Introduction to Lie groups, Lie algebras and their representations
Problems Sheet 3
To hand in: 1, 4, 6, 11* (*nonexaminable)

Due: Wednesday, November 15

1. SO(n) and homogeneous polynomials.

   Let $V(d,n)$ denote the space of real homogeneous polynomials of degree d on $\mathbb{R}^n$, with real inner product given by

   $$\langle f, g\rangle := \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-x^2} f(\mathbf{x}) g(\mathbf{x})\, d^n x,$$

   where $x = ||\mathbf{x}||$.

   (a) Let $SO(n)$ denote the matrix Lie group of real orthogonal $n \times n$ matrices with determinant equal to 1. Given $R \in SO(n)$, define

   $$\Gamma(R) : V(d,n) \to V(d,n)$$

   by

   $$(\Gamma(R)f)(\mathbf{x}) = f\left(R^{-1}\mathbf{x}\right)$$

   for $f \in V(d,n)$. Show that Γ is a representation on $V(d,n)$, and show that Γ is orthogonal, i.e.

   $$\langle \Gamma(R)f, \Gamma(R)g\rangle = \langle f, g\rangle,$$

   for all $f, g \in V(d,n)$.

   (b) The Lie algebra of SO(n), denoted so(n), is the space of $n \times n$ antisymmetric matrices. Let $A \in \mathrm{so}(n)$. Define

   $$\hat{\Gamma}(A) : V(d,n) \to V(d,n)$$

   by

   $$\left(\hat{\Gamma}(A)f\right)(\mathbf{x}) := \left.\frac{\partial}{\partial t}\right|_{t=0} \left(\Gamma(\exp(-tA))f\right)(\mathbf{x}).$$

   Show that

   $$\hat{\Gamma}(A)f = \mathbf{x}^t A \boldsymbol{\nabla} f = \sum_{j,k=1}^{n} x_j A_{jk} \frac{\partial f}{\partial x^k}.$$

(c) Verify directly that

$$[\hat{\Gamma}(A), \hat{\Gamma}(B)] = \hat{\Gamma}([A, B]).$$

2. Molien series.

Let $(\Gamma, \mathbb{R}^n)$ be a real representation of a finite group G. Let $V(n, d)$ denote the space of homogeneous polynomials on $\mathbb{R}^n$ of degree d. A homogeneous polynomial f is *invariant* under G if

$$f(\Gamma^T(g)\mathbf{x}) = f(\mathbf{x})$$

for all $g \in G$. Let N_d denote the dimension of the subspace of G-invariant polynomials $V(n, d)$. Show that

$$N_d = \frac{1}{d!} \left.\frac{d}{dt}\right|_{t=0} M(t),$$

where

$$M(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I_n - t\Gamma(g))}.$$

Formally, we may write that

$$M(t) = \sum_{d=0}^{\infty} N_d t^d.$$

$M(t)$ is called the Molien function. It can be defined for compact matrix Lie groups as well. Suggested argument:

(a) Consider the representation Γ^d of G on $(\otimes \mathbb{R}^n)^d$, the d-fold tensor product of $\mathbb{R}^n$ with itself, given by

$$\Gamma^d(g)\left(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_d\right) = \left(\Gamma(g)\mathbf{x}_1\right) \otimes \cdots \otimes \left(\Gamma(g)\mathbf{x}_d\right),$$

and the representation Δ^d of the symmetric group S_d on $(\otimes \mathbb{R}^n)^d$ given by

$$\Delta^d(\sigma)\left(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_d\right) = \mathbf{x}_{\sigma^{-1}(1)} \otimes \cdots \otimes \mathbf{x}_{\sigma^{-1}(d)}.$$

Show that the space of G-invariant homogenous polynomials of degree d may be identified with the subspace of $(\otimes \mathbb{R}^n)^d$ which is invariant under both Γ^d and Δ^d.

