## Oppenheim Conjecture

$Q(x_1, \dots x_d) = \sum_{i,j=1}^{d} a_{ij} x_i x_j$, $a_{ij} \in \mathbb{R}$, – quadratic form

Assume that $Q$ is nondegenerate (i.e. $\det(a_{ij}) \neq 0$).

If $Q$ is positive definite or if $Q$ is proportional to a rational form, then $Q(\mathbb{Z}^d) \subset \mathbb{R}$ discrete.

Conjecture (Oppenheim / proved by Margulis)

$Q$ – nondegenerate indefinite quadratic form, $d \geq 3$,
– not a multiple of a rational form.

Then $Q(\mathbb{Z}^d)$ is dense in $\mathbb{R}$.

ex. $\{x^2 + y^2 - \sqrt{2} z^2 : x, y, z \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Def $X \in M_d(\mathbb{R})$ a nilpotent matrix (i.e., $X^d = 0$).
A unipotent 1-par. subgroup $u(t) = \exp(tn)$, $t \in \mathbb{R}$.

Conjecture (Raghunathan / proved by Ratner)

Let $H$ be a closed subgroup of $SL_d(\mathbb{R})$ generated by 1-par. unipotent subgroups. Then for any $x = g\,SL_d(\mathbb{Z}) \in SL_d(\mathbb{R})$: $\exists$ closed connected subgroup $F$ such that $\overline{Hx} = Fx$ and $g^{-1} H g \subset F$.

Remark. The essence of this conjecture is the "regular" behaviour of polynomial functions.

An analogue of this conjecture fails for other group, e.g., the geodesic flow $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$.

# Raghunathan Conj. $\Rightarrow$ Oppenheim Conj.

Let $SO(Q) = \{g \in SL_d(\mathbb{R}) : Q(gx) = Q(x) \text{ for } x \in \mathbb{R}^d\}$.
Let $H$ = connected component of $SO(Q)$

Facts: when $d \geq 3$,
(1) $H$ is generated by 1-par. unipotent subgroups,
(2) $H$ is a maximal connected subgroup of $SL_d(\mathbb{R})$

By Raghunathan Conj., we have two cases:

1) $\boxed{\overline{H SL_d(\mathbb{Z})} = SL_d(\mathbb{R})}$

Then $Q(\mathbb{Z}^d) = Q(H SL_d(\mathbb{Z}) \cdot \mathbb{Z}^d)$

Hence, $\overline{Q(\mathbb{Z}^d)} \supset Q(SL_d(\mathbb{R})\mathbb{Z}^d) = Q(\mathbb{R}^d \setminus \{0\}) = \mathbb{R}$.

2) $\boxed{H SL_d(\mathbb{Z}) \text{ is closed.}}$

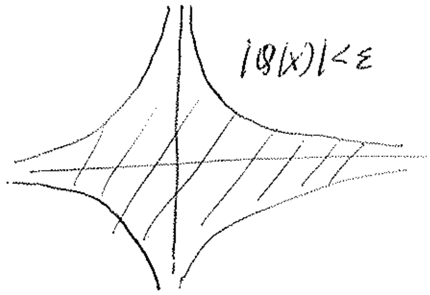
Consider set $\{A \in M_d(\mathbb{R}) : {}^tA = A,\ {}^th A h = A,\ h \in H\} = S$.
This is a solution of a system of linear equations.
Claim: $S = \mathbb{R} \cdot A_0$ where $A_0$ is the matrix of the quadratic form $Q$.

By Borel Density Thm,
$S = \{A \in M_d(\mathbb{R}) : {}^tA = A;\ {}^t\gamma A \gamma = A,\ \gamma \in H \cap SL_d(\mathbb{Z})\}$
$S$ is the set of solutions of a system of linear equations with integer coefficents.
Hence, $S$ is a rational line, and
$Q$ is proportional to a rational form.
Contradiction.

**Thm.** (weak form of Oppenheim Conjecture)

$Q = \sum_{i,j} a_{ij} x_i x_j$ – nondegenerate, indefinite quadratic form in $d \geq 3$ variables
– not a scalar multiple of a rational form.

