  $G$  do not concentrate on a proper subgroup of  $G$ , namely when  $\mathfrak{a}_S(G) > \mathfrak{a}_S(\text{Stab}_G(x^0))$  for some  $x^0 \in X(O_{(V_K \setminus S) \cup S'})$ . If  $\mathfrak{a}_S(G) \geq \mathfrak{a}_S(X)$  and  $\mathfrak{q}_{V_K \setminus S}(G) = 2$ , then Theorem 1.6(i) gives the best possible estimate (cf. (1.9)).

Comparing Theorem 1.6(i) with (1.9), we deduce the lower estimates on the integrability exponents  $\mathfrak{q}_v(G)$ .

**Corollary 1.8.** Let  $G < \text{GL}_n$  be a connected almost simple algebraic  $K$ -group and  $X \subset \mathbb{A}^n$  a quasi-affine variety on which  $G$  acts transitively. Assume that  $G$  is isotropic over  $v \in V_K$  and that  $X(O_{V_K \setminus \{v\}})$  is not empty. Then

$$\mathfrak{q}_v(G) \geq \frac{2\mathfrak{a}_{V_K \setminus \{v\}}(G)}{\mathfrak{a}_{V_K \setminus \{v\}}(X)}.$$

For example, Corollary 1.8 implies that  $\mathfrak{q}_v(\text{SL}_n) \geq 2(n-1)$ , which is known to be sharp (see Section 2).

## 2. EXAMPLES

**2.1. Diophantine approximation on spheres.** Let  $S^d$ ,  $d \geq 2$  be the unit sphere of dimension  $d$  centered at the origin, which we view as the level set of the standard quadratic form given by the sum of squares. We fix a prime  $p \equiv 1 \pmod{4}$ . Then  $S^d(\mathbb{Z}[1/p])$  is dense in  $S^d(\mathbb{R})$ , and here we derive a quantitative density estimates. We treat the cases  $d = 2$ ,  $d = 3$ , and  $d \geq 4$  separately.

For  $d = 2$ , Theorem 1.6 implies that

- For almost every  $x \in S^2(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(x, \delta))$ , there exists  $z \in S^2(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-2-\delta}.$$

We note that  $\dim(\mathbb{S}^2) = 2$  and  $\mathfrak{a}_{V_{\mathbb{Q}} \setminus \{p\}}(\mathbb{S}^2) = 1$ , so that this exponent is the best possible (see (1.9)).

- For every  $x \in \mathbb{S}^2(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(\delta))$ , there exists  $z \in \mathbb{S}^2(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_{\infty} \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-4-\delta}.$$

To deduce these estimates, we consider the group  $G \simeq D^{\times}/Z^{\times}$ , where  $D$  denotes Hamilton's quaternion algebra and  $Z$  the centre of  $D$ . This group naturally acts on the variety of pure quaternions of norm one, which can be identified with the sphere  $\mathbb{S}^2$ . Hence, we are in position to apply Theorem 1.6. Since  $p \equiv 1 \pmod{4}$ , the Quaternion algebra split over  $p$  and ramifies at  $\infty$ . In this case, we have  $\mathfrak{q}_{V_{\mathbb{Q}} \setminus \{p\}}(G) = 2$ , which is a consequence of the results of Deligne combined with Jacquet–Langlands correspondence (see [30, Appendix]),  $\dim(\mathbb{S}^2) = 2$ , and  $\mathfrak{a}_{V_{\mathbb{Q}} \setminus \{p\}}(G) = 1$ .

For  $d = 3$ , Theorem 1.6 implies that

- For almost every  $x \in \mathbb{S}^3(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(x, \delta))$ , there exists  $z \in \mathbb{S}^3(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_{\infty} \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-\frac{3}{2}-\delta}.$$

Since  $\dim(\mathbb{S}^3) = 3$  and  $\mathfrak{a}_{V_{\mathbb{Q}} \setminus \{p\}}(\mathbb{S}^3) = 2$ , this exponent is the best possible (see (1.9)).

- For every  $x \in \mathbb{S}^3(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(\delta))$ , there exists  $z \in \mathbb{S}^3(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_{\infty} \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-3-\delta}.$$

To deduce these estimates, we now consider the group  $G$  of norm one elements of Hamilton's quaternion algebra  $D$ , which can be identified with the variety  $\mathbb{S}^3$ . We have  $\mathfrak{q}_p(G) = 2$ , which is a again consequence of the results of Deligne combined with Jacquet–Langlands correspondence (see [30, Appendix]),  $\dim(G) = 3$ , and  $\mathfrak{a}_{V_{\mathbb{Q}} \setminus \{p\}}(G) = 2$ . Hence, the claimed estimates follow from Theorem 1.6.

Now let  $d \geq 4$ . We consider the natural action of the group  $G = \mathrm{SO}_{d+1}$  on  $\mathbb{S}^d$ . We note that since  $p \equiv 1 \pmod{4}$ , the group  $G$  splits over  $\mathbb{Q}_p$ . By [29, 35], we have  $\mathfrak{q}_p(G) \leq d$  for even  $d$  and  $\mathfrak{q}_{V_{\mathbb{Q}} \setminus \{p\}}(G) \leq d + 1$  for odd  $d$ . By [17], the parameter  $\mathfrak{a}_p(G)$  can be estimated in terms of volumes of the height balls, which in turn can be estimated in terms of the root system data of  $G$ . This gives the estimates  $\mathfrak{a}_{V_{\mathbb{Q}} \setminus \{p\}}(G) = d^2/4$  for even  $d$  and  $\mathfrak{a}_p(G) = (d+1)(d+3)/4$  for odd  $d$ . Therefore, Theorem 1.6 implies that for even  $d \geq 4$ ,

- For almost every  $x \in \mathbb{S}^d(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(x, \delta))$ , there exists  $z \in \mathbb{S}^d(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_{\infty} \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-2-\delta}.$$

The exponent 2 should be compared with the lower estimate  $\frac{d}{d-1}$  given by (1.9). At present we don't know whether this exponent can be improved.

- For every  $x \in S^d(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(\delta))$ , there exists  $z \in S^d(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-4-\delta}.$$

Similarly, for odd  $d \geq 4$ ,

- For almost every  $x \in S^d(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(x, \delta))$ , there exists  $z \in S^d(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-\frac{2d}{d+3}-\delta}.$$

- For every  $x \in S^d(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(\delta))$ , there exists  $z \in S^d(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-\frac{4d}{d+3}-\delta}.$$

**2.2. Diophantine approximation in the orthogonal group.** Let  $\text{SO}_{d+1}$ ,  $d \geq 4$ , be the orthogonal group and  $p$  be a prime such that  $p \equiv 1 \pmod{4}$ . Then  $\text{SO}_{d+1}(\mathbb{Z}[1/p])$  is dense in  $\text{SO}_{d+1}(\mathbb{R})$ . We have  $\dim(\text{SO}_{d+1}) = d(d+1)/2$ , and  $\mathfrak{a}_{V_{\mathbb{Q}} \setminus \{p\}}(\text{SO}_{d+1})$ ,  $\mathfrak{q}_p(\text{SO}_{d+1})$  are given above. Therefore, Theorem 1.3 gives

- For almost every  $x \in \text{SO}_{d+1}(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(x, \delta))$ , there exists  $z \in \text{SO}_{d+1}(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-(d+1)-\delta}, \quad \text{when } d \text{ is even,}$$

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-\frac{d(d+1)}{d+3}-\delta}, \quad \text{when } d \text{ is odd.}$$

This should be compared with lower estimates on the exponents given by (1.9), which is  $2(d+1)/d$  for even  $d$ , and  $2d/(d+3)$  for odd  $d$ .

- For every  $x \in \text{SO}_{d+1}(\mathbb{R})$ ,  $\delta > 0$ , and  $\epsilon \in (0, \epsilon_0(\delta))$ , there exists  $z \in \text{SO}_{d+1}(\mathbb{Z}[1/p])$  such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-2(d+1)-\delta}, \quad \text{when } d \text{ is even,}$$

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \epsilon^{-\frac{2d(d+1)}{d+3}-\delta}, \quad \text{when } d \text{ is odd.}$$

**2.3. Diophantine approximation by the Hilbert modular group.**

Let  $K$  be a totally real number field,  $O$  its ring of integers, and  $S$  a proper subset of  $V_K^\infty$ . The Hilbert modular group  $\text{SL}_2(O)$  is a dense subgroup of  $(\text{SL}_2)_S = \prod_{v \in S} \text{SL}_2(K_v)$ . We have  $\dim(\text{SL}_2) = 3$  and  $\mathfrak{a}_{S \cup V_K^f}(\text{SL}_2) = 2$ , and if we assume the Ramanujan conjecture for  $\text{SL}_2$  over  $K$ , then  $\mathfrak{q}_v(\text{SL}_2) = 2$ . Hence, Theorem 1.3 implies the following quantitative density results (under the Ramanujan conjecture):

- For almost every  $x_v \in \mathrm{SL}_2(K_v)$  with  $v \in S$ ,  $\delta > 0$ , and  $\epsilon_v \in (0, \epsilon_0(x, \delta))$ , there exists  $z \in \mathrm{SL}_2(O)$  such that

$$\|x_v - z\|_v \leq \epsilon_v \text{ for } v \in S \quad \text{and} \quad \mathrm{H}(z) \leq \prod_{v \in S} \epsilon_v^{-\frac{3}{2}-\delta}.$$

The exponent  $\frac{3}{2}$  is the best possible by (1.9).

- For every  $x_v \in \mathrm{SL}_2(K_v)$  with  $v \in S$  such that  $\|x_v\|_v \leq r$ ,  $\delta > 0$ , and  $\epsilon_v \in (0, \epsilon_0(r, \delta))$ , there exists  $z \in \mathrm{SL}_2(O)$  such that

$$\|x_v - z\|_v \leq \epsilon_v \text{ for } v \in S \quad \text{and} \quad \mathrm{H}(z) \leq \prod_{v \in S} \epsilon_v^{-3-\delta}.$$

Using the best currently known estimates towards the Ramanujan–Pettersson conjecture (see [27]), we have  $\mathfrak{q}_v(\mathrm{SL}_2) \leq \frac{18}{7}$ , and Theorem 1.3 gives unconditional solutions to the above inequalities  $\|x_v - z\|_v \leq \epsilon_v$ ,  $v \in S$ , with

$$\mathrm{H}(z) \leq \epsilon^{-\frac{27}{14}-\delta} \quad \text{and} \quad \mathrm{H}(z) \leq \epsilon^{-\frac{27}{7}-\delta}$$

respectively.

**2.4. Estimates on integrability exponents.** Let us apply Corollary 1.8 to the action of  $G = \mathrm{SL}_n$  on  $X = \mathbb{A}^n \setminus \{0\}$ . In this case, we have  $\mathfrak{a}_{V_K \setminus \{v\}}(G) = n^2 - n$  and  $\mathfrak{a}_{V_K \setminus \{v\}}(X) = n$ . Hence, we conclude that

$$\mathfrak{q}_v(\mathrm{SL}_n) \geq 2(n - 1).$$

This estimate is sharp.

Another example is given by the orthogonal group  $G = \mathrm{SO}_{d+1}$  acting on the sphere  $X = S^d$ , discussed in Section 2.1. Let  $p$  be a prime such that  $p \equiv 1 \pmod{4}$ . In this case we have  $\mathfrak{a}_{V_{\mathbb{Q}} \setminus \{p\}}(G) = d^2/4$  for even  $d \geq 4$ ,  $\mathfrak{a}_{V_{\mathbb{Q}} \setminus \{p\}}(G) = (d+1)(d+3)/4$  for odd  $d \geq 4$ , and  $\mathfrak{a}_{V_{\mathbb{Q}} \setminus \{p\}}(X) = d - 1$ . Therefore,

$$\begin{aligned} \mathfrak{q}_v(\mathrm{SO}_{d+1}) &\geq \frac{d^2}{2d-2}, \quad \text{when } d \text{ is even,} \\ \mathfrak{q}_v(\mathrm{SO}_{d+1}) &\geq \frac{(d+1)(d+3)}{2d-2}, \quad \text{when } d \text{ is odd.} \end{aligned}$$

**2.5. Comparison with previous results.** The idea of using information about the spectrum of the automorphic representation associated with an arithmetic lattice subgroup in order to establish a rate of equidistribution for orbits of the lattice on compact homogeneous spaces originated in the work [31, 32] of Lubotzky, Phillips, and Sarnak. The class of examples where this method applies has been extended in [6, 36]. Spectral estimates of Hecke operators using bounds on automorphic forms have also been used to establish equidistribution of Hecke points in  $G/\Gamma$  in [9, 8].

We also note the recent results of [33] where the case of lattices in  $SL_2(\mathbb{R})$  acting on  $\mathbb{R}^2$  is considered, and a rate of Diophantine approximation is obtained, using, in the case of  $SL_2(\mathbb{Z})$ , quantitative

estimates of ergodic averages associated with unipotent flows. Explicit Diophantine approximation rates for  $SL_2(\mathbb{Z})$  acting on  $\mathbb{R}^2$  have also been considered recently in [34] using a completely different method.

Let us also compare our results with the results on Diophantine approximation on the spheres obtained in [40] by elementary methods that use rational parametrisations of spheres. It is shown in [40] that for every  $x \in S^d(\mathbb{R})$  and  $\epsilon \in (0, 1)$ , there exists  $z \in S^d(\mathbb{Q})$  such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad H(z) \leq \text{const } \epsilon^{-2\lceil \log_2(d+1) \rceil}.$$

Note that this result allows approximation using all  $\mathbb{Q}$ -points on  $S^d$  rather than just  $\mathbb{Z}[1/p]$ -points. Nevertheless the exponent obtained is unbounded as a function of  $d$ , whereas the the exponents we present are bounded.

Establishing best possible almost sure Diophantine approximation rates does not, as far as we are aware, have any precedent within the context of semisimple groups and their homogeneous spaces. Such results require developing an efficient ergodic theory of shrinking targets in measure-preserving systems, namely estimates that do not compromise the spectral rate of decay of the underlying averaging operators. In the sequel [14] we use this strategy to establish Khintchine type theorems on homogeneous varieties. We note that the works [11] and [12] study a problem of similar flavour, namely that of estimating the Hausdorff dimension of the set of “very well approximable” vectors on circles and quadrics respectively.

### 3. SEMISIMPLE GROUPS AND SPHERICAL FUNCTIONS

Let  $K$  be a number field. We denote by  $V_K$  the set of all places of  $K$ , which consists of the finite set  $V_K^\infty$  of Archimedean places and the set  $V_K^f$  of non-Archimedean places. For  $v \in V_K$ , we denote by  $|\cdot|_v$  the suitably normalised absolute value and by  $K_v$  the corresponding completion. Also for  $v \in V_K^f$  we denote by  $O_v \subset K_v$  the ring of integers and by  $q_v$  the cardinality of the residue field.

**3.1. Structure of semisimple groups.** We recall elements of the structure theory of semisimple algebraic groups over local fields, which is discussed in details in [1, 42]. Let  $G \subset GL_n$  be a connected semisimple algebraic group defined over a number field  $K$ . For all but finitely many places in  $V_K$ ,

- (i) the group  $U_v := G(O_v)$  is a hyperspecial, good maximal compact subgroup of  $G_v := G(K_v)$ ,
- (ii) the group  $G$  is unramified over  $K_v$  (that is,  $G$  is quasi-split over  $K_v$  and split over an unramified extension of  $K_v$ ).

For the other places  $v$ , we fix a good, special maximal compact subgroup  $U_v$  of  $G_v$ . For  $S \subset V_K$ , we set  $U_S = \prod_{v \in S} U_v$ .

To simplify notation, for an algebraic group  $C_v$  defined over  $K_v$  we denote by  $C_v$  the set  $C_v(K_v)$  of  $K_v$ -points in  $C_v$ .

