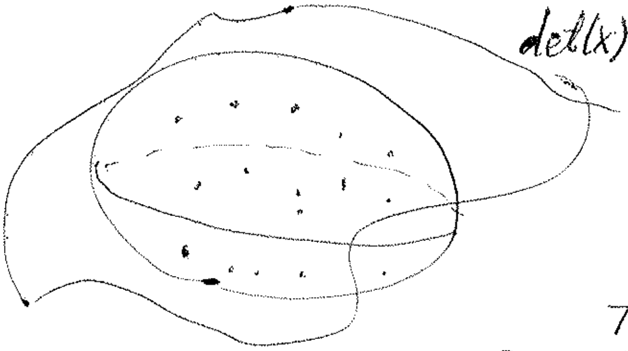
Lecture 3

Counting integral points.

$$X = \{x \in M_d(\mathbb{R}) : \det(x) = 1\}.$$

$$\|x\| = \Big(\sum_{i,j} x_{ij}^2\Big)^{1/2}.$$

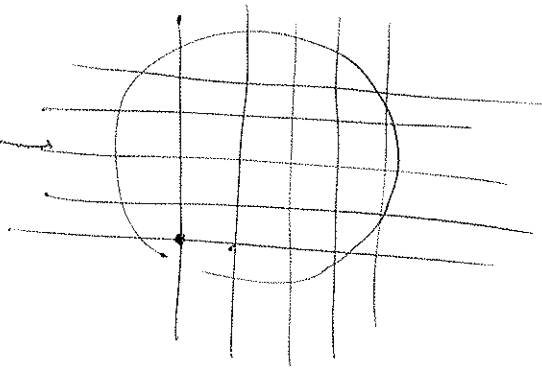


$$N_T(X) = \#\{x \in X(\mathbb{Z}) : \|x\| < T\}$$

<u>Heiristic:</u> $\det(\overbrace{\{x \in M_d(\mathbb{Z}) : \|x\| < T\}}^{T^{d^2}\text{-points}}) \subset [-cT^d, cT^d]$

$\approx T^d$-values.

$$\#\left\{x \in M_d(\mathbb{Z}) : \begin{array}{c}\det(x)=1\\ \|x\|<T\end{array}\right\} \simeq \frac{T^{d^2}}{T^d} = T^{d^2-d}.$$



$$\#\{x \in M_d(\mathbb{Z}) : \|x\| < T\} \simeq \mathrm{vol}\{x \in M_d(\mathbb{R}) : \|x\| < T\} \sim c \cdot T^{d^2}.$$

$$\#\{x \in SL_d(\mathbb{Z}) : \|x\| < T\} \overset{?}{\underset{T\to\infty}{\sim}} \mathrm{vol}(x \in SL_d(\mathbb{R}) : \|x\| < T)$$

where vol is $SL_d(\mathbb{R})$-invariant measure normalised so that $\mathrm{vol}(SL_d(\mathbb{R})/SL_d(\mathbb{Z})) = 1$.

$$B_T = \{ x \in SL_d(\mathbb{R}) : \|x\| < T \},$$
$$\mathcal{O}_\varepsilon = \{ x \in SL_d(\mathbb{R}) : \|x - id\| < \varepsilon \},$$
$$\|x^{-1} - id\| < \varepsilon.$$

<u>Lemma</u>

(1) $\mathcal{O}_\varepsilon B_T \mathcal{O}_\varepsilon \subset B_{(1+2\varepsilon)T}$ (easy: triangle inequality)

(2) $\mathrm{vol}(B_T) \underset{T\to\infty}{\sim} c \cdot T^{d^2-d}$ for some $c > 0$.

<u>Thm.</u> For $\Gamma = SL_d(\mathbb{Z})$, $\#(\Gamma \cap B_T) \sim \mathrm{vol}(B_T)$.

$\ulcorner$ $F_T(g_1, g_2) = \sum_{\gamma \in \Gamma} \chi_{B_T}(g_1 \gamma g_2^{-1})$ for $g_1, g_2 \in G = SL_d(\mathbb{R})$.

