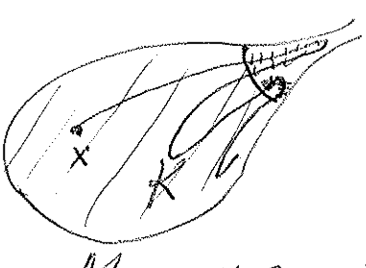
Lecture 5

Nondivergence of unipotent flows.

~~Def.~~ $x \in M_d(\mathbb{R})$ – a nilpotent matrix (i.e. $x^d = 0$).
A unipotent 1-par. subgroup is $u(t) = \exp(tn)$, $t \in \mathbb{R}$.
$\cap$ $SL_d(\mathbb{R})$

Margulis Thm on nondivergence.
Let $u(t)$ be a unipotent 1-par. subgroup of $SL_d(\mathbb{R})$.
For every $x \in \mathcal{L}_d^1 \simeq SL_d(\mathbb{R})/SL_d(\mathbb{Z})$
$\exists$ compact $K \subset \mathcal{L}_d^1$ such that
$\{t \geq 0 : u_t x \in K\}$ is unbounded.



Moreover, for every $\varepsilon > 0$, $\exists$ compact $K \subset \mathcal{L}_d^1$:
$\ell(\{t \in [0,T] : u_t x \in K\}) \geq (1-\varepsilon)T$ for all $T \geq 0$.

Proof ($d=2$) $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $x = g SL_2(\mathbb{Z})$, $g \in SL_2(\mathbb{R})$.
$P(\mathbb{Z}^2) = \{(m,n) \in \mathbb{Z}^2 : \gcd(m,n) = 1\}$
$\forall v, w \in P(\mathbb{Z}^2)$: $\mathbb{Z}v \cap \mathbb{Z}w = 0$, $\mathbb{Z}^2 = \bigcup_{v \in P(\mathbb{Z}^2)} \mathbb{Z}v$.

Let $D_R = \{x \in \mathbb{R}^2 : \|x\| < R\}$, $K_R = \{L \in \mathcal{L}_2^1 : L \cap D_R = 0\}$
By Mahler compactness criterion, K_R is compact in $\mathcal{L}_2^1$.

1) Suppose that for some $v \in P(\mathbb{Z}^2)$,
$u_t g v = g v$ for $t \in \mathbb{R}$.
Take $\gamma \in SL_2(\mathbb{Z})$: $v = \gamma e_1$. Then $(u_t g \gamma) e_1 = g \gamma e_1$
$\gamma^{-1} g^{-1} u_t g \gamma = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

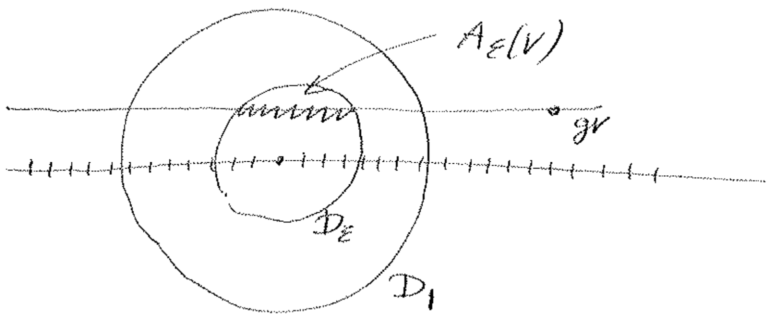
For some t_0, $u_{t_0} g \gamma = g \gamma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow u_{t_0} x = x$.

Hence, $u_t x$ is periodic $\Rightarrow$ compact.

2) Suppose that for every $v \in \mathcal{P}(\mathbb{Z}^2)$, gv is not fixed by u_t.

