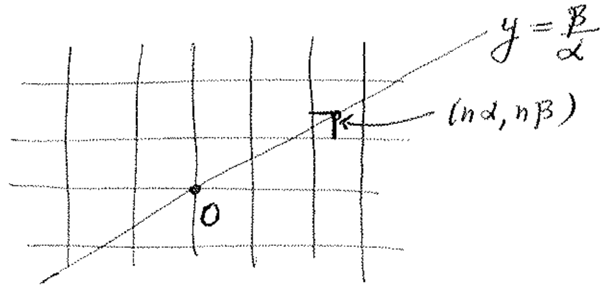
Lecture 8: Littlewood conjecture and dynamics of higher-rank abelian groups.

Littlewood Conj. $\forall \alpha, \beta \in \mathbb{R}$, $\liminf_{n\to\infty} n \cdot d(n\alpha, \mathbb{Z}) \cdot d(n\beta, \mathbb{Z}) = 0$.



Consider a cubic form $F(x_1, x_2, x_3) = x_1(\alpha x_1 + x_2)(\beta x_1 + x_3)$.

$F(\bar{x}) = F_0(g_{\alpha,\beta} \cdot \bar{x})$ where $F_0(y_1, y_2, y_3) = y_1 y_2 y_3$,

$$g_{\alpha,\beta} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix}.$$

The form F_0 is invariant under the diagonal group $A = \left\{ \begin{pmatrix} e^{-t-s} & & 0 \\ & e^t & \\ 0 & & e^s \end{pmatrix} : s, t \in \mathbb{R} \right\}$.

Prop. $\liminf_{n\to\infty} n \cdot d(n\alpha, \mathbb{Z}) \cdot d(n\beta; \mathbb{Z}) = 0$

$$\Updownarrow$$

$D \cdot g_{\alpha,\beta} \cdot \mathbb{Z}^3 \subset \mathcal{L}_3'$ is unbounded.

Here $D = \left\{ \begin{pmatrix} e^{-t-s} & & 0 \\ & e^t & \\ 0 & & e^s \end{pmatrix} : s, t \geq 0 \right\}$.

Proof.

Suppose that the D-orbit is unbounded.

Then $\forall \varepsilon > 0 \ \exists \underbrace{(p, q_1, q_2)}_{x} \in \mathbb{Z}^3, \neq 0 : \| \underbrace{d_{s,t} \cdot g_{\alpha,\beta} \cdot x}_{} \| < \varepsilon$

$$\begin{pmatrix} e^{-t-s} \cdot p \\ e^t (p\alpha + q_1) \\ e^s (p\beta + q_2) \end{pmatrix}$$

Note that $p \neq 0$ when $\varepsilon < 1$.

Then $|p| \cdot |p\alpha + q_1| \cdot |p\beta + q_2| < \varepsilon^3$.

This implies the claim.

Conversely, suppose that $\forall \varepsilon > 0$:

$\exists p \in \mathbb{Z}, \neq 0 \;\exists q_1, q_2 \in \mathbb{Z}: |p(p\alpha + q_1)(p\beta + q_2)| \leq \varepsilon$.

Suppose that $|p\alpha + q_1| \geq \varepsilon^{1/5}$. Then $|p(p\beta + q_2)| \leq \varepsilon^{4/5}$.

By Minkowski Thm, $\exists n \in \mathbb{Z}, \neq 0 \;\exists m \in \mathbb{Z}: \begin{cases} |n| \leq \varepsilon^{-1/5} \\ |n \cdot (p\alpha) + m| \leq \varepsilon^{1/5} \end{cases}$

Then $\begin{cases} |np \cdot (np \cdot \beta + nq_2)| \leq \varepsilon^{2/5} \Rightarrow |np \cdot \beta + nq_2| \leq \varepsilon^{1/5} \\ |np \cdot \alpha + m| \leq \varepsilon^{1/5} \end{cases}$

This shows that for $\forall \varepsilon > 0$:

$\exists p \in \mathbb{Z}, p \neq 0 \;\exists q_1, q_2 \in \mathbb{Z}: \begin{cases} |p(p\alpha + q_1)(p\beta + q_2)| \leq \varepsilon \\ |p\alpha + q_1| \leq \varepsilon^{1/5} \\ |p\beta + q_2| \leq \varepsilon^{1/5} \end{cases}$

Let $|p\alpha + q_1|, |p\beta + q_2| \neq 0$ (the other case is treated similarly).

