$$\boxed{\text{Lecture 2}}$$

Reduction Theory

Aim: Construct convinient "fundamental domain for $GL_d(\mathbb{R})/GL_d(\mathbb{Z})$.

Recall Iwasawa decomposition $G = KAU$

$$K = O_d(\mathbb{R}),\ A = \underset{a_i > 0}{\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix}},\ U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Def. Siegel set $\Sigma_{t,v} = K A_t U_v$ where

$$A_t = \left\{ a \in A : \frac{a_i}{a_{i+1}} \le t \right\},\ U_v = \left\{ \begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} : |u_{ij}| \le \tfrac{1}{2} \right\}.$$

Thm. For $t \ge \frac{2}{\sqrt{3}}$ and $v \ge \frac{1}{2}$, $G = \Sigma_{t,v} \cdot \Gamma$.

Proof $\{e_i\}$ - standard basis of $\mathbb{R}^d$

$\|\cdot\|$ - Euclidean norm

$\varphi : G \to \mathbb{R}^+ : \quad \varphi(g) = \|g e_1\|$.

$g = kau : \quad \varphi(g) = a_1$.

For $g \in G$, $\varphi(g\Gamma) \subset \| \underbrace{g(\mathbb{Z}e_1 + \dots + \mathbb{Z}e_d)}_{\text{discrete}} \| - \{0\}$

Hence, φ achieves minimum on $g\Gamma$.

Claim: $\exists\, h \in \Sigma_{\frac{2}{\sqrt{3}}, \frac{1}{2}} : \quad \varphi(h) = \min_{x \in g\Gamma} \varphi(x) = M$.

$h \in g\Gamma$

Step 1: $U = U_{\frac{1}{2}}\,(U\cap\Gamma)$.

$$\begin{pmatrix} 1 & \ddots & u_{ij} \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & z_1 & 0 \\ & \ddots & z_{d-1} \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_{i\,i+1}+z_i & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$\exists\, z_1, \dots z_{d-1} \in \mathbb{Z} : \quad u = \tilde{u} \cdot \begin{pmatrix} 1 & z_1 & 0 \\ & \ddots & z_{d-1} \\ & & 1 \end{pmatrix}.$$

Suppose that $u = \begin{pmatrix} 1 & \ddots & u_{ij} \\ & & 1 \end{pmatrix}$ and $|u_{ij}| \leq \frac{1}{2}$ for $j-i \leq \ell$.

$$u \cdot \begin{pmatrix} 1 & z_1 & & 0 \\ & \ddots\ 0 & \ddots & z_{d-(\ell+1)} \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_{ij} & u_{i\,i+\ell+1}+z_i & * \\ & \ddots & & \ddots \\ & & & 1 \end{pmatrix}$$

$$\exists\, z_1, \dots z_{d-(\ell+1)} \in \mathbb{Z} : \quad u_{i,\,i+\ell+1} + z_i \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

$\lrcorner$

Step 2: $g = kau\gamma$, $u \in U_{\frac{1}{2}}$, $\gamma \in U\cap\Gamma$.

$$\varphi(g) = a_1 = \varphi(kau) = M.$$

Let $w = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & \mathrm{id} \end{array}\right) \in \Gamma$.

$$\varphi(kauw) \geq M = a_1$$

$$\parallel$$

$$\|a_1 u_{12} e_1 + a_2 e_2\| = \sqrt{a_1^2 u_{12}^2 + a_2^2} \leq \left(\tfrac{1}{4} a_1^2 + a_2^2\right)^{1/2}$$

$$a_1^2 \leq \tfrac{1}{4} a_1^2 + a_2^2 \Rightarrow \boxed{a_1 \leq \tfrac{2}{\sqrt{3}}\, a_2}$$

Step 3 (induction on d)

$g = kau$. Consider $g' = \begin{pmatrix} a_2 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix} \begin{pmatrix} 1 & u_{23} & * \\ & \ddots & \\ & & 1 \end{pmatrix}$