For $v \in \mathcal{P}(\mathbb{Z}^2)$, define

$$A_\varepsilon(v) = \{ t \in \mathbb{R} : u_t g v \in D_\varepsilon \} \longrightarrow \begin{cases} |(gv)_1 + t(gv)_2| < \varepsilon \\ |gv_2| < \varepsilon . \end{cases}$$



$A_\varepsilon(v)$ is a finite interval (possibly empty)

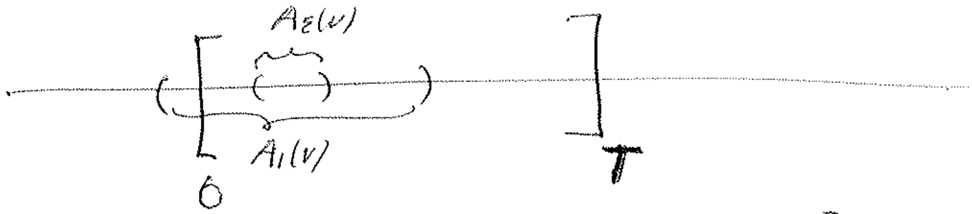
Basic properties:

1) $A_1(v) \cap A_1(w) = \emptyset$ for $v \neq w \in \mathcal{P}(\mathbb{Z}^2)$.

⌈ Suppose that $u_t g v, u_t g w \in D_1$. Then the lattice $L = \langle u_t g v, u_t g w \rangle$ has covolume < 1. On the other hand $\langle v, w \rangle$ is a sublattice of $\mathbb{Z}^2$, so that it has covolume ≥ 1 $\Longrightarrow$ contradiction. ⌋

2) $\ell\big(A_\varepsilon(v) \cap [0,T] \big) \leq 2\varepsilon \cdot \ell\big(A_1(v) \cap [0,T] \big)$

for every $\varepsilon \in (0,1)$ and $T \geq T_0$.
$v \in \mathcal{P}(\mathbb{Z}^2)$.



$$\{ t \in [0,T] : u_t x \notin K_\varepsilon \} \subset \bigcup_{v \in \mathcal{P}(\mathbb{Z}^2)} A_\varepsilon(v) \cap [0,T]$$

Hence, for $T \geq T_0$,

$$\ell\big(\{ t \in [0,T] : u_t x \notin K_\varepsilon \} \big) \leq \sum_{v \in \mathcal{P}(\mathbb{Z}^2)} \ell\big(A_\varepsilon(v) \cap [0,T] \big)$$

$$\overset{(2)}{\leq} 2\varepsilon \cdot \sum_{v \in \mathcal{P}(\mathbb{Z}^2)} \ell\big(A_1(v) \cap [0,T] \big) \overset{(1)}{\leq} 2\varepsilon \cdot T .$$

Take $K = u_{[0,T_0]} x \cup K_{\varepsilon/2}$. ⌋

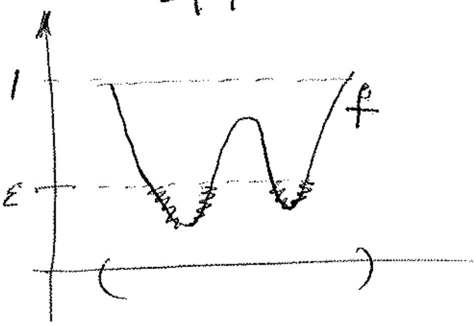
General cases follows similar strategy:

- to arrange disjointness as in (1), one needs to consider action on subgroups of $\mathbb{Z}^d$.
- The substitute of (2) is the following general property of polynomial functions.

Prop. ("good" property).

For every polynomial f of degree $\le d$ and every interval $B \subset \mathbb{R}$,

$$\ell(\{t \in B : |f(t)| < \varepsilon \cdot \max_B |f|\}) \le c_d \cdot \varepsilon^{1/d} \cdot \ell(B).$$



Proof We may assume that $\max_B |f| = 1$.

Let $\ell = \ell(\underbrace{\{t \in B : |f(t)| < \varepsilon\}}_{B_\varepsilon})$.

The set B_ε cannot be covered by $(k-1)$ intervals of length ℓ/k. This implies that $\exists t_1, \dots t_{k+1} \in B_\varepsilon$: $|t_i - t_j| \ge \ell/2k$ for $i \ne j$. Take $k = \deg(f)$.

