Functional Analysis Exercise sheet 3

1. Give an example of a closed convex set C and a point x in ℓ^∞ such that the closest point in C to x is not unique.

2. Define $f : \ell^1 \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty}(1 - 1/n)x_n$.

   (a) Show that $C = \{x \in \ell^1 : f(x) = 1\}$ is a closed convex set.

   (b) Show that there is no closest point to 0 in C.

3. Let V be an inner product space and A be an orthonormal subset of V. Show that elements of A is linearly independent.

4. Let V be an inner prouct space and let $x_1, x_2, \ldots$ be a sequence of linearly independent vectors in V. Let $e_1 = \frac{x_1}{\|x_1\|}$, and define inductively for $n \geq 2$, $f_n = x_n - \sum_{k=1}^{n-1}\langle x_n, e_k\rangle e_k$ and $e_n = f_n/\|f_n\|$. Show that $e_1, e_2, \ldots$ is an orthonormal sequence with the same span as $x_1, x_2, \ldots$. This is known as the Gram-Schmidt process.

5. Let H and K be Hilbert spaces over the same field $\mathbb{F}$. A mapping $U : H \to K$ is said to be unitary if it is a bijective linear map and it preserves inner products (that is, $\langle U(x), U(y)\rangle_K = \langle x, y\rangle_H$ for all $x, y \in H$). Two Hilbert spaces are said to be isomorphic if there exists an unitary operator bewtween them. Show that if H and K both contain complete orthonormal sequences, then H and K are isomorphic.

6. Compute the value of the sum $\sum_{n\geq 1}\frac{1}{n^2}$. (Hint: use the Fourier expansion of the function $f(x) = x$.)

   Aside: Interestingly, the exact value of $\sum_{n\geq 1}\frac{1}{n^3}$ is not known, and it is unlikely that an exact formula for this sum exists. Can you compute $\sum_{n\geq 1}\frac{1}{n^4}$? (You are not expected to hand this in.)