5 Compact linear operators

One of the most important results of Linear Algebra is that for every self-adjoint linear map A on a *finite-dimensional* space, there exists a basis consisting of eigenvectors. In particular, with respect to this basis the operator A can be represented by a diagonal matrix. The situation turns out to be much more complicated for operators on infinite dimensional spaces (in fact, self-adjoint operators may have no eigenvectors at all). Nonetheless, a satisfactory theory can be developed for so-called compact operators which we now introduce.

Definition 5.1 (Compact linear operator)**.** *A linear operator $T : X \to Y$ between normed spaces X and Y is called a* ***compact linear operator*** *if for every bounded sequence $(x_n)_{n\geq 1}$ in X, the sequence $(Tx_n)_{n\geq 1}$ has a convergent subsequence.*

We note that every compact operator T is bounded. Indeed, if $\|T\| = \infty$, then there exists a sequence $(x_n)_{n\geq 1}$ such that $\|x_n\| \leq 1$ and $\|Tx_n\| \to \infty$. Then $(Tx_n)_{n\geq 1}$ cannot have a convergent subsequence. Hence, $\|T\| < \infty$.

Example.

1. Consider the linear operator $T_N : \ell^2 \to \ell^2$ defined by $T_N x = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$. We claim that T_N is compact. Given any bounded sequence $(x^{(n)})_{n\geq 1}$ in ℓ^2, the sequence $(T_N x^{(n)})_{n\geq 1}$ is bounded in $\mathbb{F}^N \subset \ell^2$. Since every bounded sequence in $\mathbb{R}^N$ or $\mathbb{C}^N$ has a convergent subsequence, it follows that T_N is compact.

2. The identity operator $I : \ell^2 \to \ell^2$ is not compact. To prove this, consider the bounded sequence $(e_n)_{n\geq 1}$ where $e_n = (0, \ldots, 0, 1, 0, \ldots)$. Then for any $n \neq m$, $\|Ie_n - Ie_m\|_2 = \|e_n - e_m\|_2 = \sqrt{2}$. This implies that any subsequence of $(Ie_n)_{n\geq 1}$ cannot be Cauchy and, hence, cannot converge. So that I is not compact.

3. (integral operators) Let $K \in C([0,1]^2)$ and $T : L^2([0,1]) \to L^2([0,1])$ is defined by $Tf(x) = \int_0^1 K(x,y)f(y)dy$. Then the operator T is compact. We will not prove this in class, and it could be a possible topic for a level-M presentation.

We show that limits of compact operators is also compact.

Theorem 5.2. *Let $(T_n)_{n\geq 1}$ be a sequence of compact linear operators from a normed space X into a Banach spaces Y. If $T_n \to T$ (that is, $\|T_n - T\| \to 0$), then the limit operator T is compact.*

Proof. Since T_1 is a compact operator, we know that the sequence $(T_1(x_n))$ has a convergent (hence Cauchy) subsequence $(T_1(x_{1,m}))$, where $(x_{1,m})$ is a subsequence of the original sequence (x_n). The subsequence $(x_{1,m})$ is bounded, so we can repeat the argument with T_2 to produce a subsequence

$(x_{2,m})$ of $(x_{1,m})$ with the property that $(T_2(x_{2,m}))$ converges. We continue in the same way, and then define a sequence $(y_m) = (x_{m,m})$. Notice that (y_m) is a subsequence of (x_n), so it is bounded, say by $\|y_n\| \leq c$, and it has the property that for every fixed n, the sequence $(T_n(y_m))$ is convergent, and hence Cauchy.

We claim that $(T(y_m))$ is a Cauchy sequence in Y. Let $\epsilon > 0$. Since $\|T_n - T\| \to 0$, there is some $p \in \mathbb{N}$ such that $\|T_p - T\| < \frac{\epsilon}{3c}$. Also, since $(T_p(y_m))$ is Cauchy, there is some $N > 0$ such that $\|T_p(y_j) - T_p(y_k)\| < \frac{\epsilon}{3}$ whenever $j, k > N$. Therefore, for $j, k > N$, we have

$$\begin{aligned}\|T(y_j) - T(y_k)\| &\leq \|T(y_j) - T_p(y_j) + T_p(y_j) - T_p(y_k) + T_p(y_k) - T(y_k)\| \\ &\leq \|T(y_j) - T_p(y_j)\| + \|T_p(y_j) - T_p(y_k)\| + \|T_p(y_k) - T(y_k)\| \\ &< \|T - T_p\|\,\|y_j\| + \frac{\epsilon}{3} + \|T - T_p\|\,\|y_k\| \\ &< \frac{\epsilon}{3c}c + \frac{\epsilon}{3} + \frac{\epsilon}{3c}c = \epsilon,\end{aligned}$$

which proves that $(T(y_m))$ is Cauchy. Since Y is a Banach space, it is by definition complete, so $(T(y_m))$ converges. We have thus produced, for an arbitrary bounded sequence $(x_n) \subset X$, a convergent subsequence of its image under T. Therefore, T is compact. □