We need to compute $F_T(e, e) \underset{T\to\infty}{\sim} \mathrm{vol}(B_T)$.

Note that F_T is $(\Gamma \times \Gamma)$-right invariant.

$$F_T : G/\Gamma \times G/\Gamma \to \mathbb{R}.$$

<u>Step 1</u> (weak convergence) $\varphi_1, \varphi_2 : G/\Gamma \to \mathbb{C}$

$\varphi = \varphi_1 \otimes \varphi_2 : G/\Gamma \times G/\Gamma \to \mathbb{C}$

$$\langle F_T, \varphi \rangle = \int_{G/\Gamma \times G/\Gamma} F_T(g_1, g_2) \overline{\varphi_1(g_1)}\, \overline{\varphi_2(g_2)}\, dm_{G/\Gamma}(g_1)\, dm_{G/\Gamma}(g_2).$$

<u>Recall</u>: $m_{G/\Gamma}(A) = m_G(\pi^{-1}(A) \cap F)$ where $\pi : G \to G/\Gamma$ and F is the fundamental domain for Γ.

$$\int_{G/\Gamma} \left(\int_F \sum_{\gamma \in \Gamma} \chi_{B_T}(g_1 \gamma g_2^{-1}) \overline{\varphi_1(g_1)}\, \overline{\varphi_2(g_2)}\, dm_G(g_1) \right) dm_{G/\Gamma}(g_2)$$

$$= \int_{G/\Gamma} \left(\sum_{\gamma} \int_{F\gamma} \chi_{B_T}(g_1 g_2^{-1}) \overline{\varphi_1(g_1)}\, dm_G(g_1) \right) \overline{\varphi_2(g_2)}\, dm_{G/\Gamma}(g_2)$$

$$= \int_{G/\Gamma} \left(\int_G \chi_{B_T}(\underbrace{g_1 g_2^{-1}}_{B}) \overline{\varphi_1(g_1)}\, dm_G(g_1) \right) \overline{\varphi_2(g_2)}\, dm_{G/\Gamma}(g_2)$$

$$= \int_{G/\Gamma} \left(\int_{B_T} \overline{\varphi_1(b g_2)}\, dm_G(b) \right) \overline{\varphi_2(g_2)}\, dm_{G/\Gamma}(g_2)$$

$$= \int_{B_T} \underbrace{\left(\int_{G/\Gamma} \overline{\varphi_1(bg)}\, \overline{\varphi_2(g)}\, dm_{G/\Gamma}(g) \right)}_{= \langle b^{-1} \cdot \overline{\varphi}_1, \varphi_2 \rangle} dm_G(b).$$

By Howe-Moore Thm,

$$\langle b^{-1} \cdot \overline{\varphi}_1, \varphi_2 \rangle \longrightarrow \int_{G/\Gamma} \overline{\varphi}_1 \cdot \int_{G/\Gamma} \varphi_2 \quad \text{as } b \to \infty.$$

Hence, $\forall \varepsilon > 0 \ \exists T_\varepsilon$:

$$\left| \langle F_T, \varphi \rangle - \mathrm{vol}(B_T) \int_{G/\Gamma} \overline{\varphi}_1 \cdot \int_{G/\Gamma} \varphi_2 \right|$$

$$= \int_{B_T} \left| \langle b^{-1} \varphi_1, \varphi_2 \rangle - \int_{G/\Gamma} \overline{\varphi}_1 \cdot \int_{G/\Gamma} \varphi_2 \right| db$$

$$= \int_{B_{T_\varepsilon}} + \int_{B_T - B_{T_\varepsilon}} \leq c(\varphi_1, \varphi_2) \cdot \mathrm{vol}(B_{T_\varepsilon}) + \varepsilon \cdot \mathrm{vol}(B_T - B_{T_\varepsilon}).$$

$$\overline{\lim_{T \to \infty}} \left| \langle F_T, \varphi \rangle - \mathrm{vol}(B_T) \int_{G/\Gamma} \overline{\varphi}_1 \int_{G/\Gamma} \varphi_2 \right| \Big/ \mathrm{vol}(B_T) \leq \varepsilon \quad \text{for all } \varepsilon > 0.$$