Every subgroup  $U_v$  is associated to a minimal parabolic  $K_v$ -subgroup  $P_v$  of  $G_v$ , that has a decomposition  $P_v = N_v Z_v$  where  $Z_v$  is connected and reductive, and  $N_v$  is the unipotent radical of  $P_v$ . Let  $Z_v^\circ$  be the maximal compact subgroup of  $Z_v$ . Then  $Z_v/Z_v^\circ$  is a free  $\mathbb{Z}$ -module of rank equal to the  $K_v$ -rank of  $G$ . We set  $\mathcal{Z}_v = (Z_v/Z_v^\circ) \otimes \mathbb{R}$  and denote by  $\nu_v : Z_v \rightarrow \mathcal{Z}_v$  the natural map. Let  $T_v$  be the maximal  $K_v$ -split torus in  $Z_v$ . Then  $T_v^\circ := T_v \cap Z_v^\circ$  is the maximal compact subgroup of  $T_v$  and  $T_v/T_v^\circ$  has finite index in  $Z_v/Z_v^\circ$ . The group of characters of  $Z_v$  is denoted by  $X^*(Z_v)$ . For any  $\theta \in X^*(Z_v)$  we associate a linear functional  $\chi_\theta$  on  $\mathcal{Z}_v$  defined by

$$|\theta(z)|_v = q_v^{\langle \chi_\theta, \nu_v(z) \rangle}, \quad z \in Z_v, \quad (3.1)$$

where  $q_v = e$  for  $v \in V_K^\infty$ , and  $q_v$  is the order of the residue field for  $v \in V_K^f$ .

The adjoint action of  $T_v$  on  $G$  defines a root system in the dual space  $\mathcal{Z}_v^*$ . We fix the ordering on the root system defined by the parabolic subgroup  $P_v$ . Let  $\mathcal{Z}_v^-$  denote the negative Weyl chamber in  $\mathcal{Z}_v$  with respect to this ordering. The relative Weyl group  $\mathcal{W}_v := N_{G_v}(T_v)/Z_{G_v}(T_v)$  operates on  $Z_v$  and on  $\mathcal{Z}_v$ . We choose a  $\mathcal{W}_v$ -invariant scalar product on  $\mathcal{Z}_v$ . Then we regard the root system as a subset of  $\mathcal{Z}_v$ , and denote by  $\mathcal{Z}_{v,-}$  the cone consisting of negative linear combinations of simple roots. Let  $Z_v^- = \nu_v^{-1}(\mathcal{Z}_v^-)$  and  $Z_{v,-} = \nu_v^{-1}(\mathcal{Z}_{v,-})$ .

The group  $G_v$  has the *Cartan decomposition*

$$G_v = U_v Z_v^- U_v,$$

which defines a bijection between the double cosets  $U_v \backslash G_v / U_v$  and  $Z_v^- / Z_v^\circ$ , and the *Iwasawa decomposition*

$$G_v = N_v Z_v U_v,$$

which defines a bijection between  $N_v \backslash G_v / U_v$  and  $Z_v / Z_v^\circ$ . For  $g \in G_v$ , we define  $z(g) \in Z_v / Z_v^\circ$  by the property  $g \in N_v z(g) U_v$ .

For unramified  $v$ , we have  $T_v / T_v^\circ = Z_v / Z_v^\circ$  and the Cartan decomposition becomes

$$G_v = U_v T_v^- U_v, \quad (3.2)$$

where  $T_v^- = \nu_v^{-1}(\mathcal{Z}_v^-)$ .

For  $S \subset V_K$ , we denote by  $G_S$  the restricted direct product of the groups  $G_v$  for  $v \in S$  with respect to the family of subgroups  $U_v$ . Then  $G_S$  is a locally compact second countable group, and  $U_S := \prod_{v \in S} U_v$  is a maximal compact subgroup of  $G_S$ .

For every  $v \in V_K$ , we denote by  $m_{G_v}$  the Haar measure on  $G_v$  which is normalised so that  $m_{G_v}(U_v) = 1$  for non-Archimedean  $v$ . Then the restricted product measure  $\otimes_{v \in S} m_{G_v}$  defines a Haar measure on the group  $G_S$  which we denote by  $m_S$ . We also denote by  $m_{U_v}$  the probability Haar measure on  $U_v$ .

**3.2. Spherical functions.** The theory of spherical functions on semisimple groups over local fields was developed by Harish-Chandra ([19], [22, Chapter IV]) in the Archimedean case, and by Satake ([39], [5]) in the non-Archimedean case.

For  $v \in V_K$ , we introduce the *Hecke algebra*  $\mathcal{H}_v$  consisting of bi- $U_v$ -invariant continuous functions with compact support on  $G_v$  and equipped with product given by convolution. The structure of this algebra can be completely described, and in particular, it is commutative.

A continuous bi- $U_v$ -invariant function  $\eta : G_v \rightarrow \mathbb{C}$  with compact support is called *spherical* if  $\eta(e) = 1$  and one of the following equivalent conditions holds:

(a)  $\eta$  is bi- $U_v$ -invariant, and the map

$$\mathcal{H}_v \rightarrow \mathbb{C} : \phi \mapsto \int_{G_v} \phi(g)\eta(g^{-1}) dm_{G_v}(g)$$

is an algebra homomorphism.

(b) for every  $\phi \in \mathcal{H}_v$  there exists  $\lambda_\phi \in \mathbb{C}$  such that  $\phi * \eta = \lambda_\phi \eta$ .

(c) For every  $g_1, g_2 \in G_v$ ,

$$\eta(g_1)\eta(g_2) = \int_{U_v} \eta(g_1 u g_2) dm_{U_v}(u).$$

We shall use the following parametrisation of the spherical functions as well as some of their basic properties, due to Harish Chandra [19][20][21] and Satake [39].

**Theorem 3.1.** *Every spherical function on  $G_v$  is of the form*

$$\eta_\chi(g) = \int_{U_v} q_v^{\langle \chi, \nu_v(z(ug)) \rangle} \Delta_v^{-1/2}(z(ug)) dm_{U_v}(u) \quad (3.3)$$

where  $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$  and  $\Delta_v$  denotes the modular function of  $P_v$ .

Furthermore,  $\eta_\chi = \eta_{\chi'}$  if and only if  $\chi$  and  $\chi'$  are on the same orbit of the Weyl group  $\mathcal{W}_v$ .

Let  $\rho_v \in \mathcal{Z}_v^*$  denotes the character corresponding to the half-sum of positive roots. Then

$$\Delta_v(z) = q_v^{2\langle \rho_v, \nu_v(z) \rangle}, \quad z \in Z_v.$$

Let  $\Pi_v = \{\alpha\} \subset \mathcal{Z}_v^*$  denotes the system of simple roots corresponding to the parabolic subgroup  $P_v$ . We denote by  $\{\alpha'\}_{\alpha \in \Pi_v}$  the basis dual to  $\Pi_v$ . A character  $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$  is called *dominant* if  $\text{Re} \langle \chi, \alpha' \rangle \geq 0$  for all  $\alpha \in \Pi_v$ . Note that every  $\chi$  can be conjugated to a dominant one by  $\mathcal{W}_v$ . Therefore, in the discussion of spherical functions we may restrict our attention to dominant  $\chi$ 's.

We recall the following well-known properties of spherical functions:

**Lemma 3.2.** *For dominant  $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$ ,*

- the spherical function  $\eta_\chi$  is bounded if and only if

$$\operatorname{Re} \langle \chi, \alpha' \rangle \leq \langle \rho, \alpha' \rangle$$

for all  $\alpha \in \Pi_v$ .

- the spherical function  $\eta_\chi \in L^p(G_v)$  if and only if

$$\operatorname{Re} \langle \chi, \alpha' \rangle < (1 - 1/p) \langle \rho, \alpha' \rangle$$

for all  $\alpha \in \Pi_v$ .

We shall also need the following estimates.

**Lemma 3.3.** (i) For dominant  $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$ ,

$$|\eta_\chi(z)| \leq q_v^{\operatorname{Re} \langle \chi, \nu_v(z) \rangle} \eta_0(g), \quad g \in Z_v^-.$$

(ii) For every  $\epsilon > 0$ ,

$$\eta_0(g) \leq c_\epsilon \Delta_v^{-1/2+\epsilon}(g), \quad g \in Z_v,$$

with  $c_\epsilon$  bounded uniformly in  $v$ .

*Proof.* As to part (i), note that for the Archimedean case, this lemma is proved in [28, Proposition 7.15]. The proof for non-Archimedean case is similar. We have

$$|\eta_\chi(g)| = \int_{U_v} q_v^{\operatorname{Re} \langle \chi, \nu_v(z(ug)) \rangle} \Delta_v^{-1/2}(z(ug)) dm_{U_v}(u).$$

Since the double cosets  $U_v g U_v$  and  $N_v z(ug) U_v = N_v(ug) U_v$  have non-trivial intersection, it follows from [1, Proposition 4.4.4] that for dominant  $\chi$ ,

$$\operatorname{Re} \langle \chi, \nu_v(z(ug)) \rangle \leq \operatorname{Re} \langle \chi, \nu_v(z) \rangle.$$

This implies the first claim.

As to part (ii), the bound stated of the Harish Chandra  $\Xi$ -function (denoted  $\eta_0$  here) was established by Harish Chandra in both the Archimedean and non-Archimedean case (see also the discussion in [23]). The uniformity of the bound, namely the fact that  $c_\epsilon$  is independent of  $v$ , has been observed in [17, §6.2] and follows from the proof in [41, Thm. 4.2.1].  $\square$

**3.3. Unitary representations.** The spherical functions arise naturally as matrix coefficients of unitary representations. We denote by  $\hat{G}_v$  the unitary dual of  $G_v$  (i.e., the set of equivalence classes of irreducible unitary representations of the group  $G_v$ ) and by  $\hat{G}_v^1$  the spherical unitary dual (i.e., the subset consisting of spherical representations). An irreducible unitary representation is called spherical if it contains a nonzero  $U_v$ -invariant vector. Since the Hecke algebra  $\mathcal{H}_v$  is commutative, the subspace of  $U_v$ -invariant vectors in an irreducible unitary representation is at most one-dimensional. For  $\tau_v \in \hat{G}_v^1$ , we denote by  $w_{\tau_v}$  a unit  $U_v$ -invariant vector. Then the function

$$\eta_{\tau_v}(g) := \langle \tau_v(g) w_{\tau_v}, w_{\tau_v} \rangle, \quad g \in G_v,$$

is a spherical function on  $G_v$ . Moreover, different elements of  $\hat{G}_v^1$  give rise to different spherical functions. Therefore,  $\hat{G}_v^1$  can be identified with a subset of dominant  $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$  using Theorem 3.1.

More generally, for  $S \subset V_K$ , we denote by  $\hat{G}_S$  the unitary dual of  $G_S$  and by  $\hat{G}_S^1$  the spherical unitary dual (with respect to the subgroup  $U_S$ ). Every  $\tau_S \in \hat{G}_S$  is a restricted tensor product of the form  $\otimes'_{v \in S} \tau_v$  where  $\tau_v \in \hat{G}_v$  and  $\tau_v$  is spherical for almost all  $v$  (see e.g. [13]).

We define the spherical integrability exponent of a  $U_S$ -spherical unitary representation  $\tau_S : G_S \rightarrow \mathcal{U}(\mathcal{H})$  as

$$\mathfrak{q}(\tau_S, U_S) = \inf \{q > 0 : \forall U\text{-inv. } w \in \mathcal{H} : \langle \tau_S(g)w, w \rangle \in L^q(G)\}.$$

**Proposition 3.4.** *Let  $\tau_v$  for  $v \in S$  be a family consisting of irreducible spherical unitary representations of  $G_v$ . Then the integrability exponent of the restricted tensor product representation of  $G_S$  satisfies*

$$\mathfrak{q}(\otimes'_{v \in S} \tau_v, U_S) \leq (1 + \sigma_S) \sup_{v \in S} \mathfrak{q}(\tau_v, U_v),$$

where  $\sigma_S$  denotes the exponent of the subset  $S$  defined in (1.11).

*Proof.* We have to show that the spherical function  $\eta = \prod_{v \in S} \eta_{\tau_v}$  is in  $L^q(G_S)$  for  $q > (1 + \sigma_S)p$  where  $p > p(\tau_v, U_v)$  for all  $v \in S$ . To derive the required estimate we use integration calculated in terms of the Cartan decomposition (3.2). Recall the volume estimate (see [17, §6.2] for a discussion) :

$$m_{G_v}(U_v t U_v) \leq c \Delta_v(t), \quad t \in T_v^-, \quad (3.4)$$

which is uniform over  $v$ . Let the representations  $\tau_v$  correspond to dominant  $\chi_v \in \mathcal{Z}_v \otimes \mathbb{C}$ . Since the spherical function  $\eta_{\chi_v}$  is in  $L^p(G_v)$ , it follows from Lemma 3.2 that for  $t \in T_v^-$ ,

$$|q_v^{\langle \chi_v, \nu_v(t) \rangle}| \leq \Delta_v(t)^{\frac{1}{2} - \frac{1}{p}}.$$

Hence, Lemma 3.3 implies the estimate

$$|\eta_{\chi_v}(t)| \leq c_\epsilon \Delta_v(t)^{-\frac{1}{p} + \epsilon} \quad (3.5)$$

for every  $\epsilon > 0$ . Combining (3.4) and (3.5), we deduce that

$$\begin{aligned}
\int_{G_v} |\eta_{\chi_v}(g)|^q dm_{G_v}(g) &= \sum_{z \in T_v^- / T_v^\circ} |\eta_{\chi_v}(t)|^q m_{G_v}(U_v t U_v) \\
&\leq 1 + \sum_{z \in T_v^- / T_v^\circ - \{e\}} (cc_\epsilon^q) \Delta_v(t)^{-q/p+q\epsilon+1} \\
&\leq 1 + \sum_{z \in T_v^- / T_v^\circ - \{e\}} (cc_\epsilon^q) \left( \prod_{\chi \in \Pi_v} |\chi(t_v)|_v \right)^{-q/p+q\epsilon+1} \\
&\leq 1 + \sum_{i_1, \dots, i_r \in \mathbb{Z}_+, (i_1, \dots, i_r) \neq 0} (cc_\epsilon^q) q_v^{(-q/p+q\epsilon+1) \sum_{j=1}^r i_j} \\
&= 1 + O_\epsilon(q_v^{-q/p+q\epsilon+1})
\end{aligned}$$

for every  $\epsilon > 0$ . It follows from the definition of density  $\sigma_S$  that the partial Euler product  $\prod_{v \in S} (1 - q_v^{-s})^{-1}$  converges for  $s > \sigma_S$ . Therefore, we conclude that

$$\prod_{v \in S} \int_{G_v} |\eta_{\tau_v}|^q dm_{G_v} < \infty$$

provided that  $-q/p + 1 < -\sigma_S$ . This completes the proof.  $\square$

We denote by  $\pi_S$  the unitary representation of  $G_S$  on  $L_{00}^2(G_{V_K}/G(K))$ . Proposition 3.4 implies the following estimate on the integrability exponent of  $\pi_S$ .

**Corollary 3.5.** *For every  $S \subset V_K$ ,*

$$\mathfrak{q}(\pi_S, U_S) \leq \mathfrak{q}_S(\mathbb{G}),$$

where  $\mathfrak{q}_S(\mathbb{G})$  is defined in (1.12).

Moreover, we also observe that the proof of Proposition 3.4 gives the following uniform estimate, where  $\tau \prec \pi_S$  denotes weak containment (see e.g. [10] for a discussion).