Then $\forall \varepsilon > 0$: $\exists x \in \mathbb{Z}^d$, $x \neq 0$, $|Q(x)| < \varepsilon$.




$$H = SO(Q) = \{ g \in SL_d(\mathbb{R}) : Q(g \cdot x) = Q(x) \}$$

$$H \curvearrowright \mathcal{L}_d' \cong SL_d(\mathbb{R}) / SL_d(\mathbb{Z}).$$

**Lemma:** Suppose that $\exists \varepsilon_0 > 0 : \forall x \in \mathbb{Z}^d - \{0\} : |Q(x)| \geq \varepsilon_0$. Then the orbit $H \cdot \mathbb{Z}^d$ is bounded in the space of lattices.

By Mahler compactness criterion, we need to show that $\exists \varepsilon_0' > 0 : \forall h \in H \ \forall x \in \mathbb{Z}^d - \{0\} : \|hx\| \geq \varepsilon_0'$.
Suppose that, in contrary, $\exists x_n \in \mathbb{Z}^d$, $x_n \neq 0$, $h_n \in H$ such that $\|h_n x_n\| \to 0$.
Then $Q(x_n) = Q(h_n x_n) \to 0$. Contradiction

**Thm'.** Every bounded orbit of $H$ in $\mathcal{L}_d'$ is closed.

Thm' $\Longrightarrow$ Thm

If Thm fails, then $H\mathbb{Z}^d$ is bounded in $\mathcal{L}_d'$. Then $H\mathbb{Z}^d$ is closed, and $Q$ is proportional to a rational form.

Notation: $\boxed{d=3}$ $\quad G = SL_d(\mathbb{R})$, $\Gamma = SL_d(\mathbb{Z})$.

$Q_0(x) = 2x_1x_3 - x_2^2$, $\quad H = SO(Q_0)$

$$\underbrace{a(t) = \begin{pmatrix} e^t & & 0 \\ & 1 & \\ 0 & & e^{-t} \end{pmatrix}, \quad v_1(t) = \begin{pmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{pmatrix}}_{\text{one-parameter subgroups of } H.}$$

$$v_2(t) = \begin{pmatrix} 1 & 0 & t \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

$W$ = space of quadratic forms in 3 variables.

$G \to GL(W)$ - linear representation: $g \cdot Q(x) = Q(g^{-1}x)$.

For $x \in G/\Gamma$ such that $\overline{Hx}$ is compact, consider $M = M_z = \{g \in G : g^{-1} z \in \overline{Hx}\}$, $z \in \overline{Hx}$.

Idea: If $M - H$ is "large", use dynamics of $V_1 \curvearrowright W$ to show that $M - H$ has to "larger", which contradicts compactness of $\overline{Hx}$.


Lemma. Let $U = \{u(t)\}$ be a unipotent 1-par. subgroup. $Y \subset \mathbb{R}^N - Fix(U)$, $p \in Fix(U) \cap \overline{Y}$.
Then $\exists$ nonconstant polynomial map $\varphi : \mathbb{R} \to \overline{UY} \cap L$, $\varphi(0) = p$.

$\ulcorner$ $u(t) = \exp(tn)$, $n$ - nilpotent matrix.

$\{e_i^{(j)}\}$ - Jordan normal form basis for $n$.

$$u(t) \cdot e_i^{(j)} = \sum_{k=0}^{i-1} \frac{t^k}{k!} \cdot e_{i-k}^{(j)}.$$

$$y_n = \sum_{i,j} \alpha_n(i,j) \cdot e_i^{(j)} \in Y, \quad y_n \to p.$$

$$Fix(U) = \langle e_1^{(j)} \rangle, \quad \alpha_n(i,j) \to 0 \text{ for } i \geq 2.$$

$$u(st_n)y_n = \sum_{i,j}\left(\sum_{k=0}^{i-1} \alpha_n(i,j)\cdot t_n^k\cdot\frac{s^k}{k!}\right) e_{i-k}^{(j)}$$

Let $\beta_n = \min_{i\geq 2,\, j} |\alpha_n(i,j)|^{1/(i-1)}$.