Then by the Lagrange interpolation,

$$f(t) = \sum_{i=1}^{k+1} f(t_i) \cdot \frac{\prod_{j \ne i}(t - t_j)}{\prod_{j \ne i}(t_i - t_j)}.$$

For $t \in B$,

$$|f(t)| \le (k+1)\varepsilon \cdot \frac{\ell(B)^k}{(\ell/2k)^k}.$$

Hence,

$$1 \le (k+1)\varepsilon \cdot \frac{\ell(B)^k}{(\ell/2k)^k}.$$

This implies the claim.

Poincare Recurrence Thm.

Let $A \subset \mathcal{L}_d^1$, $\mathrm{vol}(A) > 0$, $g \in SL_d(\mathbb{R})$.

Then for a.e. $x \in A$, $g^n x \in A$ for infinitely many n.

Proof Let $B = \{x \in A : g^i \notin A \text{ for all } i > 0\}$.

Note that $g^i B \cap g^j B = \emptyset$ for $i \neq j \geq 0$.

Hence, $\sum_{i \geq 0} \mathrm{vol}(g^i B) = \mathrm{vol}(\bigcup_{i \geq 0} g^i B) \leftarrow$ finite

On the other hand, $\mathrm{vol}(g^i B) = \mathrm{vol}(B)$.

Hence, $\mathrm{vol}(B) = 0$.

Let $A_1 = \{x \in A : g^i x \in A \text{ for some } i > 0\} = A - B$

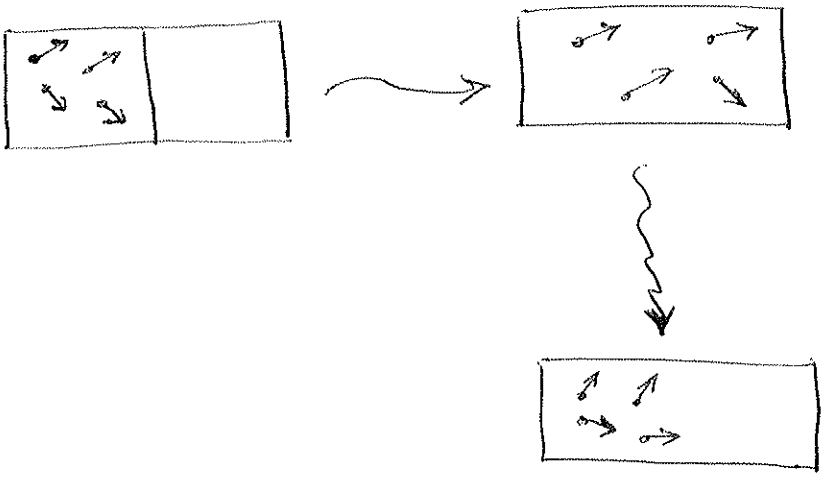
$\mathrm{vol}(A_1) = \mathrm{vol}(A)$

Let $A_k = \{x \in A_{k-1} : g^i x \in A_{k-1} \text{ for some } i > 0\}$

$\mathrm{vol}(A_k) = \mathrm{vol}(A_{k-1})$.

For every $x \in A_\infty = \bigcap_{k \geq 1} A_k$: $g^n x \in A$ infinitely often.

$\mathrm{vol}(A_\infty) = \mathrm{vol}(A)$. $\lrcorner$



Applications of nondivergence of unipotent flows.

1) Borel density Thm.

Let H be a closed subgroup of $SL_d(\mathbb{R})$ generated by unipotent 1-par. subgroups, $x \in \mathcal{L}_d'$ such that Hx is closed in $\mathcal{L}_d'$.
Then every polynomial function $F: M_d(\mathbb{R}) \to \mathbb{R}$ which is constant on $\mathrm{Stab}_H(x)$ is constant on H.
(Zariski topology $=$ closed sets are zero sets of polynomials.)
Thm says that $\mathrm{Stab}_H(x)$ is Zariski dense in H.

ex. - $SL_d(\mathbb{Z})$ is Zariski dense in $SL_d(\mathbb{R})$,
- $SO(1,n)_{\mathbb{Z}}$ is Zariski dense in $SO(1,n)_{\mathbb{R}}$ for $n \geq 2$.