**Corollary 3.6.** *For every  $S \subset V_K$  and  $q > \mathfrak{q}_S(\mathbb{G})$ ,*

$$\sup \left\{ \|\eta_\tau\|_q : \tau \in \hat{G}_S^1, \tau \prec \pi_S \right\} < \infty.$$

We remark that the foregoing uniform bound on  $L^q$ -norm of spherical functions associated with irreducible representation implies the equivalence of the two definitions given for  $\mathfrak{q}_v$  in §1.2. Indeed, a general diagonal spherical matrix coefficient of  $\pi_v$  is the convex average of spherical functions associated with irreducible spherical representations of  $G_v$ . Hence by Jensen's inequality its  $L^q$ -norm is bounded by the supremum of the  $L^q$ -norms of the spherical functions, which is finite by Corollary 3.6.

We now use the foregoing spectral considerations to obtain an operator norm estimate. First observe

**Proposition 3.7.** *Let  $\beta$  be a bi- $U_S$ -invariant finite Borel measure on  $G_S$  and  $\pi : G_S \rightarrow \mathcal{U}(\mathcal{H})$  a strongly continuous unitary representation. Then*

$$\|\pi(\beta)\| \leq \sup \left\{ \sqrt{\beta^*(\eta_\tau)\beta(\eta_\tau)} : \tau \in \hat{G}_S^1, \tau \prec \pi \right\}.$$

*Proof.* To bound  $\|\tau(\beta)\|$  for irreducible representations weakly contained in  $\pi$ , we first consider the case when the measure  $\beta$  is symmetric. Let  $\mathcal{H}_\tau^{U_S}$  denote the subspace of  $\tau(U_S)$ -invariant vectors. It is clear that

$$\tau(\beta)\mathcal{H}_\tau \subset \mathcal{H}_\tau^{U_S}.$$

In particular,  $\tau(\beta) = 0$  for  $\tau \notin \hat{G}_S^1$ . Since  $\beta$  is symmetric, the operator  $\tau(\beta)$  is self-adjoint, and hence,

$$\tau(\beta)(\mathcal{H}_\tau^{U_S})^\perp \subset (\mathcal{H}_\tau^{U_S})^\perp.$$

This implies that

$$\|\tau(\beta)\| = \int_{G_S} \langle \tau(\beta)w_\tau, w_\tau \rangle d\beta(g) = \beta(\eta_\tau)$$

where  $w_\tau \in \mathcal{H}_\tau^{U_S}$  with  $\|w_\tau\| = 1$ . This completes the proof when  $\beta$  is symmetric. In general, we have

$$\|\pi(\beta)\|^2 = \|\pi(\beta^* * \beta)\|$$

Hence, by the previous argument,

$$\|\pi(\beta)\| \leq \sup \left\{ \sqrt{(\beta^* * \beta)(\eta_\tau)} : \tau \in \hat{G}_S^1, \tau \prec \pi \right\}.$$

Since  $\beta$  is bi- $U_S$ -invariant, it follows from the properties of spherical functions that

$$(\beta^* * \beta)(\eta_\tau) = \beta^*(\eta_\tau)\beta(\eta_\tau),$$

which implies the claim.  $\square$

We can now obtain the following operator norm estimate.

**Proposition 3.8.** *Let  $\beta$  be a Haar-uniform probability measure supported on a bi- $U_S$ -invariant bounded subset  $B \subset G_S$ . Then*

$$\|\pi_S(\beta)\| \ll_{S,\delta} m_S(B)^{-\frac{1}{\mathfrak{q}_S(\mathbb{G})} + \delta}.$$

for every  $\delta > 0$ .

*Proof.* In view of Proposition 3.7 we need to establish a uniform estimate

$$\beta(\eta_\tau) = \frac{1}{m_S(B)} \int_B \eta_\tau(g) dm_S(g),$$

where  $\eta_\tau$  is the spherical function of a representation  $\tau \in \hat{G}_S^1$  which is weakly contained in  $\pi_S$ . By Hölder's inequality, for  $q > \mathfrak{q}_S(\mathbb{G})$ , we have

$$\beta(\eta_\tau) \leq \frac{1}{m_S(B)} \|\chi_B\|_{(1-1/q)^{-1}} \|\eta_\tau\|_q = m_S(B)^{-1/q} \|\eta_\tau\|_q.$$

Now the claim follows from Proposition 3.7, Corollary 3.5 and Corollary 3.6.  $\square$

#### 4. MEAN ERGODIC THEOREMS

We keep the notation from the previous section. In particular,  $G$  denotes a connected semisimple group defined over a number field  $K$ . Our aim is to prove the mean ergodic theorem for the space  $\Upsilon := G_{V_K}/G(K)$  equipped with the invariant probability measure  $\mu$ . For  $S \subset V_K$ , we consider the natural action of  $G_{V_K \setminus S}$  on  $\Upsilon$  and, given a Haar-uniform probability measure  $\beta$  on  $G_{V_K \setminus S}$ , the averaging operator

$$\pi_{V_K \setminus S}(\beta)\phi(\varsigma) = \int_{G_{V_K \setminus S}} \phi(g^{-1}\varsigma) d\beta(g), \quad \phi \in L^2(\Upsilon). \quad (4.1)$$

We first consider the case when the group  $G$  is simply connected. For simply connected groups, the mean ergodic theorem admits a simple version, but the general case requires more delicate considerations because of the presence of nontrivial automorphic characters.

Let  $\mathcal{X}_{aut}(G_{V_K})$  be the set of automorphic characters, that is, the set consisting of continuous unitary characters  $\chi$  of  $G_{V_K}$  such that  $\chi(G(K)) = 1$ .

**Lemma 4.1.** *If  $G$  is simply connected, then  $\mathcal{X}_{aut}(G_{V_K}) = 1$ .*

*Proof.* The group  $G$  is isotropic over  $K_v$  for some  $v \in V_K$  (see [37, Theorem 6.7]). Then the group  $G_v$  coincides with its commutator (see [37, §7.2]), and hence  $\chi(G_v) = 1$  for every character  $\chi$  of  $G_{V_K}$ . Since by the strong approximation property  $G(K)G_v$  is dense in  $G_{V_K}$ , the claim follows.  $\square$

Lemma 4.1 implies that for simply connected groups,  $L^2_{00}(G_{V_K}/G(K))$  is the space of functions with zero integral. Hence, Proposition 3.8 gives

**Theorem 4.2** (mean ergodic theorem). *Assume that  $G$  is simply connected, and let  $\beta$  be the Haar-uniform probability measure supported on bi- $U_{V_K \setminus S}$ -invariant subset  $B$  of  $G_{V_K \setminus S}$ . Then for every  $\phi \in L^2(\Upsilon)$ ,*

$$\left\| \pi_{V_K \setminus S}(\beta)\phi - \int_{\Upsilon} \phi d\mu \right\|_2 \ll_{S,\delta} m_{V_K \setminus S}(B)^{-\frac{1}{q_{V_K \setminus S}(G)} + \delta}$$

for every  $\delta > 0$ .

The general case of Theorem 4.2 requires two auxiliary lemmas.

**Lemma 4.3.** *Let  $p : \tilde{G} \rightarrow G$  be a simply connected cover of a connected semisimple group defined over  $K$ . Then for every  $S \subset V_K$ ,  $p(\tilde{G}_S)$  is a normal co-Abelian subgroup of  $G_S$ . Moreover, if  $S$  is finite, then  $p(\tilde{G}_S)$  has finite index in  $G_S$ .*

*Proof.* For every  $v \in V_K$ , we have the exact sequence in Galois cohomology

$$\tilde{G}_v \xrightarrow{p} G_v \rightarrow H^1(K_v, \ker(p)).$$

Since  $H^1(K_v, \ker(p))$  is Abelian,  $p(\tilde{G}_v)$  is a normal co-Abelian subgroup of  $G_v$ . This implies that  $p(\tilde{G}_v)$  contains the commutator of  $G_v$ . Since  $H^1(K_v, \ker(p))$  is finite group (see [37, §6.4]),  $p(G_v)$  has finite index in  $G_v$ . Also for almost all  $v \in V_K$ , we have the exact sequence

$$\tilde{G}(O_v) \xrightarrow{p} G(O_v) \rightarrow H^1(K_v^{ur}/K_v, \ker(p)(O_v^{ur}))$$

(see [37, Proposition 6.8]), where  $K_v^{ur}$  denotes the unramified closure of  $K_v$  and  $O_v^{ur}$  denotes its ring of integers. Therefore,  $p(\tilde{G}(O_v))$  is a normal co-Abelian subgroup of  $G(O_v)$ , and in particular,  $p(\tilde{G}(O_v))$  contains the commutator of  $G(O_v)$ .

The group  $G_S$  is a union of the subgroups  $G_{S,S'}$  defined by

$$G_{S,S'} := \left( \prod_{v \in S'} G_v \right) \left( \prod_{v \in S'} G(O_v) \right),$$

as  $S'$  runs over finite subsets of  $S$ . It follows from the first paragraph that  $p(\tilde{G}_S)$  contains the commutator of  $G_{S,S'}$  for every finite  $S' \subset S$ . Hence,  $p(\tilde{G}_S)$  contains the commutator of  $G_S$ , and  $p(\tilde{G}_S)$  is a normal co-Abelian subgroup of  $G_S$ .  $\square$

Let  $S'$  be a finite subset of  $V_K$ . For an open subgroup  $U$  of  $G_{V_K^f \setminus S'}$ , we denote by  $\mathcal{X}_{aut}(G_{V_K})^U$  the subset of  $\mathcal{X}_{aut}(G_{V_K})$  consisting of  $U$ -invariant characters. Let  $G^U$  denote the kernel of  $\mathcal{X}_{aut}(G_{V_K})^U$  in  $G_{V_K}$ .

**Lemma 4.4.** *The group  $\mathcal{X}_{aut}(G_{V_K})^U$  is finite, and  $G^U$  has finite index in  $G_{V_K}$ .*

*Proof.* Let  $p : \tilde{G} \rightarrow G$  be the simply connected cover. For every  $\chi \in \mathcal{X}_{aut}(G_{V_K})$ , we have  $\chi \circ p \in \mathcal{X}_{aut}(\tilde{G}_{V_K})$ . Hence, it follows from Lemma 4.1 that  $\chi \circ p = 1$ . We conclude that every  $\chi \in \mathcal{X}_{aut}(G_{V_K})^U$  vanishes on  $H := G(K)p(\tilde{G}_{V_K})U$ . By Lemma 4.3,  $p(\tilde{G}_{V_K})$  is a normal co-Abelian subgroup of  $G_{V_K}$ . Hence it follows that  $H$  is a normal co-Abelian subgroup as well. We claim that it has finite index in  $G_{V_K}$ .

For every  $v \in V_K$ , we have the exact sequence

$$\tilde{G}_v \rightarrow G_v \rightarrow H^1(K_v, \ker(p)),$$

where the last term is finite by [37, §6.4]. This shows that  $p(\tilde{G}_{V_K^\infty \cup S'})$  has finite index in  $G_{V_K^\infty \cup S'}$ . The number of double cosets (i.e., the class number) of the subgroups  $G(K)$  and  $G_{V_K^\infty} U_{V_K^f \cap S'} U$  in  $G_{V_K}$  is finite (see [37, §8.1]). Then the number of double cosets of  $G(K)$  and  $p(\tilde{G}_{V_K^\infty \cup S'})U$  in  $G_{V_K}$  is finite as well. From this we conclude that the number of double cosets of  $G(K)$  and  $p(\tilde{G}_{V_K})U$  in  $G_{V_K}$  is finite, and since  $p(\tilde{G}_{V_K})$  is co-Abelian, the factor group  $G_{V_K}/H$  is finite.

We have shown above that every  $\chi \in \mathcal{X}_{\text{aut}}(G_{V_K})^U$  factors through the finite factor group  $G_{V_K}/H$ . This implies that  $\mathcal{X}_{\text{aut}}(G_{V_K})^U$  is finite and  $G^U$  has finite index in  $G_{V_K}$ , as required.  $\square$

**Theorem 4.5** (mean ergodic theorem). *Let  $S$  be a subset of  $V_K$  and  $S'$  a finite subset of  $S$ . Let  $U^0$  be a finite index subgroup of  $U_{V_K^f \cap (S \setminus S')}$  and  $U = U_{V_K^f \setminus S} U^0$ . Let  $B$  be a bounded measurable subset of  $G_{V_K \setminus S} \cap G^U$  which is  $U_{V_K^f \setminus S}$ -biinvariant and  $\beta$  the Haar-uniform probability measure supported on the subset  $U^0 B$  of  $G_{(V_K \setminus S) \cup (V_K^f \setminus S')}$ . Then for every  $\phi \in L^2(\Upsilon)$  such that  $\text{supp}(\phi) \subset G^U$ , we have*

$$\left\| \pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)\phi - \left( \int_{\Upsilon} \phi d\mu \right) \xi_U \right\|_2 \ll_{U^0, S, \delta} m_{V_K \setminus S}(B)^{-\frac{1}{q_{V_K \setminus S}(G)} + \delta} \|\phi\|_2$$

for every  $\delta > 0$ , where  $\xi_U$  is the function on  $\Upsilon$  such that  $\xi_U = |G_{V_K} : G^U|$  on the open set  $G^U/G(K) \subset \Upsilon$  and  $\xi_U = 0$  otherwise.

*Proof.* We have the decomposition

$$L^2(\Upsilon) = \mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \mathcal{H}^2$$

where  $\mathcal{H}^0$  is the orthogonal complement of  $\mathcal{X}_{\text{aut}}(G_{V_K})$ ,  $\mathcal{H}^1$  is the (finite dimensional) span for  $\mathcal{X}_{\text{aut}}(G_{V_K})^U$ , and  $\mathcal{H}^2$  is the orthogonal complement of  $\mathcal{H}^0 \oplus \mathcal{H}^1$ . For  $\phi \in L^2(\Upsilon)$ , we have the corresponding decomposition

$$\phi = \phi_0 + \phi_1 + \phi_2.$$

We observe that the set  $\mathcal{X}_{\text{aut}}(G_{V_K})^U$  can be identified with the group of characters of the finite Abelian group  $G_{V_K}/G^U$ . Hence, it follows that

$$\sum_{\chi \in \mathcal{X}_{\text{aut}}(G_{V_K})^U} \chi = \xi_U.$$

Since  $\mathcal{X}_{\text{aut}}(G_{V_K})^U$  forms an orthonormal basis of  $\mathcal{H}^1$ ,

$$\phi_1 = \sum_{\chi \in \mathcal{H}^1} \langle \phi, \chi \rangle \chi = \left( \int_{\Upsilon} \phi d\mu \right) \sum_{\chi \in \mathcal{X}_{\text{aut}}(G_{V_K})^U} \chi = \left( \int_{\Upsilon} \phi d\mu \right) \xi_U.$$

Since the measure  $\beta$  is  $U$ -invariant, for every  $\chi \in \mathcal{X}_{\text{aut}}(G_{V_K})$  and  $u \in U$ , we have

$$\pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)\chi = \chi(u)\pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)\chi.$$

Therefore, if  $\chi$  is not  $U$ -invariant, then

$$\pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)\chi = 0.$$

This implies that

$$\pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)|_{\mathcal{H}^2} = 0.$$

Also, since  $\text{supp}(\beta) \subset G^U$ , we have

$$\pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)|_{\mathcal{H}^1} = \text{id}.$$

We conclude that

$$\pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)\phi - \left( \int_{\Upsilon} \phi d\mu \right) \xi_U = \pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)\phi_0.$$

By Jensen's inequality,

$$\left\| \pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)\phi_0 \right\|_2 \ll_{U^0} \left\| \pi_{V_K \setminus S}(\beta')\phi_0 \right\|_2,$$

where  $\beta'$  is the Haar-uniform probability measure supported on  $B \subset G_{V_K \setminus S}$ . Therefore, by Proposition 3.8,

$$\left\| \pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta)\phi_0 \right\|_2 \ll_{U^0, S, \delta} m_{V_K \setminus S}(B)^{-\frac{1}{q_{V_K \setminus S}(G)} + \delta} \|\phi_0\|_2$$

for every  $\delta > 0$ . This implies the claim.  $\square$

## 5. THE DUALITY PRINCIPLE

**5.1. Duality and almost sure approximation on the group variety.** In this section we develop an instance of the duality principle in homogeneous spaces, which is analyzed more generally in [18]. Heuristically speaking, the principle asserts that given a lattice subgroup  $\Gamma$  of  $G$ , and another closed subgroup  $H$ , the effective estimates on the ergodic behavior of  $\Gamma$ -orbits in  $G/H$  can be deduced from the effective estimates on the ergodic behavior of  $H$ -orbits in  $G/\Gamma$ . This accounts for the essential role that the automorphic representation and its restriction to subgroups plays in our considerations.