By induction, $\exists \gamma_1 \in GL_{d-1}(\mathbb{Z})$: $g'\gamma_1 = k'a'u'$ for $k' \in SO_{d-1}(\mathbb{R})$, $a' \in A^{d-1}_{\frac{2}{\sqrt{3}}}$, $u' \in \mathcal{U}^{d-1}_{\frac{1}{2}}$.

$$g = k \left(\begin{array}{c|c} a_1 & 0 \\ \hline 0 & id \end{array}\right) \left(\begin{array}{c|c} 1 & * \\ \hline 0 & id \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g' \end{array}\right).$$

$$g \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \gamma_1 \end{array}\right) = \underbrace{k \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & k' \end{array}\right)}_{k''} \underbrace{\left(\begin{array}{c|c} a_1 & 0 \\ \hline 0 & a' \end{array}\right)}_{a''} u''$$ for some $u'' \in \mathcal{U}$.

By steps 1-2, $\exists \gamma \in \Gamma \cap \mathcal{U}$: $u''\gamma \in \mathcal{U}_{1/2}$,

$$\varphi(k''a''u'') = \varphi(k''a''u''\gamma),$$

$$\frac{a_1''}{a_2''} \leq \frac{2}{\sqrt{3}}.$$

$$\varphi\left(g \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \gamma_1 \end{array}\right) \gamma\right) = \varphi(g) = M.$$

$$g \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \gamma_1 \end{array}\right) \gamma \in \Sigma_{\frac{2}{\sqrt{3}}, \frac{1}{2}}.$$

$GL_2(\mathbb{R})^+$ action on upper half plane

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $z \cdot g = \frac{dz+b}{cz+a}$, $z \in \mathbb{H}^2 = \{z \in \mathbb{C} : Im(z) > 0\}$

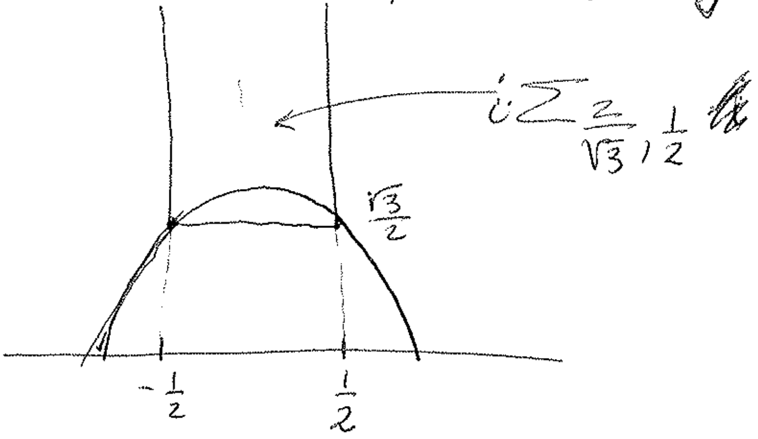
$$i \cdot g = \frac{ab+cd}{a^2+c^2} + \frac{i(ad-bc)}{a^2+c^2}$$

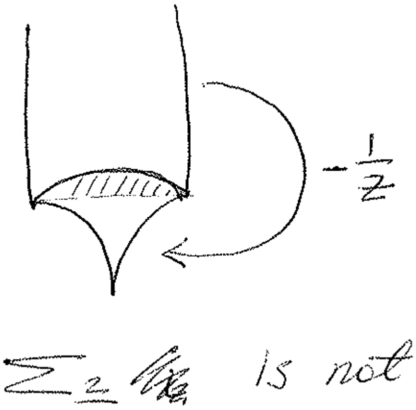
- action is transitive,
- $Stab_{GL_2(\mathbb{R})^+}(i) = \mathbb{R}^+ \cdot O_2(\mathbb{R})^+$

Hence, $\mathbb{H}^2 = \mathbb{R}^+ O_2(\mathbb{R})^+ \backslash GL_2(\mathbb{R})^+$.

Siegel sets: $\Sigma_{t,v} = \left\{ k \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : \begin{matrix} k \in O_2(\mathbb{R})^+ \\ \frac{a_1}{a_2} \le t, \ a_1, a_2 > 0 \\ |u| \le v \end{matrix} \right\}$

$$\Sigma_{t,v} \cdot i = \{x + iy : |x| \le \tfrac{1}{2}, \ y \ge t^{-1}\}.$$





$F = i \cdot \Sigma_{\frac{2}{\sqrt{3}}}$ is not a fundamental domain,

but $\#\{\gamma \in GL_2(\mathbb{Z})^+ : F\gamma \cap F \neq \emptyset\}$ is finite.