Example. Let $T : \ell^2 \to \ell^2$ be an operator defined by $Tx = (\lambda_n x_n)_{n \geq 1}$ for a sequence $\lambda_n \to 0$. We claim that T is compact. To show this, we approximate T by compact operators T_N such that $T_N x = (\lambda_1 x_1, \ldots, \lambda_N x_N, 0, \ldots)$. As in the previous example we observe that T_N is a compact operator. For $x \in \ell^2$,

$$\begin{aligned}\|T_N x - Tx\|_2 &= \left(\sum_{n>N} |\lambda_n x_n|^2\right)^{1/2} \leq \left(\sup_{n>N} |\lambda_n|\right)\left(\sum_{n>N} |x_n|^2\right)^{1/2} \\ &\leq \left(\sup_{n>N} |\lambda_n|\right)\|x\|_2.\end{aligned}$$

This shows that $\|T_N - T\| \leq \sup_{n>N} |\lambda_n|$. Since $\lambda_n \to 0$, it follows that $\|T_N - T\| \to 0$. Hence, T is compact by the previous theorem.

We note that for the operator T in the previous example there exists a basis of eigenvectors e_n with eigenvalues λ_n. Remarkably, every self-adjoint compact operator on a Hilbert space is of this form: it can be diagonalised with respect to suitable orthonormal set. The following is the main result of this lecture:

Theorem 5.3 (spectral theorem)**.** *Let H be a Hilbert space and $T : H \to H$ a compact self-adjoint operator. Then there exists an orthonormal set*

$(e_n)_{n\geq 1}$ *consisting of eigenvectors of* T *with eigenvalues* λ_n *such that*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n, \quad x \in H.$$

The proof of this theorem require several auxilary results.

Theorem 5.4. *Let* T *be a self-adjoint operator on a Hilbert space* H. *Then*

$$\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}.$$

Proof. Let $m = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$. For $x \in H$ with $\|x\| = 1$,

$$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|.$$

This shows that $m \leq \|T\|$.

For any $x, y \in H$,

$$\operatorname{Re}\langle Tx, y\rangle = \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle).$$

This formula can be checked by expanding the right hand side. We use that $|\langle Tu, u\rangle| \leq m\|u\|^2$ for all $u \in H$. Then from the above formula, we obtain

$$\begin{aligned}\operatorname{Re}\langle Tx, y\rangle &\leq \frac{1}{4}(|\langle T(x+y, x+y\rangle| + |\langle T(x-y), x-y\rangle|) \\ &\leq \frac{1}{4}(m\|x+y\|^2 + m\|x-y\|^2) = \frac{m}{2}(\|x\|^2 + \|y\|^2),\end{aligned}$$

where we used the parallelogram identity. Replacing x by λx where $|\lambda| = 1$ and λ is chosen so that $\lambda\langle Tx, y\rangle = |\langle Tx, y\rangle|$ is real and non-negative, we obtain

$$|\langle Tx, y\rangle| \leq \frac{m}{2}(\|x\|^2 + \|y\|^2).$$

Suppose that $\|Tx\| \neq 0$ and take $y = \|x\|\frac{Tx}{\|Tx\|}$. This gives $\|x\|\|Tx\| \leq m\|x\|^2$. Hence, $\|Tx\| \leq m\|x\|$. This inequality also obviously holds when $Tx = 0$. We conclude that $\|T\| \leq m$. □

Theorem 5.5. *Let* T *be a compact self-adjoint operator on a Hilbert space* H. *Then either* $\|T\|$ *or* $-\|T\|$ *is an eigenvalue.*

Proof. We assume that that $T \neq 0$.

By the previous theorem there exists $x_n \in H$ with $\|x_n\| = 1$ such that $|\langle Tx_n, x_n\rangle| \to \|T\|$. Since $T^* = T$, $\overline{\langle Tx_n, x_n\rangle} = \langle x_n, Tx_n\rangle = \langle Tx_n, x_n\rangle$, so that $\langle Tx_n, x_n\rangle$ is real. Passing to subsequence we may assume that $\langle Tx_n, x_n\rangle \to \lambda$ where $\lambda = \|T\|$ or $\lambda = -\|T\|$. Then

$$\|Tx_n - \lambda x_n\|^2 = \|Tx_n\|^2 - 2\lambda\langle Tx_n, x_n\rangle + \lambda^2\|x_n\|^2 \leq 2\lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \to 0.$$

This shows that $Tx_n - \lambda x_n \to 0$. Since T is compact, passing to a subsequence we may assume that $Tx_n \to y$ for some $y \in H$. Tnen it follows that $\lambda x_n \to y$. Since T is continuous, $\lambda Tx_n \to Ty$, but $\lambda Tx_n \to \lambda y$. Hence, we conclude that $Ty = \lambda y$.