Let  $G \subset \mathrm{GL}_n$  be a linear algebraic group defined over a number field  $K$ . In this section we relate the problem of Diophantine approximation on  $G$  and on its homogeneous spaces with “shrinking target” properties of suitable dynamical systems. More precisely, we consider the space  $\Upsilon = G_{V_K}/G(K)$  equipped with the natural translation action of the group  $G_{V_K \setminus S}$  where  $S \subset V_K$ . We show that a point in  $G_S$  can be approximated by rational points in  $G(K)$  with given accuracy provided that the corresponding orbit of  $G_{V_K \setminus S}$  in  $\Upsilon$  can be used to approximate the identity coset with a suitable accuracy.

Let  $\|\cdot\|_v$  be the maximum norm on  $K_v^n$ , that is,  $\|x\|_v = \max_i |x_i|_v$  for  $x \in K_v^n$ . We observe that for  $g \in \mathrm{GL}_n(K_v)$ ,

$$\|g \cdot x\|_v \leq c_v(g) \|x\|_v, \quad (5.1)$$

where  $c_v(g) = \max_{i,j} |g_{ij}|_v$  for  $v \in V_K^f$  and  $c_v(g) = n \max_{i,j} |g_{ij}|_v$  for  $v \in V_K^\infty$ . Note that  $c_v(g)$  is bounded on compact subsets of  $\mathrm{GL}_n(K_v)$ , and  $c_v(g) = 1$  for  $g \in \mathrm{GL}_n(O_v)$ .

The height function  $H$  on  $G_{V_K}$  is defined by

$$H(g) = \prod_{v \in V_K} \max_{i,j} (1, |g_{ij}|_v), \quad g \in G_{V_K}. \quad (5.2)$$

This extends the definition of the height function from (1.2).

We note for future reference that given a bounded  $\Omega \subset G_{V_K}$ ,

$$H(b_1 g b_2) \ll_{\Omega} H(g) \quad (5.3)$$

for every  $b_1, b_2 \in \Omega$  and  $g \in G_{V_K}$ .

For  $v \in V_K$  and  $\epsilon > 0$ , we set

$$\mathcal{O}_v(\epsilon) := \{g \in G_v : \|g - e\|_v \leq \epsilon\},$$

and for any  $S \subset V_K$ ,

$$\mathcal{O}_S(\epsilon) := \prod_{v \in S} \mathcal{O}_v(\epsilon)^3. \quad (5.4)$$

Let  $m$  be the Tamagawa measure on  $G_{V_K}$ . We refer to [43, §14] for a detailed construction of  $m$ . Given a nonzero left-invariant differential  $K$ -form of degree  $\dim(G)$  on  $G$ , one defines the corresponding local measures  $m_v$  on  $G_v$ . We assume that the product  $\prod_{v \in V_K^f} m_v(G(O_v))$  converges. Then the Tamagawa measure is given by

$$m = \prod_{v \in V_K} m_v.$$

We note that when  $G$  is a semisimple group, then the product of  $m_v(G(O_v))$  is known to converge (see [43, §14]). This property is crucial for us to prove results on Diophantine approximation that are uniform over finite subsets  $S'$  of  $V_K$ .

**Remark 5.1.** When the product  $\prod_{v \in V_K^f} m_v(G(O_v))$  diverges, then the Tamagawa measure is defined by  $m = \prod_{v \in V_K} \lambda_v m_v$  where  $\lambda_v > 0$  are suitably chosen convergence factors. The arguments of this section can be modified to deal with this case as well. However, one loses uniformity over subsets  $S'$ .

We use the following estimates for the local measures. Recall that  $I_v = \{q_v^n\}_{n>0}$  for  $v \in V_K^f$  and  $I_v = (0, 1)$  for  $v \in V_K^\infty$ .

**Lemma 5.2.** *There exist  $d_v \in (0, 1]$  such that  $d_v = 1$  for almost all  $v$  and*

$$m_v(\mathcal{O}_v(\epsilon)) \geq d_v \epsilon^{r_v \dim(G)}$$

for all  $\epsilon \in I_v$ .

*Proof.* The measures  $m_v$  in local coordinate are given by

$$|h(x)|_v (dx_1 \wedge \cdots \wedge dx_d)_v,$$

where  $h(x)$  is a convergent power series and  $d = \dim(G)$ . This implies the estimate  $m_v(\mathcal{O}_v(\epsilon)) \geq d_v \epsilon^{r_v \dim(G)}$  with some  $d_v \in (0, 1]$ . We note that for  $v \in V_K^f$ , the neighbourhoods  $\mathcal{O}_v(\epsilon)$  with  $\epsilon = q_v^{-n}$  are precisely the congruence subgroups

$$G^{(n)}(O_v) = \{g \in G(O_v) : g = e \pmod{\mathfrak{p}_v^n}\},$$

---

<sup>3</sup>Note that  $\mathcal{O}_v(\epsilon) = \mathcal{O}_{\{v\}}(\epsilon)$  and  $\mathcal{O}_v(1) = O_v \neq O_{\{v\}}$ .

where  $\mathfrak{p}_v$  denotes the maximal ideal of  $O_v$  and  $q_v$  denotes the cardinality of the residue field. The computation in [43, §14.1] shows that for almost all  $v$  and  $n \geq 1$ ,

$$m_v(\mathbb{G}^{(n)}(O_v)) = q_v^{-nd}.$$

This implies the claim.  $\square$

The measure  $m$  on  $G_{V_K}$  defines the probability Haar measure on the factor space  $\Upsilon = G_{V_K}/\mathbb{G}(K)$ , which we denote by  $\mu$ . The measure  $\mu$  is defined so that for measurable subsets  $B \subset G_{V_K}$  that project injectively on  $\Upsilon$ , one has

$$\mu(B\mathbb{G}(K)) = \frac{1}{m(G_{V_K}/\mathbb{G}(K))} m(B).$$

Now we establish a series of results that provide a connection between dynamics on the space  $\Upsilon$  and Diophantine approximation. The first two results deal with group varieties and the last two results with general homogeneous varieties.

*Convention:* We shall use the product decompositions of the adèle group defined by any subset  $Q \subset V_K$

$$G_{V_K} = G_{V_K \setminus Q} \times G_Q.$$

In order to simplify notation we identify a subset  $B \subset G_{V_K \setminus Q}$  with the subset  $B \times \{e\}$  of  $G_{V_K}$ .

In Propositions 5.3 and 5.4 below, all implicit constants may depend on the set of places  $S$  and the bounded subset  $\Omega$ , but are independent of other parameters unless stated otherwise.

**Proposition 5.3.** *Fix  $S \subset V_K$  and a bounded subset  $\Omega$  of  $G_S$ . Then there exists a family of measurable subsets  $\Phi_\epsilon$  of  $\Upsilon$  indexed by  $\epsilon = (\epsilon_v)_{v \in S'}$ , where  $S'$  is a finite subset of  $S$  and  $\epsilon_v \in I_v$ , that satisfies*

$$\mu(\Phi_\epsilon) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(\mathbb{G})} \quad (5.5)$$

and the following property holds:

if for a subset  $B \subset G_{V_K \setminus S}$ ,  $\epsilon = (\epsilon_v)_{v \in S'}$  as above,  $x \in \Omega$  and  $\varsigma = (e, x^{-1})\mathbb{G}(K) \in \Upsilon$ , we have

$$B^{-1}\varsigma \cap \Phi_\epsilon \neq \emptyset, \quad (5.6)$$

then there exists  $z \in \mathbb{G}(K)$  such that

$$\mathbb{H}(z) \ll \max_{b \in B} \mathbb{H}(b) \quad (5.7)$$

and

$$\begin{aligned} \|x_v - z\|_v &\leq \epsilon_v && \text{for all } v \in S', \\ \|x_v - z\|_v &\leq 1 && \text{for all } v \in S \setminus S'. \end{aligned} \quad (5.8)$$

*Proof.* Since  $\Omega$  is a bounded subset of  $G_S$ , there exists finite  $R \subset S$  such that

$$\Omega \subset G_R \times \prod_{v \in S \setminus R} G(O_v).$$

We fix constants  $c_v \geq 1$ ,  $v \in S$ , such that  $c_v \geq \sup_{g \in \Omega} c_v(g_v^{-1})$ , where  $c_v(\cdot)$  is the constant given by (5.1). We can take  $c_v = 1$  for  $v \in S \setminus R$ .

We set  $\epsilon_v = 1$  for  $v \in S \setminus S'$  and  $\delta_v = \epsilon_v/c_v$  for  $v \in S$ ,  $\delta_v = 1$  for  $v \in V_K \setminus S$ . Let

$$\tilde{\Phi}_\epsilon = \prod_{v \in V_K} \mathcal{O}_v(\delta_v) \subset G_{V_K} \quad \text{and} \quad \Phi_\epsilon = \tilde{\Phi}_\epsilon G(K) \subset \Upsilon.$$

Since  $\delta_v \leq 1$  for all  $v \in V_K^f$ , we have  $\mathcal{O}_v(\delta_v) \subset G(O_v)$  for every  $v \in V_K^f$ . Therefore, if  $z \in G(K)$  satisfies

$$\tilde{\Phi}_\epsilon z \cap \tilde{\Phi}_\epsilon \neq \emptyset,$$

then  $z \in G(O)$ , where  $O$  denotes the ring of integers of  $K$ . Since the image of the diagonal embedding of  $O$  in  $\prod_{v \in V_K^\infty} K_v$  is discrete, there exists  $\epsilon_0 > 0$  (depending only on the number field  $K$ ) such that

$$\mathcal{O}_{V_K^\infty}(\epsilon_0)z \cap \mathcal{O}_{V_K^\infty}(\epsilon_0) = \emptyset \quad \text{for every } z \in G(O), z \neq e.$$

Then when  $\epsilon_v \in (0, \epsilon_0)$  for all  $v \in V_K^\infty$ , we have

$$\tilde{\Phi}_\epsilon z \cap \tilde{\Phi}_\epsilon = \emptyset \quad \text{for every } z \in G(K), z \neq e. \quad (5.9)$$

This implies that for such  $\epsilon$ ,

$$\begin{aligned} \mu(\Phi_\epsilon) &\gg m(\tilde{\Phi}_\epsilon) = \prod_{v \in V_K} m_v(\mathcal{O}_v(\delta_v)) \\ &= \left( \prod_{v \in S' \cup R} m_v(\mathcal{O}_v(\epsilon_v/c_v)) \right) \left( \prod_{v \in V_K \setminus S'} m_v(\mathcal{O}_v(1)) \right). \end{aligned}$$

Since the product  $\prod_{v \in V_K^f} m_v(G(O_v))$  converges, we deduce using Lemma 5.2 that

$$\mu(\Phi_\epsilon) \gg \prod_{v \in S' \cup R} m_v(\mathcal{O}_v(\epsilon_v/c_v)) \gg \prod_{v \in S'} d_v(\epsilon_v/c_v)^{r_v \dim(G)} \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(G)}.$$

Since  $c_v = 1$  for  $v \notin R$ , the implied constants here are independent of  $S'$  (and depend only on  $\Omega$ ). This proves estimate (5.5) provided that  $\epsilon_v \in (0, \epsilon_0)$  for all  $v \in V_K^\infty$ .

To prove this estimate in general, we set  $\epsilon'_v = \epsilon_0 \epsilon_v \leq \epsilon_0$  for  $v \in V_K^\infty$  and  $\epsilon'_v = \epsilon_v$  for  $v \in V_K^f$ . Then

$$\begin{aligned} \mu(\Phi_\epsilon) &\geq \mu(\Phi_{\epsilon'}) \gg \prod_{v \in S'} (\epsilon'_v)^{r_v \dim(G)} \\ &\geq \left( \prod_{v \in S' \cap V_K^\infty} \epsilon_0^{r_v \dim(G)} \right) \prod_{v \in S'} \epsilon_v^{r_v \dim(G)}. \end{aligned}$$

This completes the proof of (5.5).

Now we suppose that (5.6) holds. Then for some  $b \in B$  and  $z \in G(K)$ ,

$$(b^{-1}z, x^{-1}z) \in \tilde{\Phi}_\epsilon.$$

Hence, for  $v \in S$ , we have  $x_v^{-1}z \in \mathcal{O}_v(\delta_v)$  and

$$\|x_v - z\|_v \leq c_v \|x_v^{-1}z - e\|_v \leq c_v \delta_v.$$

This proves (5.8).

Finally, we observe that

$$z \in \Omega(b, e) \tilde{\Phi}_\epsilon.$$

Hence, it follows from (5.3) that

$$H(z) \ll H(b).$$

This shows (5.7) and completes the proof of the proposition.  $\square$

**5.2. Duality and approximation at every point on the group variety.** In order to prove the uniform version of our main theorem (Theorem 1.3(ii)), condition (5.6) in Proposition 5.3 needs to be relaxed. This is achieved by the following proposition:

**Proposition 5.4.** *Fix  $S \subset V_K$  and a bounded subset  $\Omega$  of  $G_S$ . Then there exists a family of measurable subsets  $\tilde{\Psi}_\epsilon$  of  $G_{V_K}$  indexed by  $\epsilon = (\epsilon_v)_{v \in S'}$ , where  $S'$  is a finite subset of  $S$  and  $\epsilon_v \in I_v$ , that satisfies*

$$\mu(\tilde{\Psi}_\epsilon G(K)) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(G)}, \quad (5.10)$$

$$\mu(\tilde{\Psi}_\epsilon^{-1}(e, x^{-1})G(K)) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(G)} \quad \text{for all } x \in \Omega, \quad (5.11)$$

and the following property holds:

if for  $B \subset G_{V_K \setminus S}$ ,  $\epsilon = (\epsilon_v)_{v \in S'}$  as above,  $x \in \Omega$  and  $\varsigma = (e, x^{-1})G(K) \in \Upsilon$ , we have

$$B^{-1} \tilde{\Psi}_\epsilon^{-1} \varsigma \cap \tilde{\Psi}_\epsilon G(K) \neq \emptyset, \quad (5.12)$$

then there exists  $z \in G(K)$  such that

$$H(z) \ll \max_{b \in B} H(b)$$

and

$$\begin{aligned} \|x_v - z\|_v &\leq \epsilon_v \quad \text{for all } v \in S', \\ \|x_v - z\|_v &\leq 1 \quad \text{for all } v \in S \setminus S'. \end{aligned} \quad (5.13)$$

*Proof.* Similarly to (5.1), for every  $g \in \mathrm{GL}_n(K_v)$  and  $y \in \mathrm{M}_n(K_v)$ ,

$$\|g \cdot y \cdot g^{-1}\|_v \leq c'_v(g) \|y\|_v \quad (5.14)$$

where  $c'_v(g) \geq 1$ ,  $c'_v(g) = 1$  for  $g \in \mathrm{GL}_n(O_v)$ , and  $c'_v(g)$  is uniformly bounded over bounded subsets. We set  $c'_v = \sup_{g \in \Omega} c'_v(g)$ .

