Functional Analysis Exercise sheet 4 — selected solutions

6. Let $V = C([0,1])$ equipped with the norm $\|f\|_2 = \sqrt{\int_0^1 |f(x)|^2 dx}$. For $\phi \in V$, consider the multiplication operator $A_\phi(f) = \phi \cdot f$. Show that A_ϕ is bounded and compute its norm.

For every $f \in V$,

$$\|A_\phi(f)\|_2^2 = \int_0^1 |\phi(x)f(x)|^2 dx \leq \left(\max_{x\in[0,1]} |\phi(x)|\right)^2 \int_0^1 |f(x)|^2 dx$$
$$= \|\phi\|_\infty^2 \|f\|_2^2.$$

This implies that $\|A_\phi\| = \sup\left\{\frac{\|A_\phi(f)\|_2}{\|f\|_2} : f \neq 0\right\} \leq \|\phi\|_\infty$. In particular, A_ϕ is bounded.

We claim that in fact $\|A_\phi\| = \|\phi\|_\infty$. Let $m = \|\phi\|_\infty$. Since $|\phi|$ is continuous, there exists $x_0 \in [0,1]$ such that $|\phi(x_0)| = m$, and for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in B_\delta(x_0)$, we have $|\phi(x)| \geq m - \epsilon$. Let f be a non-zero continuous function such that $f = 0$ outside of $B_\delta(x_0)$. Then

$$\|A_\phi(f)\|_2^2 = \int_0^1 |\phi(x)f(x)|^2 dx = \int_{B_\delta(x_0)} |\phi(x)f(x)|^2 dx$$
$$\geq (m-\epsilon)^2 \int_{B_\delta(x_0)} |f(x)|^2 dx = (m-\epsilon)^2 \|f\|_2^2.$$

Hence, $\|A_\phi\| = \sup\left\{\frac{\|A_\phi(f)\|_2}{\|f\|_2} : f \neq 0\right\} \geq m - \epsilon$ for every $\epsilon > 0$.

This proves that $\|A_\phi\| = \|\phi\|_\infty$.