UNIVERSITY OF BRISTOL

Examination for the Degree of M.Sci. and M.Sc. (Level M)

Lie Groups, Lie Algebras and their Represenations
MATH M0012
(Paper Code MATH-M0012)

January 2016, 1 hours 30 minutes

This paper contains **TWO** *questions.*
Candidates should answer **BOTH** *questions.*

Calculators are **not** *permitted in this examination.*

Please note: Some questions may be organised sequentially, with subsequent parts building on previous parts. Credit will be awarded for correct answers to subsequent parts in which results from previous parts have been assumed, even if the previous parts have not been correctly answered.

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_j \sigma_k = i\epsilon_{jkl}\sigma_l$$

$$\rho_L(\mathbf{x}) = \frac{1}{\det L'_{\Phi(\mathbf{x})}}(0)$$

Do not turn over until instructed.

1. (50 marks total)

   (a) (15 marks) Let $G \subset \mathbb{C}^{n\times n}$ be a matrix Lie group. Define its Lie algebra, $\mathfrak{g}$, in terms of smooth curves $t \mapsto A(t) \in G$, and show that $\mathfrak{g}$ is a real vector space.

   (b) The Cayley transform on $n \times n$ real matrices X is given by

   $$\Phi(X) = (I - X)(I + X)^{-1},$$

   where X is restricted to have no eigenvalue equal to -1. Letting $\mathbb{R}_-^{n\times n}$ denote the space of antisymmetric matrices, you are given that $\Phi : \mathbb{R}_-^{n\times n} \to SO(n)$ is a parameterisation of $SO(n)$ that satisfies the requirements for $SO(n)$ to be a matrix Lie group. You are also given that $\Phi(\Phi(X)) = X$. Compute the left-invariant Haar measure $\rho_L(X)$ according to the following plan:

   i. (5 marks) For $R \in SO(n)$, show that the map $L_R : \mathbb{R}_-^{n\times n} \to \mathbb{R}_-^{n\times n}$, which satisfies $\Phi(L_R(Y)) = R\,\Phi(Y)$, is given by

   $$L_R(Y) = \Phi(R\,\Phi(Y)).$$

   ii. (10 marks) For $X, Y \in \mathbb{R}_-^{n\times n}$, show that

   $$L_{\Phi(X)}(\epsilon Y) = X + \epsilon(I - X)Y(I + X) + O(\epsilon^2).$$

   Hence argue that

   $$L'_{\Phi(X)}(0) \cdot Y := \frac{d}{dt} L_{\Phi(X)}(tY)\bigg|_{t=0} = (I - X)Y(I + X).$$

   iii. (10 marks) You are given that $X \in \mathbb{R}_-^{n\times n}$ is diagonalisable with imaginary eigenvalues $i\omega_j$ and corresponding eigenvectors $v_{(j)}$. Let

   $$Y = v_{(j)}v_{(k)}^T - v_{(k)}v_{(j)}^T, \quad \text{i.e., } Y_{ab} = v_{(j)a}v_{(k)b} - v_{(k)a}v_{(j)b},$$

   so that $Y \in \mathbb{C}_-^{n\times n}$. Show that

   $$L'_{\Phi(X)}(0) \cdot Y = \lambda Y,$$

   and express λ in terms of the eigenvalues of X.

   iv. (10 marks) Hence show that

   $$\rho_L(X) = \frac{1}{\det(I - X)^{n-1}}.$$

Continued...

2. (50 marks total)

(a) (10 marks) Show that

$$\exp(i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}) = \cos\theta I_2 + i\sin\theta\,\hat{\mathbf{n}}\cdot\boldsymbol{\sigma},$$

where $\hat{\mathbf{n}}$ is a unit vector in $\mathbb{R}^3$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

(b) (15 marks) Let (Γ_1, V_1), (Γ_2, V_2) be representations of a group G. Let $\phi : V_1 \to V_2$ be a (linear) intertwining map, i.e.

$$\Gamma_2(g)\,\phi = \phi\,\Gamma_1(g), \quad \forall g \in G.$$

Show the following:

i. If Γ_1 is irreducible, then either $\phi = 0$ or ϕ is injective.
ii. If Γ_2 is irreducible, then either $\phi = 0$ or ϕ is surjective.

(c) Let $\mathfrak{g} \subset \mathbb{C}^{n\times n}$ denote a compact simple Lie algebra of rank r. Let $\mathfrak{h} \subset \mathfrak{g}$ denote a Cartan subalgebra with orthonormal basis $h_1, \ldots, h_r$ (ie, orthonormal with respect to the standard inner product on $\mathbb{C}^{n\times n}$).

i. (5 marks) Let $(\hat{\Gamma}, V)$ denote a representation of $\mathfrak{g}$. Explain what it means for $\boldsymbol{\mu} \in \mathbb{R}^d$ to be a weight of $\mathfrak{g}$, and for $\boldsymbol{\alpha} \in \mathbb{R}^d$ to be a root of $\mathfrak{g}$.

ii. (5 marks) Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be roots of $\mathfrak{g}$. Given the formula

$$\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\alpha^2} = \frac{q-p}{2},$$

explain how the nonnegative integers q and p are defined in terms of the adjoint representation of $\mathfrak{g}$.

iii. (15 marks) Suppose $\mathfrak{g}$ has rank two with simple roots $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ satisfying $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = -1$, $\alpha^2 = 2$, $\beta^2 = 1$. Show that $\mathfrak{g}$ has four positive roots, which you should express in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and determine their lengths and the angles between them. Finally, determine the dimension of $\mathfrak{g}$.

End of examination.