Application to quadratic forms.

Thm. $Q(x) = \sum_{i,j} a_{ij} x_i x_j$ – positive definite quadratic form,

Then $\exists x \in \mathbb{Z}^d$, $x \neq 0$, $Q(x) \le \left(\frac{4}{3}\right)^{\frac{d-1}{2}} \cdot \det(Q)^{1/d}$.

$$\exists g \in GL_d(\mathbb{R}): \quad Q(gx) = \|x\|^2.$$

$$g^{-1} = g_1 \gamma, \quad g_1 \in \Sigma_{\frac{2}{\sqrt{3}}, \frac{1}{2}}, \quad \gamma \in GL_d(\mathbb{Z}).$$

$$Q(\gamma^{-1} e_1) = \|g^{-1} \gamma e_1\|^2 = \|g_1 e_1\|^2 = a_1^2$$

$$\det(Q) = \det(g_1)^2 = (a_1 \cdots a_d)^2.$$

$$\frac{a_1^d}{a_1 \cdots a_d} \le \frac{a_1}{a_2} \cdots \frac{a_1}{a_d} \le \frac{2}{\sqrt{3}} \left(\frac{2}{\sqrt{3}}\right)^2 \cdots \left(\frac{2}{\sqrt{3}}\right)^{d-1}$$

$$= \left(\frac{2}{\sqrt{3}}\right)^{\frac{d(d-1)}{2}}.$$

Thm. $\#\{\gamma \in GL_d(\mathbb{Z}) : \Sigma_{t,v}\gamma \cap \Sigma_{t,v} \neq 0\}$ is finite.

Proof. Maybe later.

$\mathcal{L}_d' =$ subspace of lattices with covolume 1.

$$\mathcal{L}_d' \simeq \overbrace{SL_d(\mathbb{R})}^{G} / \overbrace{SL_d(\mathbb{Z})}^{\Gamma}.$$

Iwasawa decomposition: $G = SL_d(\mathbb{R}) = KAU$

$K = SO_d(\mathbb{R})$, $A = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{pmatrix} : \begin{matrix} a_i > 0 \\ a_1 \dots a_d = 1 \end{matrix} \right\}$

$U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}$.

For $t \geq \frac{2}{\sqrt{3}}$, $v \geq \frac{1}{2}$, $G = (\Sigma_{t,v} \cap G) \cdot \Gamma$.

Thm. $\operatorname{vol}(G/\Gamma) < \infty$.

Lemma The invariant measure on $G = GL_d(\mathbb{R})$ with respect to KAU decomposition is given by

$$d\nu(k) \left(\prod_{i<j} \frac{a_j}{a_i} \right) \cdot \prod_i \frac{da_i}{a_i} \prod_{i<j} du_{ij}$$

where ν is invariant measure on K.

Proof Let the measure be $\delta(k,a,v) \cdot d\nu(k) \prod_i \frac{da_i}{a_i} \prod_{i<j} du_{ij}$.

For $f : G \to \mathbb{R}$; $k_0 \in K$, $a_0 \in A$, $u_0 \in \mathcal{U}$:

$$\int_G f(\underbrace{k_0 k a u a_0 u_0}_{\parallel}) \, d\mu(k,a,u) = \int_G f(kau) \, d\mu(k,a,u).$$

$$(k_0 k)(a a_0)(a_0^{-1} u a_0) u_0.$$

$$\int_G f(kau) \, \delta(k_0^{-1}k, a a_0^{-1}, a_0(u u_0^{-1}) a_0^{-1}) \, \mathrm{Jac}(u \mapsto (a_0^{-1} u a_0) u_0)^{-1} \times$$

$$d\nu(k) \prod_i \frac{da_i}{a_i} \prod_{i<j} du_{ij}$$

Since this holds for all test-functions f,

$$\delta(k,a,u) = \delta(k_0^{-1}k,\ a a_0^{-1},\ a_0(u u_0^{-1})a_0^{-1})\cdot \mathrm{Jac}(u \mapsto (a_0^{-1} u a_0) u_0)^{-1}.$$