<u>Step 2.</u> (pointwise convergence)

Let $\varphi_1 = \varphi_2 = \sum_{\gamma \in \Gamma} \frac{\chi_{\mathcal{O}_\varepsilon}(g\gamma)}{\mathrm{vol}(\mathcal{O}_\varepsilon)}$. $\int_{G/\Gamma} \varphi_1 = \int_{G/\Gamma} \varphi_2 = 1$.

$$\langle F_T, \varphi \rangle = \int_{G/\Gamma \times G/\Gamma} F_T(g_1, g_2)\, \varphi_1(g_1)\, \varphi_2(g_2)\, dm_{G/\Gamma}(g_1)\, dm_{G/\Gamma}(g_2)$$

$$= \int_{F \times F} \sum_{\gamma_1, \gamma_2 \in \Gamma} F_T(g_1, g_2) \frac{\chi_{\mathcal{O}_\varepsilon}(g_1 \gamma)}{\mathrm{vol}(\mathcal{O}_\varepsilon)} \cdot \frac{\chi_{\mathcal{O}_\varepsilon}(g_2 \gamma)}{\mathrm{vol}(\mathcal{O}_\varepsilon)}\, dm_G(g_1)\, dm_G(g_2)$$

$$= \mathrm{vol}(\mathcal{O}_\varepsilon)^{-2} \int_{\mathcal{O}_\varepsilon \times \mathcal{O}_\varepsilon} F_T(g_1, g_2)\, dm_G(g_1)\, dm_G(g_2).$$

$$\int_{\Theta_\varepsilon\times\Theta_\varepsilon} F_T(g_1,g_2)\,dg_1dg_2 = \sum_{\gamma\in\Gamma}\int_{\Theta_\varepsilon\times\Theta_\varepsilon}\chi_{B_T}(g_1\gamma g_2^{-1})\,dg_1dg_2$$

$$= \sum_{\gamma\in\Gamma}\int_{\Theta_\varepsilon\times\Theta_\varepsilon}\chi_{g_1^{-1}B_Tg_2}(\gamma)\,dg_1dg_2$$

$$\le \sum_{\gamma\in\Gamma}\int_{\Theta_\varepsilon\times\Theta_\varepsilon}\chi_{B_{T(1+2\varepsilon)}}(\gamma)\,dg_1dg_2 = F_{T(1+2\varepsilon)}(e,e)\,\mathrm{vol}(\Theta_\varepsilon)^2.$$

Similarly, $\int_{\Theta_\varepsilon\times\Theta_\varepsilon} F_T(g_1,g_2)\,dg_1dg_2 \ge F_{T(1-2\varepsilon)}(e,e)\cdot\mathrm{vol}(\Theta_\varepsilon)^2$

Hence, $\langle F_{T(1-2\varepsilon)},\varphi\rangle \le F_T(e,e) \le \langle F_{T(1+2\varepsilon)},\varphi\rangle$

$$\overline{\lim}\ \frac{F_T(e,e)}{\mathrm{vol}(B_T)} \le \overline{\lim}\ \frac{\mathrm{vol}(B_{T(1+2\varepsilon)})}{\mathrm{vol}(B_T)}\cdot\overline{\lim}\ \frac{\langle F_{T(1+2\varepsilon)},\varphi\rangle}{\mathrm{vol}(B_{T(1+2\varepsilon)})}$$

$$= (1+2\varepsilon)^{d^2-d}\cdot\int_{G/\Gamma\times G/\Gamma}\varphi = (1+2\varepsilon)^{d^2-d}$$

for every $\varepsilon>0$.

The lower estimate is proved similarly. $\square$