(b) Following the example of Problem Sheet 2.11, show that the projector P^d onto the subspace of $(\otimes \mathbb{R}^n)^d$ invariant under Γ^d and Δ^d is given by

$$P^d = \frac{1}{|G|} \sum_{g \in G} \frac{1}{d!} \sum_{\sigma \in S_d} \Gamma^d(g)\Delta^d(\sigma).$$

Then argue that

$$N_d = \operatorname{Tr} P^d.$$

(c) If σ has a decomposition into disjoint cycles with m_1 cycles of length 1, m_2 cycles of length 2, and so on up to m_d cycles of length d, where $m_j \geq 0$, show that

$$\operatorname{Tr}\left(\Gamma^d(g)\Delta^d(\sigma)\right) = (\operatorname{Tr}\Gamma(g))^{m_1}(\operatorname{Tr}\Gamma^2(g))^{m_2} \cdots (\operatorname{Tr}\Gamma^d(g))^{m_d}.$$

(d) Show that the number of permutations with cycle decomposition $(m_1, \ldots, m_d)$ is given by

$$\frac{d!}{1^{m_1} 2^{m_2} \cdots d^{m_d}\, m_1! \cdots m_d!}.$$

(e) Using the formula

$$\det(I - M) = \exp(\mathrm{Tr}(\log(I - M)) = \prod_{j=1}^{\infty} \exp\left(-\frac{\mathrm{Tr}\, M^j}{j}\right),$$

show that

$$\sum_{d=0}^{\infty} N_d t^d = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I_n - t\Gamma(g))}.$$

3. Show that SU(n) is not a simple group. Show that su(n) is a simple Lie algebra.

4. Suppose that $H \subset G$ is a normal subgroup and that H is a matrix Lie group in its own right, with Lie algebra $\mathfrak{h}$. Show that $\mathfrak{h} \subset \mathfrak{g}$ is an ideal.

5. Let $\Gamma : G \to GL(V)$ be a representation of a matrix Lie group $G \subset GL(n, \mathbb{C})$ on a vector space V. Show that if Γ is continuous, then it is also smooth (i.e., infinitely differentiable). More explicitly, if $\Phi : U \subset \mathbb{R}^d \to V_I$ is a smooth parameterisation of a neighbourhood V_I of the identity in $GL(n, \mathbb{C})$, then $\Gamma \circ \Phi : U \to GL(V)$ is smooth.

   (Suggested plan: Introduce a basis on V and use the matrix exponential and logarithm to define a map $\hat{\Gamma} : \mathfrak{g} \to \mathbb{C}^{r \times r}$, where $r = \dim V$. Then show that

   (a) for all t, $\Gamma(\exp ta) = \exp(t\hat{\Gamma}(a))$ (you may want to show that matrices close to the identity have a unique qth root that is also close to the identity).

   (b) Using the result of (a) along with the Baker-Campbell-Hausdorff theorem, show that $\hat{\Gamma}$ is linear (and therefore smooth).

   (c) The fact that $\Gamma \circ \Phi : U \to GL(V)$ is smooth is then a more or less immediate consequence of the preceding results.)

6. Let (Γ, V) be a representation of a matrix Lie group G on a vector space V, and let $\hat{\Gamma} : \mathfrak{g} \to L(V)$ be the representation of its Lie algebra given by (cf Theorem 7.6)

$$\hat{\Gamma}(a) = \left.\frac{d}{dt}\right|_{t=0} \Gamma(\exp ta).$$

   Prove Proposition 7.8: Show that if Γ is reducible, so is $\hat{\Gamma}$.

7. As in the previous problem, let (Γ, V) be a representation of a matrix Lie group G, and let $(\hat{\Gamma}, V)$ denote the representation of its Lie algebra given by (cf Theorem 7.6)

$$\hat{\Gamma}(a) = \left.\frac{d}{dt}\right|_{t=0} \Gamma(\exp ta).$$

   Suppose that G is connected. Show that Γ may be expressed in terms of $\hat{\Gamma}$. Then show that if $\hat{\Gamma}$ is reducible, then Γ is reducible.

8. Prove Theorem 9.10: Let $\mathfrak{g}$ be a compact simple Lie algebra of a matrix Lie group G. Show that all Cartan subalgebras of $\mathfrak{g}$ are conjugate; that is, if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Cartan subalgebras, then

$$\mathfrak{h}_2 = \mathrm{Ad}_A\, \mathfrak{h}_1 = A\mathfrak{h}_1 A^{-1}$$

   for some $A \in G$. (Note: it is *not* the case that individual elements of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are necessarily conjugate. Indeed, a necessary condition for $h_1 = \mathrm{Ad}_A\, h_2$ is that h_1 and h_2 have the same eigenvalues.)