In particular, $\delta(*,a,u) = \mathrm{const}$

$$\delta(a, u u_0^{-1})\, \mathrm{Jac}(u \mapsto u u_0)^{-1} = \delta(a,u).$$

$$(u\cdot u_0)_{ij} = u_{ij} + (u_0)_{ij} + P(u_{i\,i+1}, \dots u_{i\,j-1})$$

$$\mathrm{Jac}(u \mapsto u\cdot u_0) = 1$$

↖ triangular transformation.

Hence, $\delta(a,*) = \mathrm{const}$.

Finally, $\delta(a_0^{-1}) = \mathrm{Jac}(u \mapsto (a_0^{-1} u a_0)).$

$$= \left(\prod_{i<j} \frac{a_i^0}{a_j^0}\right)^{-1}.$$

Lemma. The invariant measure μ_1 on $G = SL_d(\mathbb{R})$ is given by

$$d\nu_1(k) \left(\prod_{i=1}^{d-1} b_i^{\,i(d-i)}\right) \prod_{i=1}^{d-1} \frac{db_i}{b_i} \cdot \prod_{i<j} du_{ij}.$$

where $b_i = \frac{a_i}{a_{i+1}}$ and ν_1 is invariant measure on $SO_d(\mathbb{R})$.

$b_0 = a_1$, $b_i = \frac{a_i}{a_{i+1}}$. $\prod_{i=1}^{d} \frac{da_i}{a_i} = \prod_{i=0}^{d-1} \frac{db_i}{b_i}$.

Invariant measure μ on $GL_d(\mathbb{R})$:

$$d\nu(k) \left(\prod_{i=1}^{d-1} b^{\,i(d-i)} \cdot \frac{db_i}{b_i}\right) \cdot \frac{db_0}{db_0} \cdot \prod_{i<j} du_{ij}.$$

$$GL_d(\mathbb{R}) = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & \mathrm{id} \end{array}\right) SL_d(\mathbb{R}).$$

$$d\mu(b_0, g) = \frac{db_0}{b_0}\, d\mu_1(g),$$

μ_1 - invariant measure on $SL_d(\mathbb{R})$.

This implies the claim.

Thm. $\mathrm{vol}\left(SL_d(\mathbb{R})/SL_d(\mathbb{Z}) \right) < \infty$

$$\mu_1(\Sigma_{t,v}) = \underbrace{\left(\int_K d\nu_1\right)}_{<\infty}\left(\prod_{i=1}^{d-1}\int_0^t b_i^{\,i(d-i)-1}\,db_i\right)\left(\prod_{i<j}\int_{-\frac{1}{2}}^{1/2} du_{ij}\right)$$

since K is compact.

Rmk. $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$ is not compact.

Howe-Moore Thm.

$G = SL_d(\mathbb{R})$, $\Gamma = SL_d(\mathbb{Z})$.

$$L^2(G/\Gamma) = \{\varphi : G/\Gamma \to \mathbb{C} : \int_{G/\Gamma} |\varphi|^2 d\mu < \infty\}.$$

$$L_0^2(G/\Gamma) = \{\varphi \in L^2(G/\Gamma) : \int_{G/\Gamma} \varphi = 0\}.$$

$$\mathcal{H} = L_0^2(G/\Gamma).$$

$$\langle \varphi_1, \varphi_2 \rangle = \int_{G/\Gamma} \varphi_1 \cdot \overline{\varphi_2}\, d\mu.$$

$\mathcal{H}$ is a Hilbet space with no G-fixed vectors.

Thm. Let $\mathcal{H}$ be a Hilbert space with $\pi : G \to \mathcal{U}(\mathcal{H})$ - unitary representation, and no G-fixed vectors.