Let  $c_v \geq 1$  be defined as in Proposition 5.3 and  $\epsilon_v = 1$  for  $v \in S \setminus S'$ . We set  $\delta_v = 1$  for  $v \in V_K \setminus S$ ,  $\delta_v = \epsilon_v / ((2n+1)c_v c'_v)$  for Archimedean  $v \in S$ , and  $\delta_v = \epsilon_v / (c_v c'_v)$  for non-Archimedean  $v \in S$ . Note  $\delta_v = 1$  for almost all  $v$ . Let

$$\tilde{\Psi}_\epsilon = \prod_{v \in V_K} \mathcal{O}_v(\delta_v).$$

Since  $\delta_v = 1$  for all non-Archimedean  $v \notin S' \cup R$ , estimate (5.10) can be established by the same argument as in the proof of Proposition 5.3. To prove (5.11), we observe that

$$\mu(\tilde{\Psi}_\epsilon^{-1}(e, x^{-1})\mathrm{G}(K)) = \mu((e, x)\tilde{\Psi}_\epsilon^{-1}(e, x^{-1})\mathrm{G}(K)),$$

and by (5.14),

$$(e, x)\tilde{\Psi}_\epsilon^{-1}(e, x^{-1}) \supset \prod_{v \in V_K} \mathcal{O}_v(\delta_v / c'_v)^{-1}.$$

Hence, estimate (5.11) can be proved similarly to (5.10).

Suppose that (5.12) holds. Then for some  $b \in B$ ,  $f' \in \prod_{v \in V_K \setminus S} \mathcal{O}_v(\delta_v)$ ,  $f \in \prod_{v \in S} \mathcal{O}_v(\delta_v)$ , and  $z \in \mathrm{G}(K)$ , we have

$$(b^{-1}(f')^{-1}z, f^{-1}x^{-1}z) \in \tilde{\Psi}_\epsilon.$$

For  $v \in S \cap V_K^\infty$ , we have  $f_v^{-1}x_v^{-1}z \in \mathcal{O}_v(\delta_v)$  and  $f_v \in \mathcal{O}_v(\delta_v)$ . We observe that  $\|f_v\|_v \leq 2$ , and for every  $a \in \mathrm{M}_n(K_v)$ ,

$$\|f_v a\|_v \leq n \|f_v\|_v \|a\|_v \leq (2n) \|a\|_v.$$

Hence,

$$\begin{aligned} \|x_v - z\|_v &\leq \|x_v - x_v f_v\|_v + \|x_v f_v - z\|_v \\ &\leq c_v (\|e - f_v\|_v + (2n) \|e - f_v^{-1} x_v^{-1} z\|_v) \leq (2n+1) c_v \delta_v. \end{aligned}$$

The case  $v \in S \cap V_K^f$  is treated similarly. We observe that  $\|f_v\|_v \leq 1$ , and for every  $a \in \mathrm{M}_n(K_v)$ ,

$$\|f_v a\|_v \leq \|f_v\|_v \|a\|_v \leq \|a\|_v,$$

so that

$$\begin{aligned} \|x_v - z\|_v &\leq \max\{\|x_v - x_v f_v\|_v, \|x_v f_v - z\|_v\} \\ &\leq c_v \max\{\|e - f_v\|_v, \|e - f_v^{-1} x_v^{-1} z\|_v\} \leq c_v \delta_v. \end{aligned}$$

This proves (5.13).

Finally, we observe that

$$z \in \Omega(f', f)(b, e)\tilde{\Psi}_\epsilon.$$

Hence, it follows from (5.3) that

$$H(z) \ll H(b).$$

This completes the proof of the proposition.  $\square$

**5.3. Duality and approximation on homogeneous varieties.** Consider now the problem of Diophantine approximation on general homogeneous varieties. The results obtained in the present subsection will be used in the proof of Theorem 1.6.

In Propositions 5.5 and 5.6 below, all implicit constants may depend on  $S' \subset S \subset V_K$ ,  $x^0 \in X_{S'}$ , and  $\Omega \subset G_{S'}$ , but are independent of other parameters unless stated otherwise.

**Proposition 5.5.** *Let  $X \subset \mathbb{A}^n$  be a homogeneous quasi-affine variety of the group  $G \subset \mathrm{GL}_n$ . Fix  $S \subset V_K$ , finite  $S' \subset S$ ,  $x^0 \in X_{S'}$ , and a bounded subset  $\Omega$  of  $G_{S'}$ . Then there exist  $\epsilon_0 \in (0, 1)$  and a family of measurable subset  $\Phi_\epsilon$  of  $\Upsilon$  indexed by  $\epsilon = (\epsilon_v)_{v \in S'}$  where  $\epsilon_v \in I_v \cap (0, \epsilon_0)$  that satisfy*

$$\mu(\Phi_\epsilon) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(X)} \quad (5.15)$$

and the following property holds:

if for  $B \subset G_{V_K \setminus S} \times \prod_{v \in V_K^f \cap (S \setminus S')} G(O_v)$ ,  $\epsilon = (\epsilon_v)_{v \in S'}$  as above, and  $\varsigma = (e, g^{-1})G(K) \in \Upsilon$  with  $g \in \Omega$ , we have

$$B^{-1}\varsigma \cap \Phi_\epsilon \neq \emptyset, \quad (5.16)$$

then there exists  $z \in G(O_{(V_K \setminus S) \cup S'})$  such that

$$H(z) \ll \max_{b \in B} H(b) \quad (5.17)$$

and for  $x = gx^0 \in X_{S'}$

$$\|x_v - zx_v^0\| \leq \epsilon_v \quad \text{for all } v \in S'. \quad (5.18)$$

*Proof.* For  $v \in S'$ , let  $p_v : G_v \rightarrow G_v x_v^0$  be the map  $g \mapsto gx_v^0$ . There exists a neighbourhood  $\mathcal{U}_v$  of  $x_v^0$  in  $G_v x_v^0$  and an analytic section  $\sigma_v : \mathcal{U}_v \rightarrow G_v$  of the map  $p_v$  such that  $\sigma_v(x_v^0) = e$ . (The section  $\sigma_v$  can be constructed using the exponential map in  $G_v$ .) Let

$$\mathcal{O}_v(x_v^0, \epsilon) := \{y \in G_v x_v^0 : \|y - x_v^0\|_v \leq \epsilon\}.$$

Let  $L_v$  be the stabiliser of  $x_v^0$  in  $G_v$ , and let  $L_v^1$  be a compact neighbourhood of identity in  $L_v$ . Let  $c_v = \sup_{g \in \Omega} c_v(g^{-1})$ , where  $c_v(\cdot)$  is the constant given by (5.1). We set

$$\tilde{\Phi}_\epsilon = \left( \prod_{v \in V_K \setminus S'} \mathcal{O}_v \right) \times \left( \prod_{v \in S'} \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c_v))L_v^1 \right),$$

where  $\epsilon_v > 0$  are sufficiently small, so that  $\mathcal{O}_v(x_v^0, \epsilon_v/c_v) \subset \mathcal{U}_v \cap \sigma_v^{-1}(G(O_v))$ , and where  $\mathcal{O}_v = G(O_v)$  for  $v \in V_K^f \setminus S'$  and  $\mathcal{O}_v$  is a fixed neighbourhood of identity in  $G_v$  for  $v \in V_K^\infty \setminus S'$ . We also set

$$\Phi_\epsilon = \tilde{\Phi}_\epsilon G(K) \subset \Upsilon.$$

As in the proof of Proposition 5.3, we can choose sufficiently small  $\epsilon_v$ ,  $L_v^1$  and  $\mathcal{O}_v$ , so that

$$\tilde{\Phi}_\epsilon z \cap \tilde{\Phi}_\epsilon = \emptyset \quad \text{for every } z \in G(K), z \neq e.$$

Then

$$\mu(\Phi_\epsilon) \gg m(\tilde{\Phi}_\epsilon) \gg \prod_{v \in S'} m_v(\sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c_v))L_v^1),$$

and estimating volumes in local coordinates, we obtain

$$\mu(\Phi_\epsilon) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(X)}.$$

This proves (5.15).

Now suppose that (5.16) holds. Then for some  $b \in B$  and  $z \in G(K)$ , we have

$$(b^{-1}z, g^{-1}z) \in \tilde{\Phi}_\epsilon.$$

Then for  $v \in S'$ , we have  $g_v^{-1}z \in \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c_v))L_v^1$ , and

$$\|g_v^{-1}z x_v^0 - x_v^0\|_v \leq \epsilon_v/c_v.$$

Therefore,

$$\|z x_v^0 - x_v\|_v \leq \epsilon_v \quad \text{for all } v \in S'.$$

Also, for  $v \in V_K^f \cap (S \setminus S')$ , we have  $z \in b_v \mathcal{O}_v$ . This implies that  $z \in G(O_{(V_K \setminus S) \cup S'})$ . Hence, we have established (5.18).

Finally, we observe that since

$$z \in \Omega(b, e)\tilde{\Phi}_\epsilon,$$

estimate (5.17) follows from (5.3).  $\square$

We now turn to proving the estimate crucial in establishing diophantine approximation at every point in a homogeneous variety.

**Proposition 5.6.** *Let  $X \subset \mathbb{A}^n$  be a homogeneous quasi-affine variety of the group  $G \subset \mathrm{GL}_n$ . Fix  $S \subset V_K$ , finite  $S' \subset S$ ,  $x^0 \in X_{S'}$ , and a bounded subset  $\Omega$  of  $G_{S'}$ . Then there exist  $\epsilon_0 \in (0, 1)$  and a family of measurable subset  $\tilde{\Psi}_\epsilon$  of  $G_{V_K}$  indexed by  $\epsilon = (\epsilon_v)_{v \in S'}$  where  $\epsilon_v \in I_v \cap (0, \epsilon_0)$  that satisfy*

$$\mu(\tilde{\Psi}_\epsilon G(K)) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(X)}, \quad (5.19)$$

$$\mu(\tilde{\Psi}_\epsilon^{-1}(e, g^{-1})G(K)) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(X)} \quad \text{for all } g \in \Omega, \quad (5.20)$$

and the following property holds:

if for  $B \subset G_{V_K \setminus S} \times \prod_{v \in V_K^f \cap (S \setminus S')} G(O_v)$ ,  $\epsilon = (\epsilon_v)_{v \in S'}$  as above, and  $\varsigma = (e, g^{-1})G(K) \in \Upsilon$  with  $g \in \Omega$ , we have

$$B^{-1} \tilde{\Psi}_\epsilon^{-1} \varsigma \cap \tilde{\Psi}_\epsilon G(K) \neq \emptyset, \quad (5.21)$$

then there exists  $z \in G(O_{(V_K \setminus S) \cup S'})$  such that

$$H(z) \ll \max_{b \in B} H(b) \quad (5.22)$$

and for  $x = gx_0 \in X_{S'}$

$$\|x_v - zx_v^0\| \leq \epsilon_v \quad \text{for all } v \in S'. \quad (5.23)$$

*Proof.* We use notation introduced in the proof of Proposition 5.5. There exists  $c''_v > 0$  such that for every  $r_v \in \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v))L_v^1$ , we have

$$\|r_v x\|_v \leq c''_v \|x\|_v, \quad x \in K_v^n.$$

Let  $c'_v = c_v(1 + c''_v)$  and

$$\tilde{\Psi}_\epsilon = \left( \prod_{v \in V_K \setminus S'} \mathcal{O}_v \right) \times \left( \prod_{v \in S'} \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c'_v))L_v^1 \right),$$

Estimate (5.19) is proved as in Proposition 5.5. Taking  $\mathcal{O}_v$ ,  $\epsilon_v$ , and  $L_v^1$  sufficiently small, we can arrange that

$$(e, g) \tilde{\Psi}_\epsilon^{-1}(e, g^{-1})z \cap (e, g) \tilde{\Psi}_\epsilon^{-1}(e, g^{-1}) = \emptyset \quad \text{for } z \in G(K), z \neq e.$$

Then

$$\mu(\tilde{\Psi}_\epsilon^{-1}(e, g^{-1})G(K)) \gg m(\tilde{\Psi}_\epsilon^{-1}(e, g^{-1})) = m(\tilde{\Psi}_\epsilon^{-1}).$$

Hence, estimate (5.20) follows from (5.19).

Suppose that (5.21) holds. Then for some  $b \in B$ ,  $f' \in \prod_{v \in V_K \setminus S'} \mathcal{O}_v$ ,  $f \in \prod_{v \in S'} \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c'_v))L_v^1$ , and  $z \in G(K)$ , we have

$$(b^{-1}(f')^{-1}z, f^{-1}g^{-1}z) \in \tilde{\Psi}_\epsilon.$$

For  $v \in S'$ ,

$$\|f_v x_v^0 - x_v^0\|_v \leq \epsilon_v/c'_v \quad \text{and} \quad \|f_v^{-1}g_v^{-1}z x_v^0 - x_v^0\|_v \leq \epsilon_v/c'_v.$$

Hence,

$$\begin{aligned} \|x_v - zx_v^0\|_v &\leq \|g_v x_v^0 - g_v f_v x_v^0\|_v + \|g_v f_v x_v^0 - zx_v^0\|_v \\ &\leq c_v (\|x_v^0 - f_v x_v^0\|_v + c''_v \|x_v^0 - f_v^{-1}g_v^{-1}z x_v^0\|_v) \leq \epsilon_v. \end{aligned}$$

Also, for  $v \in (S \setminus S') \cap V_K^f$ ,

$$z \in f'_v b_v G(O_v) \subset G(O_v).$$

Hence,  $z \in G(O_{(V_K \setminus S) \cup S'})$ . This completes the proof of (5.23).

Finally, since

$$z \in \Omega(f', f)(b, e) \tilde{\Psi}_\epsilon,$$

estimate (5.22) follows from (5.3).  $\square$

## 6. PROOF OF THE MAIN RESULTS

**6.1. Almost sure Diophantine approximation on the group variety.** In this section we complete the proofs of main results. Our strategy is to combine the duality principle from Section 5 with the mean ergodic theorem from Section 4.

Let  $S \subset V_K$  and  $G$  be a connected almost simple algebraic  $K$ -group. Our first task is to establish an estimate on the exponent  $\mathfrak{a}_S(G)$  of the variety  $G$  defined after (1.7). It is worth mentioning that  $\mathfrak{a}_S(G)$  can, in principle, be computed in terms of the root data of  $G$  using methods from [16, 17], but for the purpose of this paper the simple estimate of Lemma 6.1 below will be sufficient. We recall that

$$\mathfrak{a}_S(G) = \sup_{\Omega \subset G_S} \limsup_{h \rightarrow \infty} \frac{\log A_S(\Omega, h)}{\log h},$$

where  $\Omega$  runs over bounded subset of  $G_S$ , and

$$A_S(\Omega, h) = |\{z \in G(K) : H(z) \leq h, z \in \Omega\}|.$$

We show that  $\mathfrak{a}_S(G)$  can be estimated in terms of volumes of the sets

$$B_h = U_{V_K \setminus S} \{g \in G_{V_K \setminus S} : H(g) \leq h\} U_{V_K \setminus S} \subset G_{V_K \setminus S}. \quad (6.1)$$

**Lemma 6.1.** *Let  $\Omega$  be a bounded subset of  $G_S$ . Then there exist  $c > 1$  and  $h_0 > 0$  such that for every  $h \geq h_0$*

$$A_S(\Omega, h) \ll_{\Omega} m_{V_K \setminus S}(B_{ch}).$$

*In particular, it follows that for every  $\delta > 0$  and  $h \geq h_0(S, \delta)$ .*

$$m_{V_K \setminus S}(B_h) \gg h^{\mathfrak{a}_S(G) - \delta}.$$

*Proof.* Let

$$\mathcal{A}_S(\Omega, h) = \{\gamma \in G(K) \cap G_{V_K \setminus S} \Omega : H(\gamma) \leq h\}.$$

Then  $A_S(\Omega, h) = |\mathcal{A}_S(\Omega, h)|$ . Since  $G(K)$  is a discrete subgroup of  $G_{V_K}$ , there exists a bounded neighbourhood  $\mathcal{O}$  of identity in  $G_{V_K}$  such that  $\mathcal{O}\gamma_1 \cap \mathcal{O}\gamma_2 = \emptyset$  for  $\gamma_1 \neq \gamma_2 \in G(K)$ . Then

$$A_S(\Omega, h) = |\mathcal{A}_S(\Omega, h)| \leq \frac{m_{V_K}(\mathcal{O}\mathcal{A}_S(\Omega, h))}{m_{V_K}(\mathcal{O})}.$$

We observe that it follows from (5.3) that

$$\mathcal{O}\mathcal{A}_S(\Omega, h) \subset \mathcal{A}_S(\Omega', c_1 h) \subset B_{c_2 h} \Omega'$$

for a bounded  $\Omega' \subset G_S$  and  $c_1, c_2 > 0$ . This implies the claim.  $\square$

We will also need volume estimates for the intersections of  $B_h$  with finite index subgroups.