Note that $|\alpha_n(i,j)\cdot\beta_n^{i-1}|\leq 1$ for $i\geq 2$, and passing to a subsequence, $|\alpha_n(i_0,j)\cdot\beta_n^{i_0-1}|=1$. for some $i_0\geq 2$.

and $\alpha_n(i,j)\beta_n^{i-1}\xrightarrow[n\to\infty]{}\lambda(i,j)$.

Take $t_n=\beta_n$. Then

$$u(st_n)y_n \longrightarrow \underbrace{\sum_{i,j}\lambda(i,j)\frac{s^{i-1}}{(i-1)!}\cdot e_1^{(j)}}_{=\varphi(s)\text{ - polynomial map as required.}} \quad \text{as } n\to\infty.$$

$V_1 \curvearrowright W$, $Fix(V_1) = \langle Q_0, Q_1\rangle$ (check!)
where $Q_1(x) = x_3^2$.

$Stab_G(Q_0) = H \Rightarrow G/H \simeq \underbrace{G\cdot Q_0}_{cl}$.

Let $Y = M\cdot Q_0 - Fix(V_1)$.

$\boxed{\text{Assume that } Q_0\in\overline{Y}}$ (*)

By Lemma, $\exists$ nonconstant polynomial map $\varphi:\mathbb{R}\to\overline{V_1Y}\cap Fix(V_1)$.
$\varphi(0)=Q_0$

$\forall Q\in G\cdot Q_0$: $\det(Q)=1$.

On the other hand, $\det(\alpha Q_0+\beta Q_1)=\alpha^3$.

Hence, $\varphi(s) = Q_0+\psi(s)Q_1 \Longrightarrow Q_0+\mathbb{R}^+\cdot Q_1\subset\overline{V_1Y}$ or $Q_0+\mathbb{R}^-Q_1\subset\overline{V_1Y}$.

↑ polynomial $\neq$ const

$V_2(t)\cdot Q_0 = Q_0+2t\cdot Q_1 \Rightarrow V_2^+\cdot Q_0\subset\overline{V_1Y}$ or $V_2^-\cdot Q_0\subset\overline{V_1Y}$.

Since, $G\cdot Q_0\simeq G/H$, it follows that

$$V_2^+\subset\overline{V_1MH} \quad\text{or}\quad V_2^-\subset\overline{V_1MH}.$$

Now we finish the proof under assumption $(*)$.
Let $Z$ be a minimal $V_1$-inv. subset of $\overline{Hx}$.
Consider $S = \{g \in G : g^{-1}Z \cap \overline{Hx} \neq \emptyset\}$.
Clearly, $S$ is $(V_1 \times H)$-invariant and $M_Z \subset S$.
Since $Z$ & $Hx$ are compact, $S$ is closed.
Hence, $V_2^+$ (or $V_2^-$) $\subset \overline{V_1 M H} \subset S$.
For $v_2 \in V_2^+$, we have $v_2^{-1} z \in \overline{Hx}$ for some $z \in Z$.
Since $V_1 \subset H$ and $V_1$ & $V_2$ commute, $v_2^{-1} \cdot V_1 z \subseteq \overline{Hx}$.
By minimality $v_2^{-1} \cdot Z \subset \overline{Hx}$.
The following lemma gives a contradiction:

<u>Lemma</u> $\forall x \in G/\Gamma$: $\overline{HV_2^+ x}$ (and $\overline{HV_2^- x}$) are not compact.

Let $x = g\mathbb{Z}^3$, $g \in SL_3(\mathbb{Z})$.
$g\mathbb{Z}^3 \cap \{v \in \mathbb{R}^3 : v_3 \neq 0, Q_0(v) < 0\} \neq 0$
(check this!)
Take $v \in g\mathbb{Z}^3 : v_3 \neq 0$, $Q_0(v) < 0$.
$Q_0(V_2(t)v) = Q_0(v) + 2t\, Q_1(v)$.
Choose $t > 0$ so that $Q_0(V_2(t)v) = 0$.
The group $H = SO(Q)$ acts transitively on $\{Q_0 = 0\}$
Hence, $\exists h_n \in H$: $\|h_n V_2(t) v\| \to 0$ as $n \to \infty$.
By Mahler compactness, $HV_2^+ x$ is unbounded in $G/\Gamma$.