Then $\forall v, w \in \mathcal{H}$: $\langle \pi(g)v, w \rangle \xrightarrow[g\to\infty]{} 0$.

ex. $\pi : G \to \mathcal{U}(L^2(G/\Gamma))$:

$$\pi(g)\varphi(x) = \varphi(g^{-1}x).$$

$\pi(g)$ is unitary since μ is invariant.

$$\|\pi(g)\varphi\| = \left(\int_{G/\Gamma} |\varphi(g^{-1}x)|^2 d\mu(x)\right)^{1/2} = \|\varphi\|.$$

COR. $\forall \varphi_1, \varphi_2 \in L^2(G/\Gamma)$: $\int_{G/\Gamma} \varphi_1(gx)\varphi_2(x)\,d\mu(x) \xrightarrow[g\to\infty]{} \int_{G/\Gamma} \varphi_1 \cdot \int_{G/\Gamma} \varphi_2$

Lemma (Cartan decomposition)

$$G = KA^{+}K, \quad K = SO_d(\mathbb{R}), \quad A^{+} = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}_{a_1 \geq a_2 \geq \dots \geq a_d}.$$

Ω_d = space of positive-definite quadratic forms, det=1, in d-variables

$$q_0(x) = \sum_{i=1}^{d} x_i^2 \in \Omega_d.$$

$$\mathrm{Stab}_G(q_0) = K.$$

$q_0(g\cdot x)$ can be diagonalised by orthogonal transformation.

$$\exists k \in K: \ q_0(gkx) = \sum_{i=1}^{d} b_i x_i^2, \quad b_i > 0.$$

$$a = \begin{pmatrix} b_1^{-1/2} & & \\ & \ddots & \\ & & b_d^{-1/2} \end{pmatrix}: \quad q_0(gka\cdot x) = q_0(x).$$

Hence, $gka \in K \Rightarrow g \in KAK$.

Proof of Thm (step 1)

Suppose not: i.e., $\exists g_n \to \infty : u, v \in \mathcal{H}: \ \langle \pi(g_n)u, v\rangle \not\to 0$.

$g_n = k_n a_n k_n'$, $k_n, k_n' \in K$, $a_n \in A$, $a_n \to \infty$

$$\langle \pi(g_n)u, v\rangle = \langle \pi(a_n)\pi(l_n)u, \pi(k_n^{-1})v\rangle.$$

Passing to a subsequence, $\pi(l_n)u \to \tilde{u}$, $\pi(k_n^{-1})v \to \tilde{v}$.

$$\langle \pi(a_n)\pi(l_n)u, \pi(k_n^{-1})v\rangle - \langle \pi(a_n)\tilde{u}, \tilde{v}\rangle$$

$$= \langle \pi(a_n)(\pi(l_n)u - \tilde{u}), \pi(k_n^{-1})v\rangle + \langle \pi(a_n)\tilde{u}, \pi(k_n^{-1})v - \tilde{v}\rangle$$

$\to 0$. Hence, $\langle \pi(a_n)\tilde{u}, \tilde{v}\rangle \not\to 0$.

Weak convergence: $v_n \in \mathcal{H}$, $v_n \to v$ weakly
if $\forall w \in \mathcal{H} : \langle v_n, w \rangle \to \langle v, w \rangle$.

Alouglu Thm: Bounded sets in $\mathcal{H}$ are weakly compact.

Proof of Thm ($G = SL_2(\mathbb{R})$).

$$a_n = \begin{pmatrix} t_n & \\ & t_n^{-1} \end{pmatrix} \to \infty, \text{ say, } t_n \to \infty$$

$$\langle a_n u, v \rangle \not\to 0.$$

$$u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

By weak compactness, $a_n u \to \tilde{u}$, after passing to a subsequence.

then $\langle a_n u, v \rangle \to \langle \tilde{u}, v \rangle \neq 0$.

