Functional Analysis Exercise sheet 5 — selected solutions

6. Let $A : H \to H$ be a compact operator on a Hilbert space H such that $Ax = 0$ implies that $x = 0$.

   (a) Show that if A is self-adjoint, then there exists a sequence of operators A_n such that $A_n Ax \to x$ for all $x \in H$.

   (b) Show that the same claim is true for all compact A. (Hint: consider the operator A^*A.)

   (c) Can one choose A_n's such that $A_n A \to I$ in norm? (justify)

To prove (a), we use the spectral theorem for A. There exists an orthonormal set (e_k) such tha $Ax = \sum_{k\geq 1} \lambda_k \langle x, e_k \rangle e_k$. Moreover, we claim that (e_n) is complete. Indeed, suppose that $\langle x, e_k \rangle = 0$ for all k. Then $Ax = 0$, and by the assumption $x = 0$. Since (e_k) is a complete orthonormal set, every $x \in H$ can be written as $x = \sum_{k\geq 1} \langle x, e_k \rangle e_k$. We define the linear operator A_n as $A_n x = \sum_{k=1}^{n} \lambda_k \langle x, e_k \rangle e_k$. Then

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\begin{aligned}
A_n Ax &= \sum_{k=1}^{n} \lambda_k^{-1} \langle Ax, e_k \rangle e_k = \sum_{k=1}^{n} \lambda_k^{-1} \left\langle \sum_{i\geq 1} \lambda_i \langle x, e_i \rangle e_i, e_k \right\rangle e_k \\
&= \sum_{k=1}^{n} \langle x, e_k \rangle e_k \to x
\end{aligned}
$$

This proves (a).

We observe that $(A^*A)^* = A^*A$, so A^*A is self-adjoint. If $A^*Ax = 0$, then $\|Ax\|^2 = \langle x, A^*Ax \rangle = 0$, so that $Ax = 0$ and $x = 0$. The operator A^*A is compact by exercise 2. Hence, we can apply part (a) to this operator. We conclude that there exists a sequence of operators B_n such that $B_n(A^*A)x \to x$ for all $x \in H$. Now it is sufficient to take $A_n = B_n A^*$.

If $\dim H < \infty$, it follows from our assumption that the operator A has an inverse, and we have $A^{-1}A = I$. We claim that this is not possible if $\dim H = \infty$. Indeed, if it were true that $A_n A \to I$, then it would follow that I is a limit of a sequence of compact operators, so that it would be compact as well. However, the identity operator cannot be compact. To show this, we consider an infinite orthonormal set (e_n) in H (such a set can be constructed inductively since $\dim H = \infty$). We note that (e_n) is bounded, but it doesn't contain any convergent subsequence because $\|e_n - e_m\| = \sqrt{2}$ for $n \neq m$, so I is not compact. This contradiction shows that we cannot choose such A_n's in infinite-dimensional setting.