We have $G \cdot Q_0 \simeq G/H$ and $\mathrm{Fix}(V_1) \cap G \cdot Q_0 = Q_0 + \mathbb{R} \cdot Q_1 = V_2 \cdot Q_0$.

$$Y = M \cdot Q_0 - \mathrm{Fix}(V_1) \longleftrightarrow M - V_2 H$$

Hence, condition $(*)$ is equivalent to:
$\overline{M - V_2 H} \ni e$.

Now we assume that $e \notin \overline{M - V_2 H}$.
$\qquad\qquad\qquad\qquad\qquad \| $
$\overline{\{g \in G - V_2 H : g^{-1} z \in \overline{Hx}\}}$

Assume that the orbit $Hx$ is bounded, but not closed. Let $X \subset \overline{Hx}$ be a minimal $H$-inv. set.

Lemma 1 For every $y \in X$,

$$e \in \overline{\{g \in G - H : g^{-1} y \in \overline{Hx}\}}.$$

Suppose not. Then $\exists$ a nbhd $\mathcal{O}$ of $e$ in $G$: $\mathcal{O}y \cap \overline{Hx} \subseteq Hy$.

Since $y \in \overline{H \cdot x}$, $\mathcal{O} \cdot y \cap Hx \neq \emptyset$

$$\underset{Hy \cap Hx}{\cap\!\!\shortmid} \Longrightarrow Hy = Hx.$$

Now $\mathcal{O}y \cap \overline{X - Hy} = \emptyset \Rightarrow y \notin \overline{X - Hy}$.

$\overline{X - Hy}$ is a closed (proper) $H$-inv. subset of $X$.

Since $X$ is $H$-minimal, this set is empty

Hence, $X = Hy$. Contradiction.

Let $Y \subset X$ be a minimal $AV_1$ - inv. set.

(Recall $A = \left\{\begin{pmatrix} e^t & & 0 \\ & 1 & \\ 0 & & e^{-t}\end{pmatrix}\right\}$, $V_1 = \left\{\begin{pmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1\end{pmatrix}\right\}$)

$AV_1 \subset H = SO(Q_0)$

$V_2 = \left\{\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}\right\}$

Lemma 2 For every $y \in Y$,

$$e \in \overline{\{g \in G - AV_1V_2 : g^{-1} y \in AV_1 \cdot y\}}$$

Any discrete subgroup $\Lambda$ of $AV_1V_2$ is either contained in $V_1V_2$ or is conjugate to a subgroup of $A$.

If $a_0 v_0 \in \Lambda$, $a_0 \neq e$, then $\exists v \in V_1V_2$: $v^{-1}(a_0 v_0)v \in A$ (direct computation)

Then we may assume that $\Lambda \cap A \neq 1$ ($\ni a$). If $a_0 v_0 \in \Lambda$ and $v_0 \neq e$, then

$a^{-n}(a_0 v_0)a^n \to a_0$ as $n \to \infty$, contradiction with discreteness.

Let $\Lambda = Stab_{AV_1V_2}(y)$. It follows that

$\exists f_n \in AV_1$: $f_n \Lambda \to \infty$ in $DV_1V_2/\Lambda$.

Let $\mathcal{O}$ be a bounded nbhd of $e$ in $G$.
Since the action $AV_1$ on $Y$ is minimal,
$\exists g_n \in \mathcal{O},\ \ell_n \in AV_1 :\ \ell_n f_n \cdot y = g_n y.$
Moreover, using that $Y$ is compact, we can choose $\{\ell_n\}$ to be bounded (check!)
Then $\ell_n f_n \Lambda \to \infty$ in $DV_1V_2/\Lambda$.
If $g_n \in DV_1V_2$, then $g_n^{-1} \ell_n f_n \in \Lambda$.
Since $\{g_n\}$ is bounded, this is a contradiction.