In particular, $\tilde{u} \neq 0$.

$$\pi(u_s) \cdot \tilde{u} = \text{w-lim}\, \pi(u_s a_n) u = \text{w-lim}\, \pi(a_n u_{s/t_n^2}) u$$

$$\| \pi(a_n)\pi(u_{s/t_n^2}) u - \pi(a_n) u \| \to 0$$

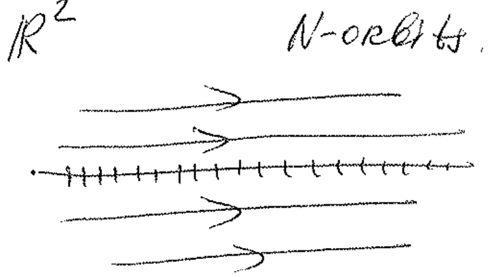
$$= \text{w-lim}\, \pi(a_n) u = \tilde{u}.$$

Hence, $\tilde{u}$ is fixed by $\pi(\underset{N}{u_s})$.

$\varphi(g) = \langle \pi(g)\tilde{u}, \tilde{u} \rangle$, φ is $\overset{N}{\{u_s\}}$-bi-invariant.

$\varphi : \underbrace{G/N}_{\simeq \mathbb{R}^2 - \{0\}} \to \mathbb{C}$, $G \curvearrowright \mathbb{R}^2 - \{0\}$: $\mathrm{Stab}(e_1) = \{u_s\} = N$.

$\varphi : \mathbb{R}^2 - \{0\} \to \mathbb{C}$ – N-invariant



$\varphi = \text{const}$ on $y = c$, $c \neq 0$.

By continuity, $\varphi = \text{const}$, $y = 0$.

$$\langle \pi(a_t)\tilde{u}, \tilde{u} \rangle = \varphi(a_t \cdot e_1) = \varphi(t e_1) = \varphi(e_1) = \|\tilde{u}\|^2.$$

Equality in Cauchy-Schwartz $\Longrightarrow \pi(a_t)\tilde{u} = \theta_t \cdot \tilde{u}$ and $\theta_t = 1$.

Now φ is (AN)-bi-invariant.

$AN \curvearrowright \mathbb{R}^2 - \{0\}$ has dense orbit $\Longrightarrow \varphi = \text{const}$.

That is, $\langle \pi(g)\tilde{u}, \tilde{u} \rangle = \|\tilde{u}\|^2$.

By Cauchy-Schwartz again $\Rightarrow$ $\pi(g)\tilde{u} = \tilde{u}$.

Contradiction.

Proof of Thm ($G = SL_d(\mathbb{R})$).

Let $\alpha_i(a) = \frac{a_i}{a_{i+1}}$, $a = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix}$.

If $a_n \to \infty$, then $\exists i : \alpha_i(a_n) \to \infty$, along a subsequence.

Indeed, suppose not, i.e., $\alpha_i(a_n) \le t$.

$$\begin{cases} a_1 \le t a_2 \le \dots \le t a_d \\ a_1 \ge a_2 \ge \dots \ge a_d \\ a_1 \dots a_d = 1 \end{cases} \Rightarrow \begin{array}{l} a_d \le 1. \\ a_1 \ge 1. \end{array}$$

We get bounded set

$$U_i = \left(\begin{array}{c|c} id & * \\ \hline & id \end{array} \right) \}k : \quad \frac{(a_n)_{i'}}{(a_n)_{j'}} \ge \frac{(a_n)_i}{(a_n)_{i+1}} \to \infty.$$

for $1 \le i' \le k$, $k+1 \le j \le d$.

We argue as in the case $G = SL_2(\mathbb{R})$, by contradiction.

$\exists u, v \in \mathcal{H}$: $a_n \in A^+$, $a_n \to \infty$: $\langle \pi(a_n)u, v \rangle \not\to 0$.

Then $\alpha_k(a_n) \to \infty$ for some k (after passing to a subsequence)

Then, as in SL_2 case, $\exists \tilde{u} \in \mathcal{H}$, $\tilde{u} \ne 0$,

$U_i \cdot \tilde{u} = \tilde{u}$.

Now we consider subgroups $G_{ij} = \begin{array}{c|ccccc} & & i & & j & \\ \hline i & 0 & * & 0 & * & 0 \\ j & & * & & * & \end{array} \simeq SL_2(\mathbb{R})$.

By SL_2-case, $\tilde{u}$ is fixed by G_{ij} for $1 \le i \le k$, $k+1 \le j \le d$.

Finally, G_{ij} generates G.

Hence, $G\tilde{u} = \tilde{u}$. Contradiction.