   <u>Remarks:</u> Here is an outline of a proof:

(a) Show that if $\mathfrak{h}$ is a Cartan subalgebra, there exists $h_* \in \mathfrak{h}$ such that $\mathfrak{h} = \ker \operatorname{ad}_{h_*}$. In other words, $a \in \mathfrak{h}$ if and only if $[a, h_*] = 0$. (Note: this is clearly not true for all $h \in \mathfrak{h}$. Trivially it is not true if $h = 0$, but it also fails to hold for certain nonzero (degenerate) elements of $\mathfrak{h}$.)

(b) With $h_{1*} \in \mathfrak{h}_1$ chosen as in (a) above, show that there exists an $A_* \in G$ such that $\operatorname{Ad}_{A_*} h_{1*} \in \mathfrak{h}_2$. Suggestion: consider the following variational argument. Choose $h_{2*} \in \mathfrak{h}_2$ as in (a) above, and consider the squared distance (defined with respect to the standard inner product) between $\operatorname{Ad}_A h_{1*}$ and h_{2*}, which is given by

$$s^2(A) = \langle \operatorname{Ad}_A h_{1*} - h_{2*}, \operatorname{Ad}_A h_{1*} - h_{2*} \rangle.$$

Explain why s^2 has a minimum, say at $A = A_*$, and show that $\operatorname{Ad}_{A_*} h_{1*} \in \mathfrak{h}_2$.

(c) Using the results of (a) and (b), show that $\operatorname{Ad}_{A_*} \mathfrak{h}_1 = \mathfrak{h}_2$.

9. Let $B(j,k)$ denote the $n \times n$ matrix with a single nonzero element, the (j,k)th element, which is equal to one. That is, if $[M]_{rs}$ denotes the (r,s)th element of M, then

$$[B(j,k)]_{rs} = \delta_{jr}\delta_{ks}.$$

Show that the $B(j,k)$'s form an orthonormal basis for $\mathbb{C}^{n\times n}$, and that

$$B(j,k)B(l,m) = \delta_{kl}B(j,m).$$

10. Dynkin diagram for su(n)

su(n) is the Lie algebra of $n \times n$ traceless antihermitian matrices.

(a) Show that a Cartan subalgebra $\mathfrak{h} \subset \mathrm{su}(n)$ may taken to be the set of $n \times n$ traceless, diagonal purely imaginary matrices. Verify that the rank of su(n) is $n-1$.

(b) Show that an orthonormal basis for $\mathfrak{h}$ is given by

$$[h(m) = iN_m \left(\sum_{a=1}^{m} B(a,a) - mB(m+1,m+1) \right), \quad N_m = (m(m+1))^{-1/2}.$$

(c) Show that su$_{\mathbb{C}}(n)$, the complexification of su(n), is the space of $n \times n$ traceless complex matrices.

(d) Show that

$$\hat{h}(m)B(j,k) = iN_m \left\{ \sum_{a=1}^{m} (\delta_{ja} - \delta_{ka}) - m\left(\delta_{j,m+1} - \delta_{k,m+1}\right) \right\} B(j,k).$$

Hence show that the $B(j,k)$'s with $j \neq k$ together with the $h(m)$'s form a complete set of orthonormal eigenvectors of the adjoint representation of $\mathfrak{h}$, and that the roots of su(n) are given

$$\boldsymbol{\alpha}(j,k) = -\left(\frac{j-1}{j}\right)^{1/2} \widehat{\mathbf{j-1}}, + \sum_{l=j}^{k-2} N(l)\,\hat{\mathbf{l}} + \left(\frac{k}{k-1}\right)^{1/2} \widehat{\mathbf{k-1}}, \quad j < k,$$

where $\hat{l}$ denotes the lth coordinate unit-vector in $\mathbb{R}^d$, and $\widehat{0}$ is defined to be 0.

$$\boldsymbol{\alpha}(j,k) = -\boldsymbol{\alpha}(k,j), \quad j > k.$$

(e) Let us introduce a different convention for positivity of roots in the case of su(n). We will say that $\boldsymbol{\alpha}$ is *positive* if the last nonzero element of $\boldsymbol{\alpha}$ is positive. Thus, $\boldsymbol{\alpha}(j,k)$ is positive if and only if $j < k$. Show that the simple roots are given by

$$\boldsymbol{\alpha}(j,j+1) = -\left(\frac{j-1}{j}\right)^{1/2}\widehat{\mathbf{j}-\mathbf{1}},+\left(\frac{j+1}{j}\right)^{1/2}\widehat{\mathbf{j}},\quad 1 \le j \le n-1.$$

(f) Show that

$$\boldsymbol{\alpha}(j,j+1)\cdot\boldsymbol{\alpha}(k,k+1) = \begin{cases} 2, & j = k, \\ -1, & |j = k| = 1, \\ 0, & \text{otherwise} \end{cases}.$$

(g) Hence show that the angle between consecutive simple roots is 120°, and that all simple roots have the same length. Thus, the Dynkin diagram of su(n) is given by Figure 1.