We claim that $y \neq 0$. Indeed, by the triangle inequality,

$$\|Tx_n\| \geq \|\lambda x_n\| - \|Tx_n - \lambda x_n\| = |\lambda| - \|Tx_n - \lambda x_n\| \to |\lambda| = \|T\| > 0.$$

Hence, $\|y\| = \lim \|Tx_n\| > 0$. We have proved that λ is an eigenvalue of T. □

Theorem 5.6. *Let T be a compact self-adjoint operator on a Hilbert space H and λ_n's are eigenvalues of T with linearly independent eigenvectors x_n. Then λ_n's are real, and for every $c > 0$ there are only finitely many n's such that $|\lambda_n| \geq c$.*

Proof. We have

$$\lambda_n \|x_n\|^2 = \langle Tx_n, x_n \rangle = \langle x_n, Tx_n \rangle = \bar{\lambda}_n \|x_n\|^2.$$

Since $x_n \neq 0$, it follows that $\lambda_n = \bar{\lambda}_n$ and λ_n is real.

Suppose that $\lambda_n \neq \lambda_m$. Then

$$\lambda_n \langle x_n, x_m \rangle = \langle Tx_n, x_m \rangle = \langle x_n, Tx_m \rangle = \lambda_m \langle x_n, x_m \rangle .$$

This implies that $\langle x_n, x_m \rangle = 0$, that is x_n and x_m are orthogonal. If λ_n's are repeating, then consider $N_\lambda = \{n : \lambda_n = \lambda\}$. We note that linear combinations of x_n's with $n \in N_\lambda$ are also eigenvectors with eigenvalue λ. Using the Gram-Schmidt orthogonalisation algorithm, we can construct orthonormal eigenvectors for λ_n with $n \in N_\lambda$.

Now we can assume that x_n's are orthonormal. Suppose that for infinitely many n's $|\lambda_n| \geq c$. Then for such indices $n \neq m$,

$$\|Tx_n - Tx_m\|^2 = \|\lambda_n x_n - \lambda_m x_m\|^2 = |\lambda_n|^2 + |\lambda_m|^2 \geq 2c^2.$$

This gives a sequence of the form (Tx_n) which does not contain any Cauchy subsequence. This contradicts compactness of the operator T. □

We note that Theorem 5.6 implies that if we order the eigenvalues of T as $|\lambda_1| \geq |\lambda_2| \geq \cdots$, then $|\lambda_n| \to 0$.

Theorem 5.7. *Let T be a self-adjoint operator on a Hilbert space H, and U is a closed subspace of H such that $T(U) \subset U$. Then $T(U^\perp) \subset U^\perp$.*

Proof. Let $x \in U^\perp$. Then for any $y \in U$, $Ty \in U$, and $\langle Tx, y \rangle = \langle x, Ty \rangle = 0$. Hence, $Tx \in U^\perp$. □

Now we are ready for the proof of the main Theorem 5.3. The idea is to apply Theorem 5.5 inductively.

Proof of Theorem 5.3. By Theorem 5.5, there exists an eigenvector e_1 with eigenvalue $\lambda_1 = \pm\|T\|$. Note that if $Tx = \lambda x$ with $x \in 0$, then $\|Tx\| = |\lambda|\|x\| \leq \|T\|\|x\|$, so that $|\lambda| \leq \|T\|$. Hence, $|\lambda_1|$ is maximal among eigenvalues. After rescaling we can assume that $\|e_1\| = 1$. The subspace $U_1 = \text{span}(e_1)$ is closed and T-invariant. By Theorem 5.7, $U_1^\perp$ is also T-invariant. Now we can apply the same construction to the operator $T_2 : U_1^\perp \to U_1^\perp$, which is the restriction of the operator T to $U_1^\perp$, to construct a unit eigenvector $e_2 \in U_1^\perp$ with eigenvalue λ_2 such that $|\lambda_2| = \|T_2\|$. Since $\|T_2\| \leq \|T\|$, we have $|\lambda_2| \leq |\lambda_1|$. Next, we consider $T_3 : U_2^\perp \to U_2^\perp$ where $U_2 = \langle e_1, e_2\rangle$, and so on ...

After n steps we produce an orthonormal set $\{e_1, \dots e_n\}$ consisting of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$ such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Setting $U_n = \langle e_1, \dots e_n\rangle$, we have the orthogonal decomposition $H = U_n \oplus U_n^\perp$. For a vector $x \in H$, we set $y_n = x - \sum_{i=1}^n \langle x, e_i\rangle e_i$. Since e_i's are orthonormal, it is easy to check that $y_n \in U_n^\perp$. Hence, we have the orthogonal sum $x = \sum_{i=1}^n \langle x, e_i\rangle e_i + y_n$, and by the Pythagoras' theorem, $\|x\|^2 = \sum_{i=1}^n |\langle x, e_i\rangle|^2 + \|y_n\|^2$. In particular, $\|y_n\| \leq \|x\|$. We observe that

$$\left\| Tx - \sum_{i=1}^n \lambda_i \langle x, e_i\rangle e_i \right\| \leq \left\| T\left(x - \sum_{i=1}^n \langle x, e_i\rangle e_i\right)\right\| = \|T_{n+1}y_n\| \leq \|T_{n+1}\|\|y_n\| \leq |\lambda_{n+1}|\|x\|.$$

Since by Theorem 5.6, $|\lambda_{n+1}| \to 0$, this proves that the operator T can be represented as $Tx = \sum_{i=1}^\infty \lambda_i \langle x, e_i\rangle e_i$, $x \in H$. □