$\exists t, s \geq 0: \begin{cases} e^t \cdot |p\alpha + q_1| = \varepsilon^{1/5} \\ e^s \cdot |p \cdot \beta + q_2| = \varepsilon^{1/5} \end{cases}$

Then $e^{-t-s} \cdot |p| \leq \varepsilon^{3/5}$.

Hence, $\| d_{s,t} \cdot g_{\alpha,\beta} \cdot x \| \leq \varepsilon^{3/5}$, where $x = \begin{pmatrix} p \\ q_1 \\ q_2 \end{pmatrix}$.

By Mahler compactness criterion, $D g_{\alpha,\beta} \mathbb{Z}^3$ is unbounded.

Conj. $F(\bar{x}) = L_1(\bar{x}) \dots L_d(\bar{x})$, L_i = linearly independent linear form in d-variables. $d \geq 3$

Assume that F is not proportional to a polynomial with rational coefficients.

Then $\inf_{x \in \mathbb{Z}^d, x \neq 0} |F(x)| = 0$.

$\exists g_F \in SL_d(\mathbb{R})$: $L_i(g_F x) = \alpha_i \cdot x_i$.

Prop. $\inf_{x \in \mathbb{Z}^d, x \neq 0} |F(x)| = 0 \iff A \cdot g_F \cdot \mathbb{Z}^d \subset \mathcal{L}_d'$ is unbounded.

(proof is similar)

Conj. (Margulis) Every orbit of A in $\mathcal{L}_d'$ is either compact or unbounded.

Compact orbits & Units in # fields

f – irreducible/$\mathbb{Q}$ polynomial.

$\alpha_1, \dots \alpha_d$ = roots of f.

$K = \mathbb{Q}(\alpha_1)$, $\sigma_i : K \hookrightarrow \mathbb{R}$: $\alpha_1 \longmapsto \alpha_i$.

$\mathcal{O}$ = ring of integers of K; $\mathcal{O} \cong \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_d$.

$\mathcal{O}^\times$ = the group of units in $\mathcal{O}$; $\mathcal{O}^\times \cong (\underset{\text{group}}{\text{finite}}) \times \mathbb{Z}^{d-1}$.

Consider the map $\rho : \mathcal{O} \to \mathbb{R}^d$: $\rho(x) = (\sigma_1(x), \dots \sigma_d(x))$.

Then $L = \rho(\mathcal{O})$ is a lattice in $\mathbb{R}^d$.

For $u \in \mathcal{O}^\times$, we set $a_u = \mathrm{diag}(\sigma_1(u), \dots \sigma_d(u))$.

This defines an embedding $\mathcal{O}^\times_{\inf} \hookrightarrow GL_d(\mathbb{R})$ such that $a_u \cdot L = L$.

Since for $u \in \mathcal{O}^\times$, $\sigma_1(u) \dots \sigma_d(u) = \pm 1$, $\det(a_u) = \pm 1$.

$\mathrm{Stab}_A(L) \supset$ cocompact group $= \{a_u : \det(a_u) = 1, u \in \mathcal{O}^\times\}$.

[Totally real number fields of degree d] $\rightsquigarrow$ [compact orbits of A in $\mathcal{L}_d^1$]

Classification of measures

<u>Question</u>: What are the A-inv. prob. measures on $\mathcal{L}_d^1$? $d \geq 3$

<u>Def</u> μ = an A-inv. measure.
μ is called <u>ergodic</u> if every A-inv. measurable subset has measure 0 or 1.

<u>Conj. 1 (Katok–Spatzier; Margulis)</u>
Every A-inv. prob. ergodic measure on $\mathcal{L}_3^1$ ——
- [a measure on compact A-orbit.
- [volume measure

<u>Conj. 2 (Furstenberg)</u> Every ergodic prob. measure on $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ which is invariant under both $z \mapsto z^2$ and $z \mapsto z^3$ is either
- [a measure supported on finitely many points in $e^{2\pi i \mathbb{Q}}$
- [the Lebesgue measure.