## Completion of the proof.

We may assume that (*) fails, i.e.
$\exists$ nbhd $\mathcal{O}$ of $e$ in $G$ such that

$$\{g \in \mathcal{O} : g^{-1} z \in \overline{Hx}\} \subset V_2 H. \quad (\times)$$

Here we choose $Z \subset Y$ a minimal $V_1$-inv. subset and $z \in Z$.
On the other hand, by Lemma 1

$$e \in \overline{\{g \in G - H :\ g^{-1} z \in Hx\}}$$

Hence, $\exists h_n \in H,\ t_n \neq 0 :\ (V_2(t_n) \cdot h_n)^{-1} z \in \overline{Hx}. \Rightarrow V_2(t_n)^{-1} \cdot z \in \overline{Hx}.$

$$V_2(t_n) h_n \to e.$$

Since $Q_0((V_2(t_n)h_n)^{-1} \cdot x) = Q_0(x) - 2t_n Q_1(x)$, we have $t_n \to 0$.

By Lemma 2, $\exists u \in \mathcal{O} \cap (G - AV_1V_2),\ p \in AV_1 ;\ u^{-1} z = p z.$
By $(\times)$, $u \in V_2 H - V_2 AV_1 = V_2 (H - AV_1)$. Write $u = v_2(t) h$.
We have $p V_2(t_n)^{-1} p^{-1} \cdot u^{-1} z = p V_2(t_n)^{-1} \cdot z \in p\overline{Hx} = \overline{Hx}.$
For large $n$, $u p V_2(t_n) p^{-1} \in \mathcal{O}$.
Hence, by $(\times)$, $u p V_2(t_n) p^{-1} \in V_2 H \Rightarrow h p V_2(t_n) \in V_2 H$
Finally, observe that $hp \in H - AV_1$. and $t_n \neq 0$.
The following lemma gives a contradiction.

<u>Lemma.</u> If $h \in H$ and $v_2 \in V_2$, $v_2 \neq e$, and $hv_2 \in V_2 H$, then $h \in AV_1$.

$\Gamma$ $$Q_0(v_2(t)^{-1} h^{-1} \cdot x) = \underbrace{Q_0(h^{-1}x)}_{Q_0(x)} - 2t\, Q_1(h^{-1} \cdot x)$$

$$Q_0(h_1^{-1} v_2(t_1)^{-1} \cdot x) = Q_0(v_2(-t_1) \cdot x) = Q_0(x) - 2t_1 Q_1(x).$$

Hence, $Q_1(h^{-1} \cdot x) = \frac{t_1}{t} Q_1(x) \Rightarrow \langle e_1, e_2 \rangle = \{Q_1 = 0\}$ is $h$-invariant

$\langle e_1 \rangle$ is the orthogonal complement of $\langle e_1, e_2 \rangle$ with respect to $Q_0$. $\Rightarrow \langle e_1 \rangle$ is $h$-invariant.

Now $h \in \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \cap H = AV_1$. $\lrcorner$

## <u>Reduction to $d = 3$</u>

Let $Q$ be a nondegenerate indefinite quadratic form in $d > 3$ variables, which is not a multiple of a rational form.

$\exists \{e_i\}$ - rational basis of $\mathbb{R}^d$: $Q|_{\langle e_1, \dots e_{d-1} \rangle}$ is indefinite and nondegenerate, and $Q|_{\langle e_1, \dots e_{d-2} \rangle} \neq 0$.

Let $L_t = \langle e_1, \dots e_{d-2}, e_{d-1} + t e_d \rangle$.

For small rational $t$, $Q|_{L_t}$ satisfies the same properties. If $Q|_{L_t}$ is a multiple of a rational form for all such $t$, and $L_t$'s generate $\mathbb{Q}^d$, one can deduce that $Q$ is a multiple of a rational form.