**Lemma 6.2.** *Let  $G_0$  be a finite index subgroup of  $G_{V_K}$ . Then there exists  $c \geq 1$  such that*

$$m_{V_K \setminus S}(B_h \cap G_0) \geq \frac{m_{V_K \setminus S}(B_{c^{-1}h})}{|G_{V_K} : G_0|}$$

for every  $h > 0$ .

*Proof.* Let  $\{g_i\}$  be a finite set of coset representatives of  $G_{V_K \setminus S} / (G_{V_K \setminus S} \cap G_0)$ . It follows from (5.3) that there exists  $c \geq 1$  such that  $g_i^{-1} B_h \subset B_{ch}$ . Hence,

$$\begin{aligned} m_{V_K \setminus S}(B_h) &= \sum_i m_{V_K \setminus S}(B_h \cap g_i G_0) = \sum_i m_{V_K \setminus S}(g_i^{-1} B_h \cap G_0) \\ &\leq |G_{V_K} : G_0| m_{V_K \setminus S}(B_{ch} \cap G_0). \end{aligned}$$

This implies the claim.  $\square$

*Convention:* In the proofs of the Theorems 1.3 and 1.5, implicit constants may depend on the set of places  $S$ , but are independent of the subset  $S'$  unless stated otherwise. In the proof of Theorem 1.6, implicit constants may depend on  $S'$  as well.

*Proof of Theorem 1.3(i).* Let  $\beta_h$  be the Haar-uniform probability measure supported on the set  $B_h$  defined in (6.1). We denote by  $\pi_{V_K \setminus S}(\beta_h)$  the corresponding averaging operator acting on  $L^2(\Upsilon)$  (see (4.1)). The main idea of the proof is to combine Theorem 4.2 with Proposition 5.3. By Theorem 4.2, for every  $\phi \in L^2(\Upsilon)$  and  $h, \delta' > 0$ ,

$$\left\| \pi_{V_K \setminus S}(\beta_h) \phi - \int_{\Upsilon} \phi d\mu \right\|_2 \ll_{\delta'} m_{V_K \setminus S}(B_h)^{-\frac{1}{q_{V_K \setminus S}(\mathbb{G})} + \delta'} \|\phi\|_2.$$

Moreover, it follows from Lemma 6.1 that this estimate can be rewritten as

$$\left\| \pi_{V_K \setminus S}(\beta_h) \phi - \int_{\Upsilon} \phi d\mu \right\|_2 \ll_{\delta'} h^{-\frac{a_S(\mathbb{G})}{q_{V_K \setminus S}(\mathbb{G})} + \delta'} \|\phi\|_2 \quad (6.2)$$

for every  $\delta' > 0$  and  $h \geq h_0(\delta')$ .

We pick a bounded subset  $\Omega$  of  $G_S$ . Let  $\Phi_\epsilon \subset \Upsilon$  be the sets introduced in Proposition 5.3 which are indexed by  $\epsilon = (\epsilon_v)_{v \in S'}$  with finite  $S' \subset S$  and  $\epsilon_v \in I_v$ . We denote by  $\phi_\epsilon$  the characteristic function of  $\Phi_\epsilon$ . Then (6.2) gives

$$\left\| \pi_{V_K \setminus S}(\beta_h) \phi_\epsilon - \mu(\Phi_\epsilon) \right\|_2 \ll_{\delta'} h^{-\frac{a_S(\mathbb{G})}{q_{V_K \setminus S}(\mathbb{G})} + \delta'} \mu(\Phi_\epsilon)^{1/2} \quad (6.3)$$

for every  $\delta' > 0$  and  $h \geq h_0(\delta')$ . We set

$$h_\epsilon = \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(\mathbb{G})}{a_S(\mathbb{G})} - \delta} \right)^{q_{V_K \setminus S}(\mathbb{G})/2} \quad (6.4)$$

with  $\delta > 0$ . Let

$$\Upsilon_\epsilon = \{\varsigma \in \Upsilon : B_{h_\epsilon}^{-1} \varsigma \cap \Phi_\epsilon = \emptyset\}. \quad (6.5)$$

Then for  $\varsigma \in \Upsilon_\epsilon$ , we have  $\pi_{V_K \setminus S}(\beta_{h_\epsilon}) \phi_\epsilon(\varsigma) = 0$ . Hence, it follows from (6.3) that for every  $\delta' > 0$ , and  $\epsilon$ , we have

$$\mu(\Upsilon_\epsilon) \ll_{\delta'} h_\epsilon^{-\frac{2a_S(\mathbb{G})}{q_{V_K \setminus S}(\mathbb{G})} + 2\delta'} \mu(\Phi_\epsilon)^{-1} \quad (6.6)$$

provided that  $h_\epsilon \geq h_0(\delta')$ . Combining this estimate with (6.4) and (5.5), we conclude that

$$\mu(\Upsilon_\epsilon) \ll_{\Omega, \delta'} \prod_{v \in S'} \epsilon_v^{\theta_v} \quad (6.7)$$

for every  $\delta' > 0$  and  $\epsilon$  satisfying  $h_\epsilon \geq h_0(\delta')$ , where

$$\begin{aligned} \theta_v &= \left( -r_v \frac{\dim(\mathbb{G})}{\mathfrak{a}_S(\mathbb{G})} - \delta \right) \frac{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})}{2} \left( -\frac{2\mathfrak{a}_S(\mathbb{G})}{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})} + 2\delta' \right) - r_v \dim(\mathbb{G}) \\ &= \mathfrak{a}_S(\mathbb{G})\delta - \delta' \cdot \mathfrak{q}_{V_K \setminus S}(\mathbb{G}) \left( r_v \frac{\dim(\mathbb{G})}{\mathfrak{a}_S(\mathbb{G})} + \delta \right). \end{aligned}$$

We pick  $\delta' = \delta'(\delta) > 0$  so that  $\theta_v > 0$ .

Let  $E_v = I_v$  for  $v \in V_K^f$ , and let  $E_v = \{2^{-n}\}_{n \geq 1}$  for  $v \in V_K^\infty$ , where we discretize the parameter in the archimedean part in order to take advantage of the Borel-Cantelli lemma. Set  $E_{S'} = \prod_{v \in S'} E_v$  for finite  $S' \subset S$ . We denote by  $\Upsilon_{S'}$  the lim sup of the sets  $\Upsilon_\epsilon$  with  $\epsilon \in E_{S'}$ . Since  $h_\epsilon \geq h_0(\delta')$  holds for all but finitely many  $\epsilon \in E_{S'}$ , estimate (6.7) holds for all but finitely many  $\epsilon \in E_{S'}$  as well, and it follows that

$$\sum_{\epsilon \in E_{S'}} \mu(\Upsilon_\epsilon) < \infty.$$

Hence, by the Borel-Cantelli lemma,  $\mu(\Upsilon_{S'}) = 0$ . Let  $\tilde{\Upsilon}_{S'} \subset G_{V_K}$  be the preimage of  $\Upsilon_{S'}$ . Then  $m(\tilde{\Upsilon}_{S'}) = 0$ . We denote by  $\tilde{\Upsilon}_0$  the union of  $\tilde{\Upsilon}_{S'}$  over finite subsets  $S'$  of  $S$ . Then  $m(\tilde{\Upsilon}_0) = 0$  as well. Moreover, using Fubini's theorem and passing to a superset of  $\tilde{\Upsilon}_0$ , we can arrange that  $\tilde{\Upsilon}_0$  additionally satisfies the property that for every finite  $S' \subset V_K$  and  $(f', f) \in G_{V_K \setminus S'} \times G_{S'}$  not belonging to  $\tilde{\Upsilon}_0$ , the set  $\{f'' : (f'', f) \notin \tilde{\Upsilon}_0\}$  has full measure in  $G_{V_K \setminus S'}$ . In particular, it follows that if for  $y \in G_{V_K \setminus S}$  and  $x \in G_S$ , we have  $(y, x) \notin \tilde{\Upsilon}_0$ , then there exists  $(y', x') \notin \tilde{\Upsilon}_0$  such that  $x'_v = x_v$  for  $v \in S'$ ,  $x'_v \in G(O_v)$  for  $v \in V_K^f \cap (S \setminus S')$ , and  $y' \in \mathcal{O}_{V_K \setminus S}(1)$ . We shall use this property below.

Let

$$\Omega' = \{x \in \Omega : \exists y \in \mathcal{O}_{V_K \setminus S}(1) : (y, x^{-1}) \notin \tilde{\Upsilon}_0\}. \quad (6.8)$$

Since

$$(\mathcal{O}_{V_K \setminus S}(1) \times (\Omega \setminus \Omega')^{-1}) \subset \tilde{\Upsilon}_0,$$

it follows that the set  $\Omega \setminus \Omega'$  has measure zero.

Let  $G_S = \cup_{j \geq 1} \Omega_j$  be an exhaustion of  $G_S$  by compact sets. Then  $Y := \cup_{j \geq 1} \Omega'_j$  is a subset of  $G_S$  of full measure. Hence, it suffices to show that given compact  $\Omega \subset G_S$ , every  $x \in \Omega'$  satisfies the claim of the theorem.

For  $x \in \Omega'$ , there exists  $\tilde{\zeta} := (y', (x')^{-1}) \notin \tilde{\Upsilon}_0$  such that  $x'_v = x_v$  for  $v \in S'$ ,  $x'_v \in G(O_v)$  for  $v \in V_K^f \cap (S \setminus S')$ , and  $y' \in \mathcal{O}_{V_K \setminus S}(1)$ .

Since  $\zeta G(K) \notin \Upsilon_0$ , it follows that  $\zeta G(K) \notin \Upsilon_\epsilon$  for all but finitely many  $\epsilon \in E_{S'}$ . That is, there exists  $\epsilon_0(x, S', \delta) > 0$  such that for  $\epsilon \in E_{S'} \cap (0, \epsilon_0(x, S', \delta))^{S'}$ , we have

$$B_{h_\epsilon}^{-1} \zeta G(K) \cap \Phi_\epsilon \neq \emptyset,$$

and

$$(\mathcal{O}_{V_K \setminus S}(1)^{-1} B_{h_\epsilon})^{-1}(e, (x')^{-1})G(K) \cap \Phi_\epsilon \neq \emptyset.$$

Now we are in position to apply Proposition 5.3. It follows that there exists  $z \in G(K)$  such that

$$H(z) \ll_\Omega \max_{b \in \mathcal{O}_{V_K \setminus S}(1)^{-1} B_{h_\epsilon}} H(b) \ll h_\epsilon$$

and

$$\begin{aligned} \|x'_v - z\|_v &\leq \epsilon_v \quad \text{for all } v \in S', \\ \|x'_v - z\|_v &\leq 1 \quad \text{for all } v \in S \setminus S'. \end{aligned}$$

Since  $x'_v \in G(\mathcal{O}_v)$  for  $v \in V_K^f \cap (S \setminus S')$ , it follows that  $z \in G(\mathcal{O}_{(V_K \setminus S) \cup S'})$ . Hence, for every  $\delta > 0$  and  $\epsilon \in E_{S'} \cap (0, \epsilon_0(x, S', \delta))^{S'}$ , we have

$$\omega_S(x, \epsilon) \ll_\Omega h_\epsilon = \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G)}{a_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2}.$$

This implies that for all  $\delta > 0$  and sufficiently small  $\epsilon$ , depending on  $\delta$  and  $\Omega$ , we also have (absorbing the constant by passing from  $\delta$  to  $2\delta$ )

$$\omega_S(x, \epsilon) \leq \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G)}{a_S(G)} - 2\delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2}.$$

Now in order to finish the proof of the theorem, we need to extend this estimate to the case when  $\epsilon_v \in (0, \epsilon_0(x, S', \delta))$  for  $v \in V_K^\infty \cap S'$ . For such  $\epsilon_v$ , there exists  $\epsilon'_v \in E_v \cap (0, \epsilon_0(x, S', \delta))$  such that  $\epsilon'_v \leq \epsilon_v \leq 2\epsilon'_v$ . For  $v \in V_K^f \cap S'$ , we set  $\epsilon'_v = \epsilon_v$ . Then

$$\begin{aligned} \omega_S(x, \epsilon) &\leq \omega_S(x, \epsilon') \leq \left( \prod_{v \in S'} (\epsilon'_v)^{-r_v \frac{\dim(G)}{a_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2} \\ &\ll_{V_K^\infty \cap S'} \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G)}{a_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2} \end{aligned} \quad (6.9)$$

for every  $\delta > 0$  and  $\epsilon = (\epsilon_v)_{v \in S'}$  with  $\epsilon_v \in I_v \cap (0, \epsilon_0(x, S', \delta))$ . This also implies that

$$\omega_S(x, \epsilon) \leq \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G)}{a_S(G)} - 2\delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2}$$

for all  $\delta > 0$  and sufficiently small  $\epsilon$  (depending on  $\delta$  and  $V_K^\infty \cap S'$ ). This completes the proof.  $\square$

*Proof of Theorem 1.5.* The proof of the theorem follows the same strategy as the proof of Theorem 1.3, but we choose the parameter  $h_\epsilon$  to be

$$h_\epsilon = \left( \prod_{v \in S'} \epsilon_v^{-\frac{r_v \dim(\mathbb{G}) + \sigma_S - \delta}{\mathfrak{a}_S(\mathbb{G})}} \right)^{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})/2}$$

with  $\delta > 0$ .

We fix a bounded subset  $\Omega$  of  $G_S$  and consider the sets

$$\Upsilon_\epsilon = \{\zeta \in \Upsilon : B_{h_\epsilon}^{-1} \zeta \cap \Phi_\epsilon = \emptyset\},$$

indexed by  $\epsilon = (\epsilon_v)_{v \in S'}$  with finite  $S' \subset S$  and  $\epsilon_v \in I_v$ , where the sets  $\Phi_\epsilon$  is defined in Proposition 5.3. Arguing as in the proof of Theorem 1.3, we obtain the estimate

$$\mu(\Upsilon_\epsilon) \ll_{\Omega, \delta'} \prod_{v \in S'} \epsilon_v^{\theta_v}, \quad (6.10)$$

where

$$\begin{aligned} \theta_v &= \left( -\frac{r_v \dim(\mathbb{G}) + \sigma_S}{\mathfrak{a}_S(\mathbb{G})} - \delta \right) \frac{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})}{2} \left( -\frac{2\mathfrak{a}_S(\mathbb{G})}{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})} + 2\delta' \right) - r_v \dim(\mathbb{G}) \\ &= \sigma_S + \mathfrak{a}_S(\mathbb{G})\delta - \delta' \cdot \mathfrak{q}_{V_K \setminus S}(\mathbb{G}) \left( \frac{r_v \dim(\mathbb{G}) + \sigma_S}{\mathfrak{a}_S(\mathbb{G})} + \delta \right). \end{aligned}$$

This estimate is valid for every  $\delta, \delta' > 0$  and  $\epsilon$  as above provided that  $h_\epsilon \geq h_0(\delta')$ . Moreover, we emphasize that this estimate is uniform over finite  $S' \subset S$ . We choose  $\delta' = \delta'(\delta) > 0$  such that  $\theta_v \geq \theta > \sigma_S$ .