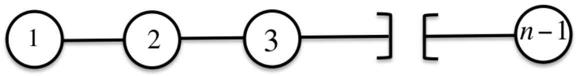

Figure 1: Dynkin diagram for su(n).

11. Dynkin diagram for so($2n$)

so($2n$) is the Lie algebra of $2n \times 2n$ real antisymmetric matrices.

It is convenient to represent $\mathbb{R}^{2n}$ as the tensor product $\mathbb{R}^n \otimes \mathbb{R}^2$. Then $\mathbb{R}^{2n\times 2n}$ corresponds to the tensor product $\mathbb{R}^{n\times n} \otimes \mathbb{R}^{2\times 2}$. For $M \otimes S \in \mathbb{R}^{n\times n} \otimes \mathbb{R}^{2\times 2}$, the transpose is defined by

$$(M \otimes S)^T = M^T \otimes S^T.$$

The trace is given by

$$\operatorname{tr}(M \otimes S) = (\operatorname{tr} M)(\operatorname{tr} S).$$

The inner product is given by

$$\langle M \otimes S, N \otimes T\rangle = \langle M, N\rangle\,\langle S, T\rangle.$$

A basis for $\mathbb{R}^{n\times n} \otimes \mathbb{R}^{2\times 2}$ is given by

$$B(j,k)\otimes I_2,\quad B(j,k)\otimes\sigma_1,\quad B(j,k)\otimes i\sigma_2,\quad B(j,k)\otimes\sigma_3,\quad \text{where } 1 \le j,k \le n,$$

where $B(j,k)$ is defined in Question 9. It follows that

$$\begin{aligned}(B(j,k)\otimes I_2)^T &= B(k,j)\otimes I_2,\\ (B(j,k)\otimes\sigma_1)^T &= B(k,j)\otimes\sigma_1,\\ (B(j,k)\otimes i\sigma_2)^T &= -B(k,j)\otimes i\sigma_2,\\ (B(j,k)\otimes\sigma_3)^T &= B(k,j)\otimes\sigma_3.\end{aligned}$$

(a) Show that a real orthonormal basis for so($2n$) is given by the following:

$$\begin{aligned}
a^1(j,k) &= \frac{1}{2}\left(B(j,k) - B(k,j)\right) \otimes \sigma_3, \quad 1 \le j < k \le n, \\
a^2(j,k) &= \frac{1}{2}\left(B(j,k) - B(k,j)\right) \otimes \sigma_1, \quad 1 \le j < k \le n, \\
b^1(j,k) &= \frac{1}{2}\left(B(j,k) + B(k,j)\right) \otimes i\sigma_2, \quad 1 \le j < k \le n, \\
b^2(j,k) &= \frac{1}{2}\left(B(j,k) - B(k,j)\right) \otimes I_2, \quad 1 \le j < k \le n, \\
h(m) &= \frac{1}{\sqrt{2}} B(j,j) \ \otimes i\sigma_2, \quad 1 \le j \le n,
\end{aligned}$$

(b) Show that a Cartan subalgebra for so($2n$) may be taken to be the span of the elements $h(m)$. Thus, so($2n$) has rank n.

(c) Show that simultaneous eigenvectors of $\hat{h}(m)$ with nonzero eigenvalues are given by

$$\begin{aligned}
a^{\pm}(j,k) &= a^1(j,k) \pm i a^2(j,k), \quad 1 \le j < k \le n, \\
b^{\pm}(j,k) &= b^1(j,k) \pm i b^2(j,k), \quad 1 \le j < k \le n,
\end{aligned}$$

and that these, together with the $h(m)$'s, span $\mathrm{so}_{\mathbb{C}}(2n)$. Show that the corresponding roots are

$$\begin{aligned}
\boldsymbol{\alpha}^{\pm}(j,k) &= \pm\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\right), \\
\boldsymbol{\beta}^{\pm}(j,k) &= \pm\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}\right),
\end{aligned}$$