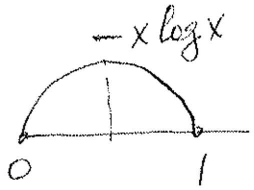
<u>Conj. 1</u> $\Longrightarrow$ Littlewood conjecture.

Proof requires notion of <u>entropy</u>.

Kolmogorov-Sinai entropy

(X, μ) = prob. space

$\mathcal{P} = \{P_1, \ldots P_n, \ldots\}$ - measurable partition
(i.e., $P_i \subset X_i$ are measurable,
$\mu(P_i \cap P_j) = 0$ for $i \neq j$,
$\mu(\bigcup_i P_i) = 1$).



Entropy of $\mathcal{P}$: $H_\mu(\mathcal{P}) = \sum_{P \in \mathcal{P}} -\mu(P) \log \mu(P)$.

For $x \in X$, $\mathcal{P}(x) = P$ such that $P \in \mathcal{P}$ and $x \in P$.

Let $T: X \to X$ be a measure preserving map.

How chaotic T is?

Suppose that we know $\mathcal{P}(x), \mathcal{P}(Tx), \ldots \mathcal{P}(T^n x)$.
Can we predict $\mathcal{P}(T^{n+1}x)$?

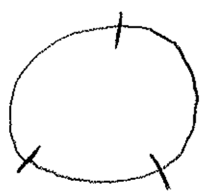
If $\mathcal{P}$ and $\mathcal{Q}$ are partitions,
$$\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : Q \in \mathcal{Q}, P \in \mathcal{P}\}.$$

Entropy of T with respect to $\mathcal{P}$

$$h_\mu(T, \mathcal{P}) = \limsup_{N \to \infty} \frac{1}{N} H_\mu\left(\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P}\right)$$
(infact, limit exists here.)

Entropy of T: $h_\mu(T) = \sup_{\mathcal{P}: H_\mu(\mathcal{P}) < \infty} h_\mu(T, \mathcal{P})$.

Examples: 1) $X = S^1$, $T: x \mapsto e^{2\pi i \alpha} \cdot x$.
$\mathcal{P}$ = partition of S^1 into k disjoint intervals



Then $\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P}$ consists of at most $N \cdot k$ intervals.

Convexity of $-x\log x$ implies that $H_\mu(\mathcal{P}) \le \log|\mathcal{P}|$.

Hence, $H_\mu(T,\mathcal{P}) = \limsup\limits_{N\to\infty} \dfrac{H_\mu(\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P})}{N} \le \limsup\limits_{N\to\infty} \dfrac{\log Nk}{N} = 0.$

2) $X = S^1$, $T: z \to z^2$, $\mathcal{P} = \{ e^{2\pi i\cdot(0,\frac{1}{2})}, e^{2\pi i(\frac{1}{2},1)} \}$

$$\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P} = \bigcup_{k=0}^{2^N-1} e^{2\pi i\left(\frac{k}{2^N}, \frac{k+1}{2^N}\right)}$$

Hence, $H_\mu(T,\mathcal{P}) = \limsup\limits_{N\to\infty} \dfrac{\sum_{k=0}^{2^N-1} -\frac{1}{2^N}\log\frac{1}{2^N}}{N} = \log 2.$

<u>Def</u> $\mathcal{P}$ is a <u>generating partition</u> if the partition $\bigvee_{n\in\mathbb{Z}} T^{-n}(\mathcal{P})$ seperates points.

<u>Prop.</u> If $\mathcal{P}$ is a generating partition, then
$$H_\mu(T,\mathcal{P}) = H_\mu(T).$$

<u>Convergence of measures:</u>
μ_n, μ – probability measures on $X = \mathcal{L}_d^1$.

$\mu_n \to \mu$ (in weak*-topology) if $\forall f \in C_c(X)$: $\int_X f\,d\mu_n \to \int f\,d\mu$.