Let

$$E_S = \{(\epsilon_v)_{v \in S} : \epsilon_v = q_v^{-n_v} \text{ with } n_v \geq 0, n_v = 0 \text{ for a.e. } v\},$$

where we set  $q_v = 2$  for  $v \in V_K^\infty$ . We note that  $h_\epsilon \geq h_0(\delta')$  for all but finitely many  $\epsilon \in E_S$ . Therefore, estimate (6.10) holds for all but finitely many  $\epsilon \in E_S$ , and the sum  $\sum_{\epsilon \in E_S} \mu(\Upsilon_\epsilon)$  can be estimated (except possibly finitely many terms) by

$$\sum_{\epsilon \in E_S} \left( \prod_{v \in S} \epsilon_v^{\theta_v} \right) \leq \prod_{v \in S} (1 - q_v^{-\theta})^{-1},$$

where the last product converges because  $\theta > \sigma_S$ .

Let  $\Upsilon_0$  be the lim sup of the sets  $\Upsilon_\epsilon$  as  $\epsilon \in E_S$ . Then by the Borel-Cantelli lemma,  $\mu(\Upsilon_0) = 0$ . Let  $\tilde{\Upsilon}_0 \subset G_{V_K}$  be the preimage of  $\Upsilon_0$ . Then  $m(\tilde{\Upsilon}_0) = 0$ . Moreover, as in the proof of Theorem 1.3(i), we can pass to a superset of  $\tilde{\Upsilon}_0$  to arrange that if for  $y \in G_{V_K \setminus S}$  and  $x \in G_S$ , we have  $(y, x) \notin \tilde{\Upsilon}_0$ , then there exists  $(y', x') \notin \tilde{\Upsilon}_0$  such that  $x'_v = x_v$  for  $v \in S'$ ,  $x'_v \in G(O_v)$  for  $v \in V_K^f \cap (S \setminus S')$ , and  $y' \in \mathcal{O}_{V_K \setminus S}(1)$ .

Given bounded  $\Omega \subset G_S$ , we define  $\Omega'$  as in (6.8). Then  $\Omega \setminus \Omega'$  has measure zero, and it suffices to show that every  $x \in \Omega'$  satisfies the claim of the theorem.

For  $x \in \Omega'$ , there exists  $\tilde{\zeta} := (y', (x')^{-1}) \notin \tilde{\Upsilon}_0$  such that  $x'_v = x_v$  for  $v \in S'$ ,  $x'_v \in G(O_v)$  for  $v \in V_K^f \cap (S \setminus S')$ , and  $y' \in \mathcal{O}_{V_K \setminus S}(1)$ . Since  $\tilde{\zeta}G(K) \notin \Upsilon_0$ , we conclude that  $\tilde{\zeta}G(K) \notin \Upsilon_\epsilon$  for all but finitely many  $\epsilon \in E_S$ . In particular, there exists  $\epsilon_v^0(x, \delta) \in (0, 1]$ ,  $v \in S$ , such that  $\epsilon_v^0(x, \delta) = 1$  for almost all  $v$  and for all  $\epsilon = (\epsilon_v)_{v \in S'}$  with finite  $S' \subset S$  and  $\epsilon_v \in E_v \cap (0, \epsilon_v^0(x, \delta))$ , we have

$$B_{h_\epsilon}^{-1} \tilde{\zeta}G(K) \cap \Phi_\epsilon \neq \emptyset.$$

Applying Proposition 5.3, we deduce that there exists  $z \in G(K)$  such that

$$H(z) \ll_\Omega h_\epsilon$$

and

$$\begin{aligned} \|x'_v - z\|_v &\leq \epsilon_v \quad \text{for all } v \in S', \\ \|x'_v - z\|_v &\leq 1 \quad \text{for all } v \in S \setminus S', \end{aligned}$$

provided that  $\epsilon_v \in \{q_v^{-n}\}_{n \geq 0} \cap (0, \epsilon_v^0(x, \delta))$ . Since  $x'_v \in G(O_v)$  for  $v \in V_K^f \cap (S \setminus S')$ , it follows that  $z \in G(O_{(V_K \setminus S) \cup S'})$ .

We conclude that

$$\omega_S(x, \epsilon) \ll_\Omega h_\epsilon = \left( \prod_{v \in S'} \epsilon_v^{-\frac{r_v \dim(G) + \sigma_S - \delta}{a_S(G)}} \right)^{q_{V_K \setminus S}(G)/2} \quad (6.11)$$

for every finite  $S' \subset S$  and  $\epsilon \in E_{S'}$  such that  $\epsilon_v \in \{q_v^{-n}\}_{n \geq 0} \cap (0, \epsilon_v^0(x, \delta))$ . Moreover, this implies that for all  $\delta > 0$  and sufficiently small  $\epsilon$  (depending on  $\delta$  and  $\Omega$ ), we have

$$\omega_S(x, \epsilon) \leq \left( \prod_{v \in S'} \epsilon_v^{-\frac{r_v \dim(G) + \sigma_S - 2\delta}{a_S(G)}} \right)^{q_{V_K \setminus S}(G)/2}$$

Finally, it remains to extend this estimate to  $\epsilon_v \in (0, \epsilon_v^0(x, \delta))$  when  $v \in V_K^\infty \cap S'$ . This can be done as in (6.9).  $\square$

## 6.2. Diophantine approximation at every point on the group variety.

*Proof of Theorem 1.3(ii).* Without loss of generality, we may assume that  $\Omega = \prod_{v \in S} \Omega_v$  where  $\Omega_v$  is a bounded subset  $G_v$  and  $\Omega_v \supset G(O_v)$  for all  $v \in V_K^f$ .

Let the sets  $\tilde{\Psi}_\epsilon \subset G_{V_K}$  be as defined in Proposition 5.4 indexed by  $\epsilon = (\epsilon_v)_{v \in S'}$  with finite  $S' \subset S$  and  $\epsilon_v \in I_v$ . We set

$$h_\epsilon = \left( \prod_{v \in S'} \epsilon_v^{-\frac{r_v \dim(G)}{a_S(G)} - \delta} \right)^{q_{V_K \setminus S}(G)}$$

with  $\delta > 0$  and

$$\Upsilon_\epsilon = \{\varsigma \in \Upsilon : B_{h_\epsilon}^{-1} \varsigma \cap \tilde{\Psi}_\epsilon G(K) = \emptyset\}.$$

Let  $\psi_\epsilon$  be the characteristic function of the set  $\Psi_\epsilon := \tilde{\Psi}_\epsilon G(K) \subset \Upsilon$ . As in the proof of Theorem 1.6, we get

$$\left\| \pi_{V_K \backslash S}(\beta_h) \psi_\epsilon - \mu(\Psi_\epsilon) \right\|_2 \ll_{\delta'} h^{-\frac{\mathfrak{a}_S(G)}{\mathfrak{q}_{V_K \backslash S}(G)} + \delta'} \mu(\Psi_\epsilon)^{1/2}$$

for every  $\delta' > 0$  and  $h \geq h_0(\delta')$ . Since  $\pi_{V_K \backslash S}(\beta_{h_\epsilon}) \psi_\epsilon = 0$  on  $\Upsilon_\epsilon$ , it follows that

$$\mu(\Upsilon_\epsilon) \ll_{\delta'} h_\epsilon^{-\frac{2\mathfrak{a}_S(G)}{\mathfrak{q}_{V_K \backslash S}(G)} + 2\delta'} \mu(\Psi_\epsilon)^{-1},$$

and using (5.10), we conclude that

$$\mu(\Upsilon_\epsilon) \ll_{\Omega, \delta'} \prod_{v \in S'} \epsilon_v^{\theta_v}, \quad (6.12)$$

where

$$\begin{aligned} \theta_v &= \left( r_v \frac{\dim(G)}{\mathfrak{a}_S(G)} + \delta \right) \mathfrak{q}_{V_K \backslash S}(G) \left( \frac{2\mathfrak{a}_S(G)}{\mathfrak{q}_{V_K \backslash S}(G)} - 2\delta' \right) - r_v \dim(G) \\ &= r_v \dim(G) + \delta \cdot \mathfrak{q}_{V_K \backslash S}(G) \left( \frac{2\mathfrak{a}_S(G)}{\mathfrak{q}_{V_K \backslash S}(G)} - 2\delta' \right) - \delta' \cdot 2r_v \frac{\dim(G)}{\mathfrak{a}_S(G)}. \end{aligned}$$

This estimate holds provided that  $h_\epsilon \geq h_0(\delta')$ , so that it holds for all but finitely many  $\epsilon \in E_S$ .

We choose  $\delta' = \delta'(\delta)$  such that  $\theta_v > r_v \dim(G)$ .

For  $x \in \Omega$ , there exists  $x' \in \Omega$  such that  $x'_v = x_v$  for  $v \in S'$  and  $x'_v \in G(O_v)$  for  $v \in V_K^f \cap (S \setminus S')$ . For  $\varsigma := (e, (x')^{-1})G(K)$ , we have by (5.11)

$$\mu(\tilde{\Psi}_\epsilon^{-1} \varsigma) \gg_\Omega \prod_{v \in S'} \epsilon_v^{r_v \dim(G)}. \quad (6.13)$$

Comparing (6.12) and (6.13), we conclude that the inequality

$$\mu(\tilde{\Psi}_\epsilon^{-1} \varsigma) > \mu(\Upsilon_\epsilon) \quad (6.14)$$

holds for all but finitely many  $\epsilon \in E_S$ . If (6.14) holds, then  $\tilde{\Psi}_\epsilon^{-1} \varsigma \not\subseteq \Upsilon_\epsilon$ , and

$$B_{h_\epsilon}^{-1} \tilde{\Psi}_\epsilon^{-1} \varsigma \cap \tilde{\Psi}_\epsilon G(K) \neq \emptyset.$$

Therefore, by Proposition 5.4, for all but finitely many  $\epsilon \in E_S$ , there exists  $z \in G(K)$  such that

$$H(z) \ll_\Omega h_\epsilon$$

and

$$\begin{aligned} \|x'_v - z\|_v &\leq \epsilon_v \quad \text{for all } v \in S', \\ \|x'_v - z\|_v &\leq 1 \quad \text{for all } v \in S \setminus S'. \end{aligned}$$

Since  $x'_v \in G(O_v)$  for  $v \in V_K^f \cap (S \setminus S')$ , it follows that  $z \in G(O_{(V_K \backslash S) \cup S'})$ .

This shows that there exists  $\epsilon_v^0(\Omega, \delta) \in (0, 1]$  satisfying  $\epsilon_v^0(\Omega, \delta) = 1$  for almost all  $v$  such that for every  $x \in \Omega$  and  $\epsilon \in E_S$  with  $\epsilon_v \in (0, \epsilon_v^0(\Omega, \delta))$ , we have the estimate

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \ll_{\Omega} h_{\epsilon} = \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G)}{a_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)}.$$

Therefore, for all  $\delta > 0$  and sufficiently small  $\epsilon$  (depending on  $\delta$  and  $\Omega$ ), we also have

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \leq \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G)}{a_S(G)} - 2\delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)}.$$

Now it remains to extend this estimate to the case when  $\epsilon_v \in (0, \epsilon_v^0(\Omega, \delta))$  for  $v \in V_K^{\infty} \cap S'$ , which can be achieved as in (6.9). This completes the proof of the theorem.  $\square$

**6.3. Diophantine approximation on homogeneous varieties.** Let  $X$  be an quasi-affine algebraic variety defined over  $K$  equipped with a transitive action of a connected almost simple algebraic  $K$ -group  $G$ . Let  $S$  be a subset of  $V_K$  and  $S'$  a finite subset of  $S$ . Before we start the proof of Theorem 1.6, we need to describe the structure of the topological closure  $\overline{X(O_{(V_K \setminus S) \cup S'})}$  in  $X_{S'}$ .

**Lemma 6.3.** *Assume that  $G$  is isotropic over  $V_K \setminus S$ , and let  $p : \tilde{G} \rightarrow G$  denote the simply connected cover of  $G$ . Then the closure  $\overline{X(O_{(V_K \setminus S) \cup S'})}$  in  $X_{S'}$  is open and*

$$\overline{X(O_{(V_K \setminus S) \cup S'})} = p(\tilde{G}_{S'})X(O_{(V_K \setminus S) \cup S'}).$$

Moreover, it is a union of finitely many orbits of  $p(\tilde{G}_{S'})$ .

*Proof.* Given  $x^0 \in X(O_{(V_K \setminus S) \cup S'})$ , we consider the map

$$P : \tilde{G} \rightarrow X : g \mapsto p(g)x^0.$$

Let

$$\Gamma = \tilde{G}(K) \cap \left( \bigcap_{v \in (S \setminus S') \cap V_K^f} p^{-1}(G(O_v)) \right).$$

Then  $P(\Gamma) \subset X(O_{(V_K \setminus S) \cup S'})$ . Since  $\tilde{G}$  is isotropic over  $V_K \setminus S$ , the subgroup  $\Gamma$  is dense in  $\tilde{G}_{S'}$ , and

$$P(\tilde{G}_{S'}) \subset \overline{P(\Gamma)} \subset \overline{X(O_{(V_K \setminus S) \cup S'})}.$$

Since the variety  $X$  is a homogeneous space of  $\tilde{G}$  with the action  $g \cdot x = p(g)x$ ,  $x \in X$ , it follows from [37, §3.1] that every orbit of  $\tilde{G}_{S'}$  is open and closed in  $X_{S'}$ . This implies the first claim.

The last claim follows from finiteness of Galois cohomology over local fields (see [37, §6.4]).  $\square$

*Proof of Theorem 1.6(i).* According to Lemma 6.3, the set  $\overline{X(O_{(V_K \setminus S) \cup S'})}$  is a finite union of open (and closed) orbits of  $p(\tilde{G}_{S'})$ . We intend to show that given  $x^0 \in X(O_{(V_K \setminus S) \cup S'})$  there exists a set  $Y$  of full measure in  $p(\tilde{G}_{S'})x^0$  whose points satisfies the claim of the theorem. In fact, we prove that for every bounded  $Y \subset p(\tilde{G}_{S'})x^0$  there exists a conull subset  $Y' \subset Y$  whose points satisfy the claim of the theorem. This will complete the proof.

Let  $\Omega$  be a bounded subset of  $p(\tilde{G}_{S'})$  such that  $\Omega x^0 = Y$ .

We set

$$U^0 = \prod_{v \in V_K^f \cap (S \setminus S')} (U_v \cap G(O_v)) \quad \text{and} \quad U = U_{V_K^f \setminus S} U^0.$$

Note that that  $U_v = G(O_v)$  for almost all  $v$ . Since both  $U_v$  and  $G(O_v)$  are open and compact, it follows that the subgroup  $U_v \cap G(O_v)$  has finite index in  $U_v$ . Hence,  $U^0$  has finite index in  $U_{V_K^f \cap (S \setminus S')}$ .