(d) Show that the simple roots of so($2n$) are given by

$$\begin{aligned}
\boldsymbol{\alpha}^{+}(n-1,n) &= \frac{1}{\sqrt{2}}\left(\widehat{\mathbf{n-1}} + \hat{\mathbf{n}}\right), \\
\boldsymbol{\beta}^{+}(j,j+1) &= \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\jmath}} - \widehat{\mathbf{j+1}}\right), \quad 1 \le j \le n-1.
\end{aligned}$$

(e) Show that the inner products of the simple roots are given by

$$\begin{aligned}
\boldsymbol{\beta}^{+}(j,j+1)\cdot\boldsymbol{\beta}^{+}(k,k+1) &= \begin{cases} 1, & j = k \\ -1/2, & |j-k| = 1, \\ 0, & \text{otherwise,} \end{cases}, \\
\boldsymbol{\alpha}^{+}(n-1,n)\cdot\boldsymbol{\alpha}^{+}(n-1,n) &= 1, \\
\boldsymbol{\alpha}^{+}(n-1,n)\cdot\boldsymbol{\beta}^{+}(j-1,j) &= \begin{cases} -1/2, & j = n-1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

(f) Show that the Dynkin diagram for so($2n$) is given by Figure 2.

12. Dynkin diagram for so($2n+1$)

so($2n+1$) is the Lie algebra of $(2n+1)\times(2n+1)$ real antisymmetric matrices.

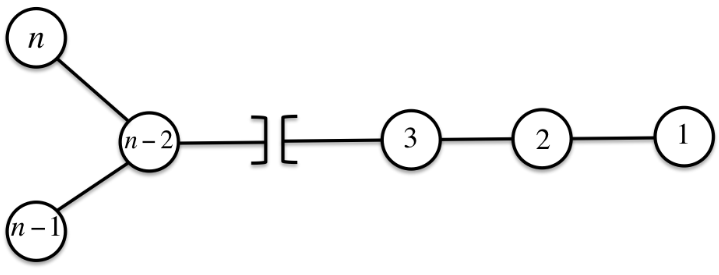

Figure 2: Dynkin diagram for so($2n$)

(a) Let $\mathbf{f_{(1)}}, \ldots, \mathbf{f_{(2n+2)}}$ denote the standard orthonormal basis on $\mathbb{R}^{2n+2}$. Show that so$(2n+1)$ may be identified with the subset of so$(2n+2)$ which has $\mathbf{f_{(2n+2)}}$ as a null vector.

(b) Show that the space of real 2×2 matrices that have $(0,1)^T$ as a null vector is spanned by

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Show that

$$P\, i\sigma_2 = Q^T,$$
$$Q\, i\sigma_2 = -P,$$

so that

$$i\sigma_2\, P = -Q,$$
$$i\sigma_2\, Q^T = P.$$

(c) With reference to the preceding identification, show that a real orthonormal basis for so$(2n+1)$ is given by the following:

$$a^1(j,k) = \frac{1}{2}\left(B(j,k) - B(k,j)\right) \otimes \sigma_3, \quad 1 \le j < k \le n,$$
$$a^2(j,k) = \frac{1}{2}\left(B(j,k) - B(k,j)\right) \otimes \sigma_1, \quad 1 \le j < k \le n,$$
$$b^1(j,k) = \frac{1}{2}\left(B(j,k) + B(k,j)\right) \otimes i\sigma_2, \quad 1 \le j < k \le n,$$
$$b^2(j,k) = \frac{1}{2}\left(B(j,k) - B(k,j)\right) \otimes I_2, \quad 1 \le j < k \le n,$$
$$c^1(j) = \frac{1}{\sqrt{2}}\left(B(j,n+1) - B(n+1,j)\right) \otimes P, \quad 1 \le j \le n,$$
$$c^2(j) = \frac{1}{\sqrt{2}}\left(B(j,n+1) \otimes Q - B(n+1,j) \otimes Q^T\right), \quad 1 \le j \le n,$$
$$h(m) = \frac{1}{\sqrt{2}} B(j,j) \ \otimes i\sigma_2, \quad 1 \le j \le n.$$

(d) Show that a Cartan subalgebra for so$(2n+1)$ may be taken to be the span of the elements $h(m)$. Thus, so$(2n+1)$ has rank n.