Equivalently, $\forall$ compact $K \subset X$ with $\mu(\partial K) = 0$:
$$\mu_n(K) \to \mu(K).$$

<u>Weak* compactness</u>: if $\mathrm{supp}(\mu_n) \subset K$ – compact, $\{\mu_n\}$ has a convergent subsequence

<u>Prop.</u> (semicontinuity of entropy).
$X = \mathcal{L}_d^1$, $T = a \in A$.
μ_n, μ = T-inv. prob. measures on X.
$\mu_n \to \mu$ in weak*-topology.
Then $h_\mu(T) \geq \overline{\lim\limits_{i\to\infty}}\, h_{\mu_i}(T)$.

Proof (sketch)

Crutial step: construct a partition $\mathcal{P}$ such that

1) $\forall P \in \mathcal{P}: \mu(\partial P) = 0$, P-compact

2) $h_\mu(T) \le h_\mu(T, \mathcal{P}) + \varepsilon$,

$h_{\mu_n}(T) \le h_{\mu_n}(T, \mathcal{P}) + \varepsilon$.

Then we have $\mu_n(P) \to \mu(P)$, and

$$\frac{1}{N} H_\mu\left(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P}\right) = \lim_{n\to\infty} \frac{1}{N} H_{\mu_n}\left(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P}\right)$$

$$\ge \overline{\lim_{n\to\infty}}\, h_{\mu_n}(T, \mathcal{P}) \ge \overline{\lim_{n\to\infty}}\, h_{\mu_n}(T) - \varepsilon.$$

Hence,

$$h_\mu(T) \ge h_\mu(T, \mathcal{P}) - \varepsilon = \overline{\lim_{N\to\infty}} \frac{1}{N} H_\mu\left(\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P}\right) - \varepsilon$$

$$\ge \overline{\lim_{n\to\infty}}\, h_{\mu_n}(T) - 2\varepsilon.$$

Thm 1 Let $\{d(t) : t \ge 0\}$ be a 1-par. semigroup of D.
Suppose that $\underline{\lim}_{n\to\infty} n \cdot d(n\alpha, \mathbb{Z}) d(n\beta, \mathbb{Z}) > 0$.
Then for every measure μ supported on
$Y = \overline{\{d(t) g_{\alpha,\beta} \mathbb{Z}^3 : t \ge 0\}}$, $h_\mu(d(t)) = 0$.
and $d(t)$-invariant.

Thm 2 (classification of measures) (Einsidler – Katok – Lindenstrauss)
If μ is ergodic A-inv. prob. measure on $\mathcal{L}_3$, such that $h_\mu(a) > 0$ for some $a \in A$,
Then μ is the volume measure.

Proof (sketch) Suppose that for μ and $d_0 = d(t)$.

$$h_\mu(d_0) > 0.$$

We know that $Z = \overline{D g_{\alpha,\beta} \mathbb{Z}^3}$ is compact.

Consider the sequence of measures

$$\mu_T = \frac{1}{T^2} \int_0^T \int_0^T (d_{s,t} \cdot \mu) \cdot ds\,dt.$$

Since μ_T is supported on Z (-compact), $\mu_{T_n} \to \mu_\infty$ where μ_∞ is a prob. measure supported on Z.

One can check that μ is A-invariant, and by semicontinuity of the entropy, $h_{\mu_\infty}(d) > 0$. Then by classification of measures, μ_∞ cannot be supported on a compact set. —$\times$—

Thm. 1 $\Rightarrow$ the set of possible exceptions to the Littlewood conj. is "thin".

The best known result towards Littlewood Conj. is

Thm (Einsidler–Katok–Lindenstrauss)

Let $\mathcal{E} = \{(\alpha, \beta) \in \mathbb{R}^2 : \liminf_{n \to \infty} n\, d(n\alpha, \mathbb{Z}) \cdot d(n\beta, \mathbb{Z}) > 0\}$.

Then $\mathcal{E}$ has zero Hausdorff dimension

(i.e., $\forall \varepsilon > 0\ \forall \delta > 0$: $\exists$ open cover $\mathcal{E} \subset \bigcup_{i=1}^{\infty} U_i$ such that $\sum_i \operatorname{diam}(U_i)^\delta < \varepsilon$).