Recall that  $G^U$  denotes the kernel of  $U$ -invariant automorphic characters of  $G_{V_K}$  (see Section 4). We note that  $G(K) \subset G^U$ , and  $G^U$  is a finite index subgroup of  $G_{V_K}$  by Lemma 4.4. Let  $\beta'_h$  be the Haar-uniform probability measure supported on  $B'_h := U^0(B_h \cap G^U)$ . By Theorem 4.5, for every  $\phi \in L^2(\Upsilon)$  such that  $\text{supp}(\phi) \subset G^U/G(K)$  and  $h, \delta' > 0$ ,

$$\left\| \pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta'_h)\phi - \left( \int_{\Upsilon} \phi d\mu \right) \xi_U \right\|_2 \ll_{\delta'} m_{V_K \setminus S}(B_h \cap G^U)^{-\frac{1}{q_{V_K \setminus S}(G)} + \delta'} \|\phi\|_2.$$

By Lemmas 6.1 and 6.2,

$$m_{V_K \setminus S}(B_h \cap G^U) \gg m_{V_K \setminus S}(B_h) \gg_{\Omega} h^{a_S(G) - \delta'}$$

for every  $\delta' > 0$  and  $h \geq h_0(\delta')$ . Hence, combining these estimates, we conclude that for every  $\delta' > 0$  and  $h \geq h_0(\delta')$ , we have

$$\left\| \pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta'_h)\phi - \left( \int_{\Upsilon} \phi d\mu \right) \xi_U \right\|_2 \ll_{\delta'} h^{-\frac{a_S(G)}{q_{V_K \setminus S}(G)} + \delta'} \|\phi\|_2. \quad (6.15)$$

We apply (6.15) in the case when  $\phi$  is the characteristic function  $\phi_{\epsilon}$  of the set  $\Phi_{\epsilon}$  introduced in Proposition 5.5, which is indexed by  $\epsilon = (\epsilon_v)_{v \in S'}$  with finite  $S' \subset S$  and  $\epsilon_v \in I_v \cap (0, \epsilon_0)$ . Note that it follows from the construction of the sets  $\Phi_{\epsilon}$  that we can arrange that for sufficiently small  $\epsilon$ , we have  $\Phi_{\epsilon} \subset G^U/G(K)$ , and hence (6.15) is applicable to  $\phi = \phi_{\epsilon}$ . We obtain

$$\left\| \pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta'_h)\phi_{\epsilon} - \mu(\Phi_{\epsilon})\xi_U \right\|_2 \ll_{\delta'} h^{-\frac{a_S(G)}{q_{V_K \setminus S}(G)} + \delta'} \mu(\Phi_{\epsilon})^{1/2} \quad (6.16)$$

for every  $\delta' > 0$  and  $h \geq h_0(\delta')$ . Let

$$h_{\epsilon} = \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(X)}{a_S(G)} - \delta} \right)^{q_{V_K \setminus S}(G)/2} \quad (6.17)$$

with  $\delta > 0$ , and

$$\Upsilon_\epsilon = \{\varsigma \in G^U/G(K) : (B'_{h_\epsilon})^{-1}\varsigma \cap \Phi_\epsilon = \emptyset\}.$$

For  $\varsigma \in \Upsilon_\epsilon$ , we have

$$\pi_{(V_K \backslash S) \cup (V_K^f \backslash S')}(\beta'_{h_\epsilon})\phi_\epsilon(\varsigma) = 0.$$

Hence, (6.16) implies the estimate

$$\mu(\Upsilon_\epsilon) \ll_{\delta'} |G_{V_K} : G^U|^{-1} h_\epsilon^{-\frac{2\mathfrak{a}_S(\mathbf{G})}{\mathfrak{q}_{V_K \backslash S(\mathbf{G})}} + 2\delta'} \mu(\Phi_\epsilon)^{-1},$$

and using (6.17) and (5.15), we conclude that

$$\mu(\Upsilon_\epsilon) \ll_{\Omega, \delta'} \prod_{v \in S'} \epsilon_v^{\theta_v}, \quad (6.18)$$

where

$$\theta_v = \mathfrak{a}_S(\mathbf{G})\delta - \delta' \cdot \mathfrak{q}_{V_K \backslash S}(\mathbf{G}) \left( r_v \frac{\dim(X)}{\mathfrak{a}_S(\mathbf{G})} + \delta \right).$$

This estimate holds provided that  $h_\epsilon \geq h_0(\delta')$ . Hence, it holds for all but finitely many  $\epsilon \in E_{S'} \cap (0, \epsilon_0)^{S'}$ .

We choose  $\delta' = \delta'(\delta) > 0$  sufficiently small, so that  $\theta_v > 0$ .

Now we may argue as in the proof of Theorem 1.3(i). We denote by  $\Upsilon_0$  the lim sup of the sets  $\Upsilon_\epsilon$  with  $\epsilon \in E_{S'} \cap (0, \epsilon_0)^{S'}$ . Since (6.18) holds for all but finitely many  $\epsilon \in E_{S'}$ , it follows that

$$\sum_{\epsilon \in E_{S'} \cap (0, \epsilon_0)^{S'}} \mu(\Upsilon_\epsilon) < \infty,$$

and by the Borel–Cantelli lemma,  $\mu(\Upsilon_0) = 0$ . We fix a bounded neighbourhood  $V$  of identity in  $G_{V_K^\infty \backslash S'}$  contained in  $G^U$ . Then  $UV$  is a neighbourhood of identity in  $G_{V_K \backslash S'}$ . If we set

$$\Omega' = \{g \in \Omega : \exists y \in UV : (y, g^{-1})G(K) \notin \Upsilon_0\},$$

then

$$(UV \times (\Omega \setminus \Omega')^{-1})G(K) \subset \Upsilon_0,$$

and, hence, the set  $\Omega \setminus \Omega'$  has measure zero. This implies that the subset  $Y' := \Omega'x^0$  has full measure in  $Y = \Omega x^0$ .

For  $g \in \Omega'$ , there exists  $y \in UV$  such that  $\varsigma := (y, g^{-1})G(K) \notin \Upsilon_0$ . This implies that there exists  $\epsilon_0(g, \delta) > 0$  such that for  $\epsilon \in E_{S'} \cap (0, \epsilon_0(g, \delta))^{S'}$ , we have

$$(B'_{h_\epsilon})^{-1}\varsigma \cap \Phi_\epsilon \neq \emptyset,$$

and hence,

$$((UV)^{-1}B'_{h_\epsilon})^{-1}(e, x^{-1}) \cap \Phi_\epsilon \neq \emptyset.$$

Therefore, it follows from Proposition 5.5 that there exists  $z \in G(O_{(V_K \backslash S) \cup S'})$  such that

$$H(z) \ll_{\Omega} \max_{b \in (UV)^{-1}B'_{h_\epsilon}} H(b) \ll h_\epsilon,$$

and for  $x = gx^0 \in Y'$ ,

$$\|x_v - zx^0\|_v \leq \epsilon_v \quad \text{for all } v \in S'.$$

We have  $zx^0 \in X(O_{(V_K \setminus S) \cup S'})$  and  $H(zx^0) \ll H(z)$ .

Since these estimates are valid for arbitrary bounded  $Y$ , we conclude that for every  $\delta > 0$ , almost every  $x \in \overline{X(O_{(V_K \setminus S) \cup S'})}$ , and  $(\epsilon_v)_{v \in S'}$  with finite  $S' \subset S$  and  $\epsilon_v \in E_v \cap (0, \epsilon_0(x, S', \delta))$ , we have

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \ll_{\Omega} h_{\epsilon} = \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(X)}{a_S(G)} - \delta} \right)^{q_{V_K \setminus S}(G)/2}.$$

Moreover, this implies that

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \leq \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(X)}{a_S(G)} - 2\delta} \right)^{q_{V_K \setminus S}(G)/2}$$

for every  $\delta > 0$  and sufficiently small  $\epsilon$  (depending on  $\delta$  and  $\Omega$ ). Finally, it remains to extend this estimate to  $\epsilon_v \in (0, \epsilon_0(x, S', \delta))$  for  $v \in V_K^{\infty} \cap S'$ , which can be done as in (6.9).  $\square$

*Proof of Theorem 1.6(ii).* By Lemma 6.3, the set  $\overline{X(O_{(V_K \setminus S) \cup S'})}$  is a finite union of open (and closed) orbits of  $p(\tilde{G}_{S'})$ . Hence, it suffices to prove the theorem for every compact subset  $Y$  contained in  $p(\tilde{G}_{S'})x^0$  with  $x^0 \in X(O_{(V_K \setminus S) \cup S'})$ .

Let  $\Omega$  be a bounded subset of  $p(\tilde{G}_{S'})$  such that  $\Omega x^0 = Y$ . Take  $g \in \Omega$  and set  $\varsigma := (e, g^{-1})G(K) \in \Upsilon$ . We note that  $p(\tilde{G}_{S'}) \subset G^U$ . In particular,  $g \in G^U$ .

Let  $B'_h$  and  $U$  be defined as in the proof of Theorem 1.6(i), and let  $\tilde{\Psi}_{\epsilon} \subset G_{V_K}$  be the sets defined as in Proposition 5.6 which are indexed by  $\epsilon = (\epsilon_v)_{v \in S'}$  with finite  $S' \subset S$  and  $\epsilon_v \in I_v \cap (0, \epsilon_0)$ . As in the proof of Theorem 1.6(i), we will work on the space  $G^U/G(K)$ . We note that taking the components of  $\epsilon$ ,  $L_v^0$ , and  $\mathcal{O}_v$  sufficiently small, we can arrange that  $\tilde{\Psi}_{\epsilon} \subset G^U$ . Then Theorem 4.5 is applicable to the function  $\psi_{\epsilon}$  which is the characteristic function of the set  $\Psi_{\epsilon} := \tilde{\Psi}_{\epsilon}G(K) \subset G^U/G(K)$ . Hence, as in the proof of Theorem 1.6(i), we have

$$\left\| \pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta'_h) \psi_{\epsilon} - \mu(\Psi_{\epsilon}) \xi_U \right\|_2 \ll_{\delta'} h^{-\frac{a_S(G)}{q_{V_K \setminus S}(G)} + \delta'} \mu(\Psi_{\epsilon})^{1/2} \quad (6.19)$$

for every  $\delta' > 0$  and  $h \geq h_0(\delta')$ . We set

$$h_{\epsilon} = \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(G)}{a_S(G)} - \delta} \right)^{q_{V_K \setminus S}(G)}$$

with  $\delta > 0$ , and

$$\Upsilon_{\epsilon} = \{\varsigma \in G^U/G(K) : (B'_{h_{\epsilon}})^{-1}\varsigma \cap \tilde{\Psi}_{\epsilon}G(K) = \emptyset\}.$$

Since  $\pi_{(V_K \setminus S) \cup (V_K^f \setminus S')}(\beta'_{h_\epsilon})\psi_\epsilon = 0$  on  $\Upsilon_\epsilon$ , inequality (6.19) implies that

$$\mu(\Upsilon_\epsilon) \ll_{\delta'} |G_{V_K} : G^U|^{-1} h_\epsilon^{-\frac{2\mathfrak{a}_S(\mathbb{G})}{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})} + 2\delta'} \mu(\Psi_\epsilon)^{-1}$$

when  $h_\epsilon \geq h_0(\delta')$ . Combining this estimate with (5.19), we conclude that

$$\mu(\Upsilon_\epsilon) \ll_{\Omega, \delta'} \prod_{v \in S'} \epsilon_v^{\theta_v}, \quad (6.20)$$

where

$$\theta_v = r_v \dim(X) + \delta \cdot \mathfrak{q}_{V_K \setminus S}(\mathbb{G}) \left( \frac{2\mathfrak{a}_S(\mathbb{G})}{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})} - 2\delta' \right) - \delta' \cdot 2r_v \frac{\dim(\mathbb{G})}{\mathfrak{a}_S(X)}.$$

This estimate holds for all  $\epsilon$  satisfying  $h_\epsilon \geq h_0(\delta')$ , and hence for all  $\epsilon$  with sufficiently small components (depending on  $\delta'$ ).

We pick  $\delta' = \delta'(\delta) > 0$  such that  $\theta_v > r_v \dim(\mathbb{G})$ .

Let  $g \in \Omega$  and  $\varsigma := (e, g^{-1})G(K)$ . Then by (5.20),

$$\mu(\tilde{\Psi}_\epsilon^{-1}\varsigma) \gg_{\Omega} \prod_{v \in S'} \epsilon_v^{r_v \dim(X)},$$

Comparing this estimate with (6.20), we conclude that

$$\mu(\tilde{\Psi}_\epsilon^{-1}\varsigma) > \mu(\Upsilon_\epsilon)$$

provided that  $\epsilon_v \leq \epsilon_0(Y, S', \delta)$ ,  $v \in S'$ . Then  $\tilde{\Psi}_\epsilon^{-1}\varsigma \not\subseteq \Upsilon_\epsilon$ , and

$$(B'_h)^{-1}\tilde{\Psi}_\epsilon^{-1}\varsigma \cap \Psi_\epsilon \neq \emptyset.$$

Hence, Proposition 5.6 implies that there exists  $z \in G(O_{(V_K \setminus S) \cup S'})$  such that

$$H(z) \ll_{\Omega} \max_{b \in B'_{h_\epsilon}} H(b) \ll h_\epsilon = \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(\mathbb{G})}{\mathfrak{a}_S(\mathbb{G})} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})},$$

and for  $x = gx_0$ , we have

$$\|x_v - zx^0\| \leq \epsilon_v \quad \text{for all } v \in S'.$$

Since  $H(zx^0) \ll H(z)$ , we deduce that for every  $\delta > 0$  and  $(\epsilon_v)_{v \in S'}$  with  $\epsilon_v \in I_v \cap (0, \epsilon_0(Y, \delta))$ , we have

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \ll_{\Omega} \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(\mathbb{G})}{\mathfrak{a}_S(\mathbb{G})} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})}.$$

Moreover, this implies that for all sufficiently small  $\epsilon$  (depending on  $\delta$  and  $\Omega$ ), we have

$$\omega_S(x, (\epsilon_v)_{v \in S'}) \ll_{\Omega} \left( \prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(\mathbb{G})}{\mathfrak{a}_S(\mathbb{G})} - 2\delta} \right)^{\mathfrak{q}_{V_K \setminus S}(\mathbb{G})}.$$

This completes the proof of the theorem.  $\square$

## NOTATION

$K$	a number field,
$V_K$	the set of places of $K$ ,
$V_K^f$	the subset of non-Archimedean places,
$V_K^\infty$	the subset of Archimedean places,
$O$	the ring of integers of $K$ ,
$O_S$	the ring of $S$ -integers of $K$ ,
$K_v$	the completion of $K$ at $v \in V_K$ ,
$O_v$	the ring of integers of $K_v$ ,
$q_v$	the cardinality of the residue field of $O_v$ ,
$r_v$	2 if $v$ is a complex place and 1 otherwise,
$I_v$	$(0, 1)$ for $v \in V_K^\infty$ and $\{q_v^{-n}\}_{n \geq 1}$ for $v \in V_K^f$ ,
$G_v$	the set of $K_v$ -points of $G$ ,
$U_v$	the maximal compact subgroup of $G_v$ chosen in Section 3,
$G_S$	the restricted direct product of $(G_v, U_v)$ with $v \in S$ ,
$m_{G_v}$	the Haar measure on $G_v$ normalised, so that $m_{G_v}(U_v) = 1$ for non-Archimedean $v$ ,
$m_S$	the product of $m_{G_v}$ over $v \in S$ ,
$m_v$	the Haar measure on $G_v$ defined by $G$ -invariant differential form,
$m$	the Tamagawa measure on $G_{V_K}$ which is the product of $m_v$ 's,
$H$	the height function, see (1.2), (5.2),
$\omega_S$	the Diophantine approximation function, see (1.6),
$\mathfrak{a}_S(X)$	the exponent of a variety $X$ , see (1.7),
$\mathfrak{q}_v(G)$	the integrability exponent of automorphic representations, see (1.10),
$\sigma_S$	the exponent of a subset $S$ of $V_K$ , see (1.11),
$\mathfrak{q}_S(G)$	see (1.12),
$\Upsilon$	$G_{V_K}/G(K)$ ,
$\mu$	the probability Haar measure on $\Upsilon$ ,
$\mathcal{X}_{\text{aut}}(G_{V_K})$	the set of automorphic characters of $G_{V_K}$ ,
$\mathcal{X}_{\text{aut}}(G_{V_K})^U$	the set of automorphic characters invariant under $U$ ,
$G^U$	the kernel of $\mathcal{X}_{\text{aut}}(G_{V_K})^U$ in $G_{V_K}$ ,
$\mathcal{O}_S(\epsilon)$	the neighborhood of identity in $G_S$ , see (5.4),
$B_h$	see (6.1).

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