Proof. We set $H_x = \mathrm{Stab}_H(x)$.
Let $\rho: H \to GL_N(\mathbb{R})$ be a linear representation of H.
We, first, show that if $v \in \mathbb{R}^N$ is fixed by H_x, then it is fixed by H.
Let $f_\ell(h) = (hv)_\ell$: $f: H/H_x \to \mathbb{R}$, $\ell = 1, \dots N$.
Let $u(t)$ be a unipotent 1-par. subgroup of H.
By Margulis nondivergence, $\exists$ compact $K \subset \mathcal{L}_d'$ such that $u(t_i)x \in K$ for $t_i \to \infty$.
Let $\tilde{K} = \{h \in H/H_x : hx \in K\}$. Since the orbit Hx is closed, $\tilde{K}$ is compact in H/H_x.
We have $f_\ell(u(t_i)) = f_\ell(\tilde{K})$ - bounded.
On the other hand, $f_\ell(u(t))$ is polynomial.
Hence, it is constant. This shows that v is fixed by $u(t)$, and thefore by H.

Similar argument also implies that if $V \subset \mathbb{R}^N$ is a H_x-invariant subspace, then it is H-invariant.

Now consider $I = \{F \in \mathbb{C}[x_{ij} : i=1,\dots d] : F(\gamma) = 0$ for $\gamma \in H_x\}$.

We have linear representation of H on $\mathbb{C}[x_{ij} : i=1,\dots d]$ defined by $(h \cdot F)(x) = F(h^{-1}x)$.

Since I is H_x-invariant, it is H-invariant.

For $F \in I$, $F(h \cdot H_x) = 0$ for every $h \in H$.

Hence, $F(H) = 0$ as required. $\lrcorner$

2) <u>Minimal sets of unipotent flows</u>

$u(t)$ = 1-par. unipotent subgroup of $Sl_d(\mathbb{R})$

<u>Def</u> A closed subset F of $\mathcal{L}_d'$ is called <u>minimal</u> if $\forall x \in F$: $\overline{u(\mathbb{R}) \cdot x} = F$ (every orbit is dense).

<u>Lemma</u> If F is compact $u(t)$-invariant subset, then F contains a minimal subset $\neq \emptyset$.

$\ulcorner$ If $F_1 \supset F_2 \dots \supset F_\alpha \supset \dots$ are closed $u(t)$-invariant sets then $\bigcap F_\alpha$ is $\neq \emptyset$ and $u(t)$-invariant. Apply Zorn lemma. $\lrcorner$

<u>Thm</u> Every minimal set of $u(t)$ is compact.

$\ulcorner$ For $x \in X$ ($=$ a minimal set of $u(t)$), let $S(x)$ be a compact disc centered at x and transversal to the flow. Let $V \subset S$ be open, and $f(x)$ = the first time $u_t x$ enters V, $x \in S$.

If $f(x) < \infty$, then by continuity $T = \sup_{x \in S} f(x) < \infty$.

Then $u_{\mathbb{R}^+} \cdot S = u_{[0,T]} \cdot S$ − compact. Since X is minimal, $X = u_{[0,T]} S$ − compact, which is a contradiction.

Hence, $\exists x \in S$: $u_{\mathbb{R}^+} \cdot x \cap V = \emptyset$. Then $u_{\mathbb{R}^+} \cdot x \cap \underbrace{u_{\mathbb{R}} \cdot V}_{\text{open}} = \emptyset$

Let $X' = \{\lim u_{t_i} x : t_i \to \infty\}$ − closed $u(t)$-invariant.

Hence, $X' = \emptyset$, i.e., $u_t \cdot x \to \infty$ as $t \to \infty$ $\lrcorner$