(e) Show that simultaneous eigenvectors of $\hat{h}(m)$ with nonzero eigenvalues are given by

$$\begin{aligned} a^{\pm}(j,k) &= a^1(j,k) \pm i a^2(j,k), \quad 1 \le j < k \le n, \\ b^{\pm}(j,k) &= b^1(j,k) \pm i b^2(j,k), \quad 1 \le j < k \le n, \\ c^{\pm}(j) &= c^1(j) \pm i c^2(j), \quad 1 \le j \le n, \end{aligned}$$

and that these, together with the $h(m)$'s, span $\mathrm{so}_{\mathbb{C}}(2n+1)$. Show that the corresponding roots are

$$\begin{aligned} \boldsymbol{\alpha}^{\pm}(j,k) &= \pm\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\right), \\ \boldsymbol{\beta}^{\pm}(j,k) &= \pm\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}\right), \\ \boldsymbol{\gamma}^{\pm}(j) &= \pm\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}. \end{aligned}$$

(f) Show that the simple roots of $\mathrm{so}(2n+1)$ are given by

$$\begin{aligned} \boldsymbol{\beta}^{+}(j,j+1) &= \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\jmath}} - \widehat{\mathbf{j}+\mathbf{1}}\right), \quad 1 \le j \le n-1, \\ \boldsymbol{\gamma}^{+}(n) &= \frac{1}{\sqrt{2}}\,\widehat{\mathbf{n}}. \end{aligned}$$

(g) Show that the inner products of the simple roots are given by

$$\begin{aligned} \boldsymbol{\beta}^{+}(j,j+1)\cdot\boldsymbol{\beta}^{+}(k,k+1) &= \begin{cases} 1, & j = k \\ -1/2, & |j-k| = 1, \\ 0, & \text{otherwise,} \end{cases}, \\ \boldsymbol{\gamma}^{+}(n)\cdot\boldsymbol{\gamma}^{+}(n) &= 1/2, \\ \boldsymbol{\gamma}^{+}(n)\cdot\boldsymbol{\beta}^{+}(j-1,j) &= \begin{cases} -1/2, & j = n-1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(h) Show that the Dynkin diagram for $\mathrm{so}(2n+1)$ is given by Figure 3.

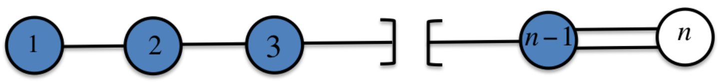

Figure 3: Dynkin diagram for $\mathrm{so}(2n+1)$

13. Dynkin diagram for $\mathrm{usp}(2n)$

It is convenient to represent $\mathbb{C}^{2n}$ as the tensor product $\mathbb{C}^n \otimes \mathbb{C}^2$. Then $\mathbb{C}^{2n\times 2n}$ corresponds to the tensor product $\mathbb{C}^{n\times n}\otimes\mathbb{C}^{2\times 2}$. For $M\otimes S \in \mathbb{C}^{n\times n}\otimes\mathbb{C}^{2\times 2}$, the hermitian conjugate is defined by

$$(M\otimes S)^{\dagger} = M^{\dagger}\otimes S^{\dagger}.$$

The transpose is defined by

$$(M\otimes S)^{T} = M^{T}\otimes S^{T}.$$

The trace is given by by

$$\operatorname{tr}(M \otimes S) = (\operatorname{tr} M)(\operatorname{tr} S).$$

The inner product is given by

$$\langle M \otimes S, N \otimes T \rangle = \langle M, N \rangle \langle S, T \rangle.$$

A basis for $\mathbb{C}^{n\times n} \otimes \mathbb{C}^{2\times 2}$ is given by

$$B(j,k) \otimes I_2, \quad B(j,k) \otimes \sigma_t, \quad \text{where } 1 \le j,k \le n, \quad 1 \le t \le 3,$$

where $B(j,k)$ is defined in Question 9. It follows that

$$\begin{aligned}
(B(j,k) \otimes I_2)^\dagger &= \left(B(j,k)^T \otimes I_2^T\right) = B(k,j) \otimes I_2, \\
(B(j,k) \otimes \sigma_t)^\dagger &= \left(B(j,k)^T \otimes \sigma_t^\dagger\right) = B(k,j) \otimes \sigma_t.
\end{aligned}$$

Let $J \in C^{n\times n} \otimes \mathbb{C}^{2\times 2}$ be given by

$$J = I_n \otimes i\sigma_2 = I_n \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The unitary symplectic Lie algebra, usp$(2n)$, consists of antihermitian matrices $s \in C^{n\times n} \otimes \mathbb{C}^{2\times 2}$ that satisfy

$$sJ + Js^T = 0,$$

(a) Let $a \in \mathbb{C}^{2\times 2}$. Show that $a(i\sigma_2) + (i\sigma_2)a^T = 0$ if and only if a is traceless, while $a(i\sigma_2) - (i\sigma_2)a^T = 0$ if and only if a is a multiple of the identity.

(b) Show that a (real) orthonormal basis for usp$(2n)$ is given by the following:

$$\begin{aligned}
&\frac{1}{2}\left(B(j,k) + B(k,j)\right) \otimes i\sigma_t, \quad 1 \le j < k \le n, \quad 1 \le t \le 3, \\
&\frac{1}{\sqrt{2}} B(m.m) \otimes i\sigma_t, \quad 1 \le m \le n, \quad 1 \le t \le 3, \\
&\frac{1}{2}\left(B(j,k) - B(k,j)\right) \otimes I_2 \quad 1 \le j < k \le n.
\end{aligned}$$

(c) Show that a Cartan subalgebra for usp$(2n)$ may be taken to be the span of the elements

$$h(m) = \frac{1}{\sqrt{2}} B(m,m) \otimes i\sigma_3, \quad 1 \le m \le n,$$

so that the rank of usp$(2n)$ is n. Verify that the $h(m)$'s are orthonormal.

(d) Show that simultaneous eigenvectors of $\hat{h}(m)$ with nonzero eigenvalues are given by

$$\begin{aligned}
a^\pm(j,k) &= \frac{1}{2\sqrt{2}}\left(B(j,k) + B(k,j)\right) \otimes \left(i\sigma_1 \mp \sigma_2\right), \quad 1 \le j < k \le n, \\
b^\pm(j,k) &= \frac{i}{2\sqrt{2}}\left(B(j,k) \otimes (i\sigma_3 \pm I_2) + B(k,j) \otimes (i\sigma_3 \mp I_2)\right), \quad 1 \le j < k \le n, \\
c^\pm(j) &= \frac{i}{2} B(j,j) \otimes \left(i\sigma_1 \mp \sigma_2\right),
\end{aligned}$$

and that these, together with the $h(m)$'s, span $\mathrm{usp}_{\mathbb{C}}(2n)$. Show that the corresponding roots

$$\boldsymbol{\alpha}^{\pm}(j,k) = \pm\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\right),$$
$$\boldsymbol{\beta}^{\pm}(j,k) = \pm\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}\right),$$
$$\boldsymbol{\gamma}^{\pm}(j) = \pm\sqrt{2}\hat{\boldsymbol{\jmath}}.$$

(e) Show that the simple roots of usp$(2n)$ are given by

$$\boldsymbol{\beta}^{+}(j,j+1) = \pm\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\jmath}} - \widehat{\mathbf{j}+\mathbf{1}}\right), \quad 1 \leq j \leq n-1,$$
$$\boldsymbol{\gamma}^{+}(n) = \sqrt{2}\hat{\mathbf{n}}.$$

(f) Show that the inner products of the simple roots are given by

$$\boldsymbol{\beta}^{+}(j,j+1)\cdot\boldsymbol{\beta}^{+}(k,k+1) = \begin{cases} 1, & j = k \\ -1/2, & |j-k| = 1, \\ 0, & \text{otherwise}, \end{cases},$$
$$\boldsymbol{\beta}^{+}(j,j+1)\cdot\boldsymbol{\gamma}^{+}(n) = \begin{cases} -1, & j = n-1 \\ 0, & \text{otherwise}, \end{cases},$$
$$\boldsymbol{\gamma}^{+}(n)\cdot\boldsymbol{\gamma}^{+}(n) = 2.$$

(g) Show that the Dynkin diagram for usp$(2n)$ is as shown in Figure 4.

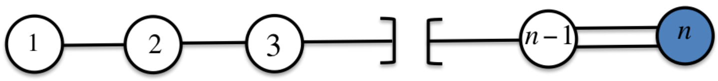

Figure 4: Dynkin diagram for usp$(2n)$

14. The exceptional Lie algebra G_2 has two simple roots, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, which satisfy

$$\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^2}{\alpha^2\,\beta^2} = \frac{3}{4}.$$

Determine the positive roots of G_2 and show that G_2 has